Far East Journal of Mathematical Sciences (FJMS)
Volume 33, Issue 1, 2009, Pages 33-40
Published Online: April 16, 2009
This paper is available online at http://www.pphmj.com

ON C_2^t-CONSTRUCTION

YOUNGKWON SONG

Department of Mathematics
Research Institute of Basic Science
Kwangwoon University
Seoul 139-701, Korea

Abstract

Let (R, m, k) be a local commutative k-subalgebra of $M_n(k)$ with nilpotent maximal ideal m and residue class field k. In this paper, we introduce an equivalent condition for R to be an algebra of the C_2^t-construction which produces an algebra in $MC_{n+t}(k)$ from an algebra in $MC_n(k)$.

1. Introduction

In this paper, k denotes an arbitrary field and (R, m, k) denotes a local commutative k-subalgebra of $M_n(k)$ with nilpotent maximal ideal m and residue class field k. We denote the set of all local maximal commutative k-subalgebras of $M_n(k)$ by $MC_n(k)$.

Brown and Call [1] introduced C_1-construction and Brown [2] introduced C_2-construction.

2000 Mathematics Subject Classification: 15A27, 15A33, 13M05.

Keywords and phrases: C_1-construction, C_2-construction, C_2^t-construction.

The present research has been conducted by the Research Grant of Kwangwoon University in 2008.

Received November 14, 2008

In [8], C_2^t-construction is introduced which is useful to construct an algebra R in $MC_{n+t}(k)$ from an algebra B in $MC_n(k)$. Using the C_2^t-construction, an algebra B in $MC_n(k)$ with $\dim_k(B) = s$ can be embedded in an algebra R in $MC_{n+t}(k)$ with $\dim_k(R) = s + t$. Moreover, if $s < n$, then we can construct infinitely many algebras R in $MC_{n+t}(k)$ whose dimensions are less than the size of the matrix.

In this paper, we shall introduce an equivalent condition to be an algebra of the C_2^t-construction.

Furthermore, we shall show the relation between C_2^t-construction and C_i-construction for $i = 1, 2$.

2. Theorems Prerequisite to the Main Results

A commutative k-algebra R is a C_1-construction if R has an ideal I satisfying the equivalence condition in the following theorem.

Theorem 2.1 [1]**.** *Let (R, m, k) be a commutative k-algebra. Then R is a C_1-construction if and only if there is an ideal I satisfying the following conditions*:

(1) $Ann_R(I) = I$,

(2) $0 \to I \to R \to R/I \to 0$ *splits as k-algebras.*

Theorem 2.2 [2, 3]**.** *Let (B, m_B, k) be a finite dimensional commutative k-algebra with identity and N be a finitely generated faithful B-module. Suppose B is isomorphic to $Hom_B(N, N)$ via the regular representation. Then there exists an element $w \in soc(B)$ with $\dim_k(Nw) = 1$.*

Theorem 2.3 is an equivalent condition for a k-algebra R to be an algebra of the C_2-construction. The proof can be found in [3].

Theorem 2.3 [3]**.** *Let (R, m, k) be a finite dimensional commutative k-algebra with identity. Then R is a C_2-construction if and only if R*

contains a k-subalgebra (B, m_B, k) *and an element* $x \in m$ *satisfying the following conditions*:

(1) $0 \neq x^p \in soc(B)$ *for some positive integer* $p > 1$,

(2) $m_B x = (0)$,

(3) $\dim_k(R) = \dim_k(B) + (p-1)$.

The k-algebra R of the following Theorem 2.4 is called a C_2^t-*construction* that can be found in [8].

Theorem 2.4 [8]**.** *Let* (B, m_B, k) *be a finite dimensional commutative k-algebra with identity. Let N be a finitely generated faithful B-module of dimension n. Suppose B is isomorphic to* $Hom_B(N, N)$ *via the regular representation. Let t be a positive integer and*

$$R = B[X_1, X_2, ..., X_t]/I,$$

where I is an ideal generated by the following:

$$m_B X_1, ..., m_B X_t, X_1^2 - w, ..., X_t^2 - w, X_i X_j \ (1 \leq i \neq j \leq t).$$

Here, $w \in soc(B) - \{0\}$ with $\dim_k(Nw) = 1$ in Theorem 2.2. If we let $M = N \oplus (\oplus_{i=1}^t) Nw$, then the k-algebra R is isomorphic to $Hom_R(M, M)$ via the regular representation. In other words, R is isomorphic to a maximal commutative subalgebra of $M_{n+t}(k)$, where $\dim_k(M) = n + t$.

3. C_2^t-construction

The following theorem is the main result of this paper which is an equivalent condition to be a C_2^t-construction.

Theorem 3.1. *Let* (R, m, k) *be a finite dimensional local commutative algebra and t be a positive integer. Then R is a* C_2^t-*construction if and only if there exist a commutative subalgebra* (B, m_B, k) *of R and elements* $x_i \in m$, $i = 1, 2, ..., t$ *satisfying the following properties*:

(1) $x_i^2 = x_j^2 \in soc(B) - \{0\}$ *for all* $1 \leq i,\ j \leq t$,

(2) $x_i x_j = 0$ *for all* $1 \leq i \neq j \leq t$,

(3) $m_B x_i = (0)$ *for all* $1 \leq i \leq t$,

(4) $\dim_k(R) = \dim_k(B) + t$.

Proof. Suppose R is a C_2^t-construction. Then, by the definition of C_2^t-construction, R has a commutative subalgebra (B, m_B, k) and elements $x_i \in m$ satisfying the four conditions (1), (2), (3) and (4).

Conversely, suppose there exist a subalgebra B and elements $x_i \in m$ such that the four conditions are satisfied. Let $x_i^2 = w \in soc(B)$ and I be the ideal generated by the following elements:

$$m_B X_1, \ldots, m_B X_t, X_1^2 - w, \ldots, X_t^2 - w, X_i X_j \ (1 \leq i \neq j \leq t).$$

Define a map

$$\psi : B[X_1, X_2, \ldots, X_t]/I \to R$$

by

$$\psi(b + I) = b, \quad \psi(X_i + I) = x_i, \quad 1 \leq i \neq j \leq t,$$

where $b \in B$. Then ψ is a k-algebra homomorphism. Suppose $\psi(a + a_1 X_1 + a_2 X_2 + \cdots + a_t X_t + I) = 0$. Then $a + a_1 x_1 + \cdots + a_t x_t = 0$. Here, we may assume $a_i \in k$ since $m_B x_i = (0)$ for all $i = 1, 2, \ldots, t$. Assume $a \neq 0$. Then $a \notin m$. If $a \in m$, then for x_j

$$a x_j = 0, \quad a_i x_i x_j = 0, \quad i = 1, 2, \ldots, t.$$

Since $x_j^2 = w$ and $0 = a x_j + a_1 x_1 x_j + \cdots + a_j x_j^2 + \cdots + a_t x_t x_j$, we have $a_j w = 0$. Thus, we should have $a_j = 0$, and so, $a_i = 0$, for all $i = 1, 2, \ldots, t$. But then $a = 0$ which is impossible. Thus, $a \notin m$ and hence $a + a_1 x_1 + a_2 x_2 + \cdots + a_t x_t$ is a unit which is impossible. Thus, we have $a = 0$. If $a_j \neq 0$ for some j, then

$$(a_j^{-1} a_1) x_1 + (a_j^{-1} a_2) x_2 + \cdots + (a_j^{-1} a_t) x_t = 0.$$

By multiplying x_j each side, we get

$$0 = (a_j^{-1}a_1)x_1x_j + (a_j^{-1}a_2)x_2x_j + \cdots + (a_j^{-1}a_t)x_tx_j = x_j^2 = w$$

which is impossible and so $a_j = 0$ for all $j = 1, 2, ..., t$. This implies ψ is monomorphism. Note that

$$\dim_k(im(\psi)) = \dim_k(B[x_1, x_2, ..., x_t]) = \dim_k(B) + t = \dim_k(R).$$

Therefore, ψ is an isomorphism and we can conclude that the algebra R is a C_2^t-construction.

Here, we have an example of C_2^t-construction. We shall let E_{ij} be the (i, j)-th matrix unit.

Example 3.2. Let $R = m \oplus kI_{t+2}$ be a k-algebra in $MC_{t+2}(k)$ such that $r \in m$ is of the following form:

$$r = a_1(E_{21} + E_{t'2}) + a_2(E_{31} + E_{t'3}) + \cdots + a_t(E_{t''1} + E_{t't''}) + cE_{t'1},$$

where $a_i, c \in k$ for $i = 1, 2, ..., t$ and $t' = t + 2, t'' = t + 1$.

If we let $B = k[E_{t'1}]$, then $soc(B) = kE_{t'1} = m_B$. Thus, the elements

$$x_{i-1} = E_{i1} + E_{t'i}, \quad i = 2, 3, ..., t + 1$$

satisfy the conditions in Theorem 3.1 and so R is a C_2^t-construction.

The socle and the index of nilpotency of R and B in Theorem 3.1 have the following relations:

Corollary 3.3. *If R and B are k-algebras in Theorem* 3.1, *then* $soc(R) = soc(B)$ *and* $i(m) = i(m_B) + 1$.

Now, we want to prove the relation between C_1-construction, C_2-construction and C_2^t-construction.

Corollary 3.4. *C_1-construction does not imply C_2^t-construction.*

Proof. Let $R = m \oplus kI_{t+1}$ be a k-algebra in $MC_{t+1}(k)$ such that the element $r \in m$ is of the following form:

$$r = a_1E_{12} + a_2E_{13} + \cdots + a_tE_{1(t+1)},$$

where $a_i \in k, i = 1, 2, ..., t$.

Then $m^2 = (0)$ and so, the algebra R is a C_1-construction. But, the algebra R has no element whose square is not zero and hence R cannot be a C_2^t-construction by Theorem 3.1.

Corollary 3.5. *C_2^t-construction does not imply C_1-construction.*

Proof. Let k be the real number field and $R = m \oplus kI_{t+2}$ be a k-algebra in Example 3.2. Then, R is a C_2^t-construction. Suppose R is a C_1-construction. Then there exists an ideal I of R such that $Ann_R(I) = I$ by Theorem 2.1. If we let $r \in Ann_R(I)$, then for some real numbers a_i, the element r is of the following form:

$$r = a_1(E_{21} + E_{t'2}) + a_2(E_{31} + E_{t'3}) + \cdots + a_t(E_{t''1} + E_{t't''}) + aE_{t'1},$$

where $t' = t + 2$, $t'' = t + 1$. Since $Ann_R(I) = I$, we have

$$0 = r^2 = \sum_{i=1}^{t} a_i^2 E_{t'1}$$

and hence $a_i = 0$ for all $i = 1, 2, \ldots, t$. Thus, $r = aE_{t'1}$ and so $I = Ann_R(I) = kE_{t'1}$. But, then $E_{21} + E_{t'2} \in Ann_R(I) = I$ which is impossible. Thus, the algebra R in Example 3.2 is a C_2^t-construction but not a C_1-construction.

Corollary 3.6. *C_2-construction does not imply C_2^t-construction.*

Proof. Let k be the real number field and $R = m \oplus kI_{t+2}$ be a k-algebra in $MC_{t+2}(k)$ such that $r \in m$ is of the following form:

$$r = a_1(E_{21} + \cdots + E_{t''t}) + a_2(E_{31} + \cdots + E_{t''t-1}) + \cdots + a_t E_{t''1} + a_{t+1}E_{t'1},$$

where $a_i \in k$ for all $i = 1, 2, \ldots, t$ and $t' = t + 2$, $t'' = t + 1$.

Now, let

$$B = k[E_{t''1}, E_{t'1}].$$

Then for an element $r = E_{21} + E_{32} + \cdots + E_{t''t}$ in m, we have the following properties:

(1) $E_{t''1} = r^t \in soc(B)$.

(2) $rm_B = (0)$.

(3) $\dim_k(R) = \dim_k(B) + (t-1)$.

This implies R is a C_2-construction.

Now, suppose R is a C_2^t-construction. Then R contains a k-subalgebra B such that for some $x_i \in m$,

(1) $x_i^2 \in soc(B) - \{0\}$, $i = 1, 2, \ldots, t$,

(2) $x_i x_j = 0$, for all $1 \le i \ne j \le t$.

For some $a_{ij} \in k$, the elements $x_i \in m$ can be written as follows:

$$
\begin{aligned}
x_1 &= a_{11}(E_{21} + \cdots + E_{t''t}) + a_{12}(E_{31} + \cdots + E_{t''t-1}) + \cdots + a_{1t}E_{t''1} + a_{1t''}E_{t'1},\\
x_2 &= a_{21}(E_{21} + \cdots + E_{t''t}) + a_{22}(E_{31} + \cdots + E_{t''t-1}) + \cdots + a_{2t}E_{t''1} + a_{2t''}E_{t'1},\\
&\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots\\
x_t &= a_{t1}(E_{21} + \cdots + E_{t''t}) + a_{t2}(E_{31} + \cdots + E_{t''t-1}) + \cdots + a_{tt}E_{t''1} + a_{tt''}E_{t'1}.
\end{aligned}
$$

Then for all i we have the following identity:

$$x_i^2 = a_{i1}^2(E_{31} + \cdots + E_{t''t-1}) + a_{i1}a_{i2}(E_{41} + \cdots + E_{t''t-2}) + \cdots + a_{i1}a_{it-1}E_{t''1}.$$

Especially, for $i = 1$, we have

$$x_1^2 = a_{11}^2(E_{31} + \cdots + E_{t''t-1}) + a_{11}a_{12}(E_{41} + \cdots + E_{t''t-2}) + \cdots + a_{11}a_{1t-1}E_{t''1}.$$

Since $x_1^2 \ne 0$, there exists some j with $1 \le j \le t-1$ such that $a_{11}a_{1j} \ne 0$. That is, $a_{11} \ne 0$. Moreover,

$$x_1x_2 = a_{11}a_{21}(E_{31} + \cdots + E_{t''t-1}) + \cdots + a_{11}a_{2t-1}E_{t''1}.$$

Since $x_1x_2 = 0$, we should have $a_{11}a_{2\ell} = 0$ for all ℓ with $1 \le \ell \le t-1$. Furthermore, $a_{11} \ne 0$ implies $a_{2\ell} = 0$ for all ℓ with $1 \le \ell \le t-1$. But then, $x_2 = a_{2t}E_{t''1} + a_{2t''}E_{t'1}$ and so $x_2^2 = 0$ which is impossible and so we can conclude that R is not a C_2^t-construction. Therefore, C_2-construction does not imply a C_2^t-construction.

Corollary 3.7. *C_2^t-construction implies C_2-construction.*

Proof. Obvious by the definition.

References

[1] W. C. Brown and F. W. Call, Maximal commutative subalgebras of $n \times n$ matrices, Communications in Algebra 21(12) (1993), 4439-4460.

[2] W. C. Brown, Two constructions of maximal commutative subalgebras of $n \times n$ matrices, Communications in Algebra 22(10) (1994), 4051-4066.

[3] W. C. Brown, Constructing maximal commutative subalgebras of matrix rings in small dimensions, Communications in Algebra 25(12) (1997), 3923-3946.

[4] R. C. Courter, The dimension of maximal commutative subalgebras of K_n, Duke Math. J. 32 (1965), 225-232.

[5] T. J. Laffey, The minimal dimension of maximal commutative subalgebras of full matrix algebras, Linear Algebra and its Applications 71 (1985), 199-212.

[6] D. A. Suprunenko and R. I. Tyshkevich, Commutative Matrices, Academic Press, 1968.

[7] Youngkwon Song, Maximal commutative subalgebras of matrix algebras, Communications in Algebra 27(4) (1999), 1649-1663.

[8] Youngkwon Song, A construction in $MC_n(k)$, Far East J. Math. Sci. (FJMS) 25(3) (2007), 585-592.