Far East Journal of Mathematical Sciences (FJMS)
Volume 33, Issue 1, 2009, Pages 1-17
Published Online: April 16, 2009
This paper is available online at http://www.pphmj.com

SEMI-$w\Delta$ SPACES AND METRIZATION

(*To the Memory of My Mother*)

G. R. HIREMATH

Department of Mathematics and Computer Science
University of North Carolina Pembroke
Pembroke, NC 28372, U. S. A.

Abstract

The semi-$w\Delta$ spaces are defined in terms of semi-open covers and they generalize simultaneously the $w\Delta$ spaces and the semi-developable spaces. We prove the following results in this paper:

- A topological space is a semi-$w\Delta$ space if and only if it is a q-space and a β-space.
- The semi-$w\Delta$ property is invariant under continuous, finite to one, pseudo-open maps.
- An isocompact semi-$w\Delta$ space with a semi-(C_2) property is θ-refinable.
- A topological space is semi-metrizable if and only if it is a semi-developable space with a quasi (α_1)-diagonal.
- A topological space is a Hausdorff semi-metrizable space if and only if it is a semi-$w\Delta$ space with an (α_2)-diagonal.
- A topological space is a regular semi-metrizable space if and only if it is a semi-$w\Delta$ space with a semi-(C_2) property and a quasi (α_1)-diagonal.

The following results are corollaries of the main results about the semi-$w\Delta$ spaces:

2000 Mathematics Subject Classification: 54E25, 54E30, 54E35.

Keywords and phrases: semi-$w\Delta$ spaces, semimetric spaces, Moore spaces, wM spaces, metrization.

Received April 13, 2005

- A topological space is a Moore space if and only if it is a $w\Delta$-space with a semi-(C_2) property and a quasi (α_1)-diagonal.

- A regular topological space is a Moore space if and only if it is a $w\Delta$-space with an (α_2)-diagonal.

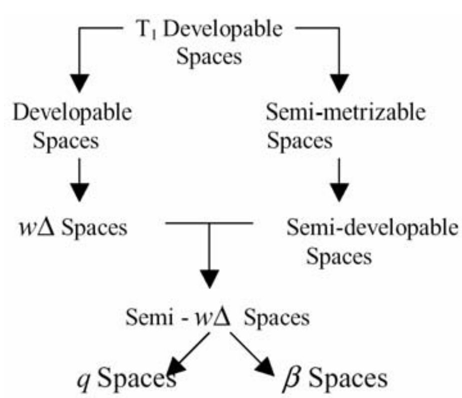
- A topological space is a metrizable space if and only if it is a wM space with a semi-(C_2) property and a quasi (α_1)-diagonal.

- A topological space is a metrizable space if and only if it is a wM space with an (α_2)-diagonal.

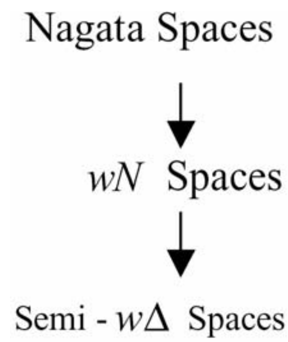
0. Introduction

By a space X we mean a topological space X without assuming any separation axiom it satisfies unless specifically mentioned. Let X be a space and $A \subset X$. Let A^0 denote the interior of A in X and $\overline{A}$ denote the closure of A in X. We use the standard notations and definitions in [6]. Let N denote the set of natural numbers. Let ξ and η be collections of subsets of a space X. Let $\xi^* = \bigcup\xi$. ξ is called a *refinement* of η (written $\xi \prec \eta$), if $\xi^* = \eta^*$ and for each $O \in \xi$, there is a $G \in \eta$ such that $O \subset G$. Let $\{\xi_n\}_n$ be a sequence of covers of a space X. $\{\xi_n\}_n$ is called a *decreasing sequence* provided $\xi_{n+1} \prec \xi_n$ for each $n \in N$. Let ξ be a collection of subsets of a space X. Let $St(x, \xi) = \bigcup\{G \in \xi \,|\, x \in G\}$. ξ is called a *semi-open collection* in X provided $x \in St(x, \xi)^0$ for each $x \in \xi^*$. A space X is called a *semi-$w\Delta$ space* provided it admits a sequence of semi-open covers $\{\xi_n\}_n$ such that if $x_n \in St(x, \xi_n)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters [12]. The sequence $\{\xi_n\}_n$ is called a *semi-$w\Delta$ sequence* and without loss of generality we may assume that $\{\xi_n\}_n$ is a decreasing sequence. A space X is called a *$w\Delta$ space* provided it admits a sequence of open covers $\{\xi_n\}_n$ such that if $x_n \in St(x, \xi_n)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters [4]. The sequence $\{\xi_n\}_n$ is called a *$w\Delta$ sequence* and without loss of generality we may assume that $\{\xi_n\}_n$ is a decreasing sequence. A space X is called a *semi-developable* *space* provided it admits a sequence of semi-open covers $\{\xi_n\}_n$ such that for each $x \in X$, the collection $\{St(x, \xi_n) | n \in N\}$ is a neighborhood (in short, nhd) base at x [1]. The sequence $\{\xi_n\}_n$ is called a *semi-development* and may be assumed to be decreasing. The class of semi-$w\Delta$ spaces includes the $w\Delta$ spaces and the semi-developable spaces. A space X is said to have a *semi-C_2 property* (resp. C_2), if every semi-open cover (resp. open cover) ξ admits a sequence of semi-open covers (resp. open covers) $\{\xi_n\}_n$ such that for each $x \in X$ there is $n_x \in N$ such that $\overline{St(x, \xi_{n_x})} \subset St(x, \xi)$. The sequence $\{\xi_n\}_n$ is called a *semi-C_2* (resp. C_2) *refinement* for ξ and without loss of generality we may assume that the sequence $\{\xi_n\}_n$ is decreasing [8, 11]. A space X is called a *q space* provided each $x \in X$ admits a sequence of its open neighborhoods $\{q_n(x)\}_n$ such that if $x_n \in q_n(x)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters in X [20]. The sequence $\{q_n(x)\}_n$ is called a *q sequence* of x and may be assumed to be decreasing. A space X is called a *β space* provided each $x \in X$ admits a sequence of its open neighborhoods $\{\beta_n(x)\}_n$ such that if $x \in \beta_n(x_n)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters in X [14]. The sequence $\{\beta_n(x)\}_n$ is called a *β sequence* and may be assumed to be a decreasing sequence. A space X is called an *isocompact space* provided the closed countably compact sets in X are compact in X [3]. A space X has a G_δ diagonal if and only if the space X admits a sequence of open covers $\{\xi_n\}_n$ such that $\bigcap_{n=1}^{\infty} St(x, \xi_n) = \{x\}$ for each $x \in X$ [5]. In a similar manner, we define the concept of α_i diagonal for $i = 1, 2$ in terms of semi-open covers. A space X is said to have an α_1- (resp. α_2-) *diagonal* provided it admits a sequence of semi-open covers $\{\xi_n\}_n$ such that $\bigcap_{n=1}^{\infty} St(x, \xi_n) = \{x\}$ $\left(\text{resp. } \bigcap_{n=1}^{\infty} \overline{St(x, \xi_n)} = \{x\}\right)$ for each $x \in X$ [7]. The sequence $\{\xi_n\}_n$ is called an α_i *diagonal sequence* for X and may be assumed to be a decreasing sequence. The α_1-diagonal property generalizes both the α_2-diagonal and the G_δ diagonal which are different generalizations of G_δ^* diagonal [14]. A space X is said to have a *quasi* G_δ *diagonal* provided it admits a sequence of open collections $\{\xi_n\}_n$ in X such that $\bigcap\{St(x, \xi_n) \mid n \in N(x)\} = \{x\}$, where $N(x) = \{n \mid x \in \xi_n^*\}$, for each $x \in X$ [16]. The sequence $\{\xi_n\}_n$ is called a *quasi* G_δ *diagonal sequence*. The quasi G_δ diagonal property generalizes the G_δ diagonal property. A space X is said to have a *quasi* α_1*-diagonal* provided it admits a sequence of semi-open collections $\{\xi_n\}_n$ such that $\bigcap\{St(x, \xi_n) \mid n \in N(x)\} = \{x\}$, where $N(x) = \{n \mid x \in \xi_n^*\}$, for each $x \in X$. The quasi α_1-diagonal generalizes the α_1-diagonal property as well as the quasi G_δ diagonal property, but the quasi G_δ diagonal property and an α_1-diagonal property are different generalizations of the G_δ diagonal property. A space X is called a *wM space* provided it admits a sequence of open covers $\{\xi_n\}_n$ such that if $x_n \in St^2(x, \xi_n)$ for each $n \in N$, where $St^2(x, \xi_n) = St(St(x, \xi_n), \xi_n)$, then the sequence $\{x_n\}_n$ clusters [17]. The definitions of the Nagata spaces and the wN spaces are available in [15]. A map $f : X \to Y$ is called a *pseudo-open map* if for each $y \in Y$, whenever G is a nhd of $f^{-1}(y)$, $f(G)$ is a nhd of y [2].

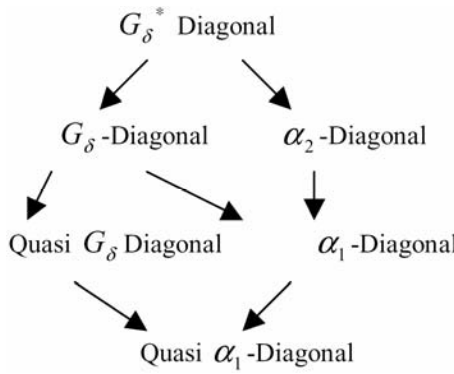
Implication Diagram of Developable Spaces



Implication Diagram of Nagata Spaces



Implication Diagram of Diagonal Properties



1. Main Section

1.1. Semi-$w\Delta$ spaces

Lemma 1.1.1. *A space X is a semi-$w\Delta$ space if and only if it admits a sequence of semi-open covers* $\{\xi_n\}_n$ *such that if* $x \in St(x_n, \xi_n)$ *for each* $n \in N$, *then the sequence* $\{x_n\}_n$ *clusters.*

Theorem 1.1.2. *A space X is a semi-$w\Delta$ space if and only if it is a q space as well as a* β *space.*

Proof. Suppose X is a semi-$w\Delta$ space and $\{\xi_n\}_n$ is a semi-$w\Delta$ sequence for X. Define $g_n(x) = St(x, \xi_n)^0$ for each $x \in X$ and each $n \in N$. Assign the sequence $\{g_n(x)\}_n$ to x for each $x \in X$. From the definition of a semi-$w\Delta$ space and Lemma 1.1.1, it follows that if $x_n \in g_n(x)$ for each $n \in N$ or if $x \in g_n(x_n)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters. Therefore X is a q space as well as a β space.

Conversely, suppose X is a q space as well as a β space. Let $\{q_n(x)\}_n$ be a decreasing q assignment to x and $\{\beta_n(x)\}_n$ be a decreasing β assignment to x for each $x \in X$. Let $h_n(x) = q_n(x) \cap \beta_n(x)$ for each $x \in X$ and each $n \in N$. Then, if $x_n \in h_n(x)$ for each $n \in N$ or if $x \in h_n(x_n)$ for each $n \in N$, the sequence $\{x_n\}_n$ clusters. Let $S_n(x) = \{\{x, y\} \mid y \in h_n(x)\}$ for each $x \in X$ and each $n \in N$. Let $\xi_n = \bigcup\{S_n(x) \mid x \in X\}$ for each $n \in N$. Since $h_n(x) = \cup S_n(x) \subset St(x, \xi_n)$ for each $x \in X$ and each $n \in N$, each ξ_n is a semi-open cover. If $x_n \in St(x, \xi_n)$ for each $n \in N$, then either $x_n \in S_n(x)$ or $x \in S_n(x_n)$. It follows that $x_n \in h_n(x)$ for infinitely many $n \in N$ or $x \in h_n(x_n)$ for infinitely many $n \in N$. Since $\{h_n(x)\}$ is decreasing, it follows that the sequence $\{x_n\}_n$ clusters. Therefore $\{\xi_n\}_n$ is a semi-$w\Delta$ sequence for X.

Corollary 1.1.2.1. *A space X is a semi-$w\Delta$ space if and only if it admits a map g on $N \times X$ into the topology on X such that if $x_n \in g(n, x)$ for each $n \in N$ or if $x \in g(n, x_n)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters.*

Lemma 1.1.3 [11]**.** *Let a space X have a semi-C_2 property. Then every sequence of semi-open covers $\{\xi_n\}_n$ admits a sequence of semi-open covers $\{\eta_n\}_n$ such that*

(1) $\eta_{n+1} \prec \eta_n \prec \xi_n$ *for each n and*

(2) *for $x \in X$, $n \in N$ there is an m such that* $\overline{St(x, \eta_m)} \subset St(x, \eta_n)$.

We call the sequence $\{\eta_n\}_n$ a semi-C_2 regular refinement of the sequence $\{\xi_n\}_n$.

Proof. This is Lemma 9 in [11].

Theorem 1.1.4. *An isocompact semi-$w\Delta$ space X with a semi-C_2 property is θ-refinable.*

Proof. Let ξ be an open cover of X. Let $\{\xi_n\}_n$ be a decreasing semi-$w\Delta$ sequence such that $\xi_1 \prec \xi$. By Lemma 1.1.3, there is a semi-C_2 regular refinement $\{\eta_n\}_n$ for $\{\xi_n\}_n$. Define $C(x) = \bigcap_{n=1}^{\infty} St(x, \eta_n)$ for $x \in X$. Then $C(x) = \bigcap_{n=1}^{\infty} \overline{St(x, \eta_n)} = \bigcap_{n=1}^{\infty} St(x, \eta_n)$ for $x \in X$. It follows that $C(x)$ is a closed countably compact set and hence, by the isocompactness of X, $C(x)$ is a compact set in X. Also, $\{St(x, \eta_n) | n \in N\}$ is a nhd base for $C(x)$. For, let G be an open nhd for $C(x)$. Suppose there is $x_n \in St(x, \eta_n) - G$ for each $n \in N$. Then the sequence $\{x_n\}_n$ clusters in $C(x)$ and hence $x_n \in G$ for some n, a contradiction. Since $C(x) \subset St(x, \eta_n) \subset St(x, \xi)$ for some $n \in N$ and $C(x)$ is a compact set, it follows that there is a finite subcollection $\xi(x)$ of ξ such that $C(x) \subset \cup\xi(x)$ and $x \in \cap\xi(x)$. Therefore there is an n such that $St(x, \eta_n) \subset \cup\xi(x)$. Thus, for any given open cover ξ of X there is a sequence of semi-open covers $\{\eta_n\}_n$ such that for each $x \in X$ there exists n_x such that $St(x, \eta_{n_x}) \subset \bigcup \xi(x)$ for some finite $\xi(x) \subset \xi$ such that $x \in \bigcap \xi(x)$. By Theorem 3.2 of Junnila in [18], X is θ-refinable.

Lemma 1.1.5 [13]**.** *A first countable space X is a T_1 space if and only if it has an α_1-diagonal.*

Lemma 1.1.6. *A space X is first countable if and only if each $x \in X$ is assigned a sequence of its open neighborhoods $\{G_n(x)\}_n$ such that if $x_n \in G_n(x)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters at x.*

Theorem 1.1.7 [13]**.** *A regular space X is a T_1 first countable space if and only if it is a q space with a quasi α_1-diagonal.*

Lemma 1.1.8. *Let X be a T_1 semi-$w\Delta$ space that satisfies* $(*)$: *Each countable closed discrete set D admits a locally finite open collection* $\{G_d \mid d \in G_d,\ d \in D\}$. *Then X admits a semi-$w\Delta$ sequence $\{\xi_n\}_n$ such that*

(1) *if $x_n \in \overline{St(x, \xi_n)}$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters*;

(2) *if* $C(x) = \bigcap_{n=1}^{\infty} \overline{St(x, \xi_n)}$, *then $C(x)$ is countably compact and* $\{\overline{St(x, \xi_n)} \mid n \in N\}$ *is a network at $C(x)$.*

Proof. Let $\{\xi_n\}_n$ be a decreasing semi-$w\Delta$ sequence. To prove (1), suppose $x_n \in \overline{St(x, \xi_n)}$ for each $n \in N$ and the sequence $\{x_n\}_n$ does not cluster. Then $\{x_n \mid n \in N\}$ is a closed discrete set. Let $\{G_n \mid x_n \in G_n, n \in N\}$ be a locally finite open collection. Choose $y_n \in G_n \cap St(x, \xi_n)$ for each $n \in N$ such that the y_n's are distinct. Then the sequence $\{y_n\}_n$ does not cluster, a contradiction.

To prove (2), let $C(x) = \bigcap_{n=1}^{\infty} \overline{St(x, \xi_n)}$. Then from (1) it follows that $C(x)$ is countably compact. Let G be an open neighborhood of $C(x)$. Suppose $x_n \in \overline{St(x, \xi_n)} - G$ for each $n \in N$. Then, since the sequence $\{\xi_n\}_n$ is assumed to be decreasing, from (1) it follows that the sequence $\{x_n\}_n$ clusters in $C(x)$, a contradiction.

Theorem 1.1.9. *A Hausdorff isocompact semi-$w\Delta$ space X with* $(*)$ (*as defined in Lemma* 1.1.8) *is a regular space.*

Proof. Let $\{\xi_n\}_n$ be a decreasing semi-$w\Delta$ sequence and $C(x) = \bigcap_{n=1}^{\infty} \overline{St(x, \xi_n)}$ for $x \in X$. From Lemma 1.1.8, $C(x)$ is countably compact and hence it is compact. Let $x \notin F = \overline{F}$. If $C(x) \cap F = \varnothing$, then $\overline{St(x, \xi_n)} \subset (X - F)$ for some n. Hence, $St(x, \xi_n)^0$ and $X - \overline{St(x, \xi_n)}$ are the disjoint open neighborhoods of x and F, respectively. If $C(x) \cap F = L \neq \varnothing$, then since X is Hausdorff, there are disjoint open neighborhoods G and H of x and L, respectively. Let $M = F - H$. Since $M \cap C(x) = \varnothing$, by the above argument, there are disjoint open neighborhoods U and V of x and M, respectively. Then $G \cap U$ and $H \cup V$ are the disjoint open neighborhoods of x and F, respectively. Therefore X is regular.

Definitions. A space X is called a *strongly semi-*$w\Delta$ (resp. a *point-star-open semi-*$w\Delta$) *space* if it admits a semi-$w\Delta$ sequence $\{\xi_n\}_n$ such that $y \in St(x, \xi_n)^0$ implies $x \in St(y, \xi_n)^0$ for each $n \in N$ and any $x, y \in X$ (resp. $St(x, \xi_n)$ is open for each $n \in N$ and each $x \in X$). In case of a strongly semi-$w\Delta$ space, $\{\xi_n\}_n$ is called a *strongly semi-*$w\Delta$ *sequence* and in case of a point-star-open semi-$w\Delta$ space, $\{\xi_n\}_n$ is called a *point-star-open semi-*$w\Delta$ *sequence.*

Lemma 1.1.10. *Let* ξ *be a strongly semi-open cover of a space* X *and* $A \subset X$. *Then* A *admits a subset* B *such that*

(1) $B \cap St(x, \xi_n)^0 = \{x\}$ *for each* $x \in B$,

(2) $A \subset \bigcup \{St(x, \xi_n)^0 \mid x \in B\}$,

(3) $\{\{x\} \mid x \in B\}$ *is a discrete collection in* X.

If X *is a* T_1 *space, then* B *is a closed set in* X.

Theorem 1.1.11. *A* T_1, $\aleph_1$*-compact, isocompact strongly semi-*$w\Delta$ *space* X *satisfying* $(*)$ *is a Lindelöf space.*

Proof. Let ξ be an open cover of X and $\{\xi_n\}_n$ be a strongly semi-$w\Delta$ sequence. For each $x \in X$, define $C(x) = \bigcap_{n=1}^{\infty} \overline{St(x, \xi_n)}$. From Lemma 1.1.8 and isocompactness, it follows that for each $x \in X$ there is n such that $x \in C(x) \subset St(x, \xi_n) \subset \cup \xi^f$ for some finite subcollection ξ^f of ξ. Let $A_n = \{x \in X \mid St(x, \xi_n) \subset \cup \xi^f$ for some finite subcollection ξ^f of $\xi\}$ for each $n \in N$. Then $X = \bigcup_{n=1}^{\infty} A_n$. From Lemma 1.1.10, A_n admits a closed discrete subset B_n such that $A_n \subset St(B_n, \xi_n)$ for each $n \in N$. Since X is $\aleph_1$-compact, each B_n is countable. Hence, ξ admits a countable subcover for X.

Theorem 1.1.12. *Let* $f : X \to Y$ *be a continuous, pseudo-open, finite to one map of a semi-*$w\Delta$ *space* X *onto a space* Y. *Then* Y *is a semi-*$w\Delta$ *space.*

Proof. Let $\{\xi_n\}_n$ be a decreasing semi-$w\Delta$ sequence in X and $\eta_n = f(\xi_n) = \{f(G) \mid G \in \xi_n\}$ for each $n \in N$. Observe the following:

(i) For each $y \in Y$ and $n \in N$, $St(y, \eta_n) = f(St(f^{-1}(y), \xi_n))$.

(ii) If a sequence $\{x_n\}_n$ clusters in X, then the sequence $\{f(x_n)\}_n$ clusters in Y.

We prove the following:

(iii) For any finite non-empty set A in X, if $x_n \in St(A, \xi_n)$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters: Since A is finite, there is $x \in A$ such that $x_{n_k} \in St(x, \xi_{n_k})$ for each k for some subsequence $\{n_k\}_k$ of the sequence $\{1, 2, 3, \ldots\}$. Since $\{\xi_n\}_n$ is decreasing, we can define a sequence $\{z_n\}_n$ such that $z_n \in St(x, \xi_n)$ for each $n \in N$, where $z_n = x_{n_1}$ for each $n \leq n_1$ and $z_n = x_{n_{i+1}}$ for each n such that $n_i < n \leq n_{i+1}$ for each integer $i \in N$. Then the sequence $\{z_n\}_n$ clusters and hence the sequence $\{x_n\}_n$ clusters. Now we prove that the sequence $\{\eta_n\}_n$ is a semi-$w\Delta$ sequence in Y: Let $y_n \in St(y, \eta_n)$ for each $n \in N$. From (i), there is a sequence $\{x_n\}_n$ in X such that $x_n \in St(f^{-1}(y), \xi_n)$ and $f(x_n) = y_n$ for each $n \in N$. From (iii), the sequence $\{x_n\}_n$ clusters in X. From (ii), the sequence $\{y_n = f(x_n)\}_n$ clusters in Y. Therefore the sequence $\{\eta_n\}_n$ is a semi-$w\Delta$ sequence in Y and Y is a semi-$w\Delta$ space.

1.2. Semi-metrizable spaces

Since a space is semi-metrizable if and only if it is a T_0 semi-developable space and quasi (α_1) diagonality implies T_1, it follows that semi-metrizable spaces are exactly semi-developable spaces with a quasi (α_1)-diagonal. We generalize this result:

(1) A space is a Hausdorff semi-metrizable space if and only if it is a semi-$w\Delta$ space with an (α_2)-diagonal.

(2) A space is a regular semi-metrizable space if and only if it is a semi-$w\Delta$ space with a semi-(C_2) property and a quasi (α_1)-diagonal.

Lemma 1.2.1. *A semi-$w\Delta$ space X with an α_2-diagonal is a semi-metrizable space.*

Proof. By an α_2-diagonal property, X is Hausdorff. Let $\{\xi_n\}_n$ be a decreasing semi-$w\Delta$ sequence and $\{\eta_n\}_n$ be a decreasing α_2-diagonal sequence. Let $\varsigma_n = \xi_n \wedge \eta_n = \{G \cap H \mid G \in \xi_n, H \in \eta_n\}$ for each $n \in N$. We claim that the sequence $\{\varsigma_n\}_n$ is a semi-development. Clearly, each ς_n is a semi-open cover of X and $\bigcap_{n=1}^{\infty} St(x, \xi_n) = \bigcap_{n=1}^{\infty} \overline{St(x, \xi_n)} = \{x\}$ for each $x \in X$. Let $x \in X$ and G be an open nhd of x. If $x_n \in St(x, \varsigma_n) - G$ for each $n \in N$, then the sequence $\{x_n\}_n$ clusters at x, a contradiction because G is an open set containing x and no x_n belongs to G. Therefore $St(x, \varsigma_n) \subset G$ for some n and $\{\varsigma_n\}_n$ is a semi-development for X. Since a Hausdorff semi-developable space is semi-metrizable [1], X is semi-metrizable.

Theorem 1.2.2. *A space X is a Hausdorff semi-metrizable space if and only if it is a semi-$w\Delta$ space with an α_2-diagonal.*

Proof. Suppose X is a Hausdorff semi-metrizable space. X is a semi-$w\Delta$ space (by Theorem 1.1.2) and X has an α_2-diagonal [10]. The converse part of the theorem follows from Lemma 1.2.1.

Corollary 1.2.2.1. *If X is a Hausdorff semi-$w\Delta$ space with a G_δ diagonal, which is θ-refinable and satisfies* (*), *then X is a semi-metrizable space.*

Proof. By Theorem 1.1.9, X is regular. X is a θ-refinable, T_3 space and so, it has (C_2) [8]. A space with a G_δ diagonal and (C_2) has a G_δ^* diagonal [8]. So, X is semi-metrizable (by Theorem 1.2.2).

Remark 1.2.2.2. We cannot replace an (α_2)-diagonal condition by a quasi (α_2)-diagonal condition in Lemma 1.2.1 and Theorem 1.2.2.

Corollary 1.2.2.3. *A regular space X is a Moore space if and only if it is a $w\Delta$-space with (α_2)-diagonal.*

Remark 1.2.2.4. We cannot replace an (α_2)-diagonal condition by a quasi (α_2)-diagonal condition in Corollary 1.2.2.3.

Corollary 1.2.2.5. *A space X is a metrizable space if and only if it is a wM-space with an (α_2)-diagonal.*

Proof. Since a Hausdorff semi-metrizable wM-space is metrizable [17], the result is a corollary to Theorem 1.2.2.

Remark 1.2.2.6. We cannot replace an (α_2)-diagonal condition by a quasi (α_2)-diagonal condition in Corollary 1.2.2.5.

Lemma 1.2.3 [13]**.** *A regular semi-metrizable space X has a semi-C_2 property.*

Theorem 1.2.4. *A space X is a regular-semi-metrizable space if and only if it is a semi-$w\Delta$ space with a semi-C_2 property and a quasi (α_1)-diagonal.*

Proof. Suppose X is a semi-$w\Delta$ space with a semi-C_2 property and a quasi (α_1)-diagonal. By the quasi (α_1)-diagonal property and the semi-C_2 property, X is a T_1 regular space [11]. It follows from Theorem 1.1.7 that X is a first countable space and hence, by Lemma 1.1.5, it has an

(α_1)-diagonal. Let $\{\eta_n\}_n$ be a decreasing (α_1)-diagonal sequence and $\{\varsigma_n\}_n$ be a decreasing semi-$w\Delta$ sequence. Let $\xi_n = \eta_n \wedge \varsigma_n = \{G \cap H \mid G \in \eta_n, H \in \varsigma_n\}$ for each n. Then $\{\xi_n\}_n$ is a decreasing semi-$w\Delta$ sequence such that $\bigcap_{n=1}^{\infty} St(x, \xi_n) = \{x\}$ for each $x \in X$. Since X has a semi-C_2 property, by Lemma 1.1.3 there is a semi-C_2 regular refinement $\{\tau_n\}_n$ for $\{\xi_n\}_n$. Then $\bigcap_{n=1}^{\infty} St(x, \tau_n) = \bigcap_{n=1}^{\infty} \overline{St(x, \tau_n)} = \{x\}$ for each $x \in X$. We claim that $\{St(x, \tau_n) \mid n \geq 1\}$ is a nhd base at x: Let $x \in X$ and G be an open nhd of x in X. If $x_n \in St(x, \tau_n) - G$ for each n, then the sequence $\{x_n\}_n$ clusters in $\bigcap_{n=1}^{\infty} \overline{St(x, \tau_n)} = \{x\}$ and hence the sequence $\{x_n\}_n$ clusters at x. Since G is an open nhd of x, $x_n \in G$ for some n, a contradiction. Therefore $St(x, \tau_n) \subset G$ for some n. Thus, the sequence $\{\tau_n\}_n$ is semi-development for X and since X is a T_1 regular space it follows that X is a semi-metrizable space [1]. The converse follows from Theorem 1.1.2 and Lemma 1.2.3.

Corollary 1.2.4.1 [13]**.** *A space X is a Moore space if and only if it is a $w\Delta$-space with a semi-C_2 property and a quasi (α_1)-diagonal.*

Proof. Suppose X is a Moore space. Then X is a $w\Delta$-space and a semi-metrizable space. Therefore X has a semi-C_2 property and a quasi (α_1)-diagonal. Conversely, suppose X is a $w\Delta$-space with a semi-C_2 property and a quasi (α_1)-diagonal. Then X is a T_1 regular semi-metrizable space (from Theorem 1.2.4). Therefore X is a Moore space [14].

Corollary 1.2.4.2 [13]**.** *A space X is metrizable if and only if it is a wM-space with a semi-C_2 property and a quasi (α_1)-diagonal.*

Proof. Suppose X is a wM space with a semi-C_2 property and a quasi (α_1)-diagonal. X is a Moore space (from Corollary 1.2.4.1). Therefore X is a metrizable space [7], [17]. The converse part is clear.

Corollary 1.2.4.3 [13]**.** *A countably compact space is metrizable if it has a semi-*C_2 *property and a quasi* (α_1)*-diagonal.*

Theorem 1.2.5. *A Hausdorff space X is a semi-metrizable space if it is a* θ*-refinable, semi-*$w\Delta$ *space satisfying* $(*)$ *and having a quasi* G_δ *diagonal.*

Proof. Suppose X is a θ-refinable, semi-$w\Delta$ space satisfying $(*)$ and has a quasi G_δ diagonal. It follows from Theorems 1.1.7 and 1.1.9 that X is a first countable, T_1 regular space. By Theorem 3.2 in [16], X is semi-stratifiable and hence it is semi-metrizable.

Remark 1.2.6. The converse of Theorem 1.2.5 is not true. The Niemytzki plane defined in [6], which is a Moore space, does not satisfy $(*)$.

Theorem 1.2.7. *Let* $f : X \to Y$ *be a pseudo-open map of a Hausdorff semi-metrizable space X onto a space Y. Let* $y_0 \in Y$ *and K be a compact set in X. Let* $f^{-1}(y)$ *be finite for each* $y \in Y - \{y_0\}$ *and* $f^{-1}(y_0) = K$. *Then Y has an* (α_1)*-diagonal.*

Proof. Let $\{\xi_n\}_n$ be a decreasing semi-development of X and $\eta_n = f(\xi_n) = \{f(G) | G \in \xi_n\}$ for each n. We claim that the sequence $\{\eta_n\}_n$ is an (α_1)-diagonal sequence of Y. Suppose the contrary holds. There exist distinct points z and y in Y such that $z \in \bigcap_{n=1}^{\infty} St(y, \eta_n)$. Then for each n there is a $G_n \in \xi_n$ such that $z, y \in f(G_n)$. If neither of z and y is y_0, then $f^{-1}(\{z, y\})$ is finite implies there are distinct points $x_1, x_2 \in G_{n_k}$ such that $x_1 \in f^{-1}(y)$ and $x_2 \in f^{-1}(z)$ for each k for some subsequence $\{n_k\}_k$. Since the sequence $\{\xi_n\}_n$ is decreasing, it follows that $x_1, x_2 \in$ some $G \in \xi_n$ for each n. But then $x_1 = x_2$, a contradiction. If one of z and y is y_0, say $y = y_0$. Then there exist sequences $\{p_n\}_n$ and $\{q_n\}_n$ such that $p_n \in f^{-1}(z)$, $q_n \in K$, and $p_n, q_n \in$ some $G_n \in \xi_n$ for each n.

Then we can find a constant sequence $\{p_{n_k}\}_k$ with $p_{n_k} = p \in f^{-1}(z)$ for each k and a convergent sequence $\{q_{n_k}\}_k$ that converges to some $q \in K$ for some subsequence $\{n_k\}_k$. Since $p \neq q$, let $G(p)$ and $G(q)$ be disjoint open nhds of p and q. There is an n such that $St(p, \xi_n) \subset G(p)$ and since $\{\xi_n\}_n$ is decreasing, $q_{n_k} \in St(p, \xi_n)$ for all $n_k \geq n$. Since the sequence $\{q_{n_k}\}_k$ converges to q, $q_{n_k} \in G(q)$ for all $n_k \geq$ some m. Therefore $q_{n_k} \in G(p) \cap G(q)$ for all $n_k \geq$ maximum of n and m, a contradiction. Therefore $\{\eta_n\}_n$ is an (α_1)-diagonal sequence.

2. Examples

(1.1) The ordinal space $[0, \omega_1]$ has no quasi (α_1)-diagonal; otherwise by Theorem 1.1.7 the space becomes a first countable space.

(1.2) There is a Hausdorff compact space that does not have a semi-C_2 property. The Alexandroff double circle $A(C)$ is a first countable space and hence by Lemma 1.1.5 it has an (α_1)-diagonal. This space cannot have a semi-C_2 property otherwise by Corollary 1.2.4.3 this space becomes a metrizable space. Also, this space has a quasi (α_2)-diagonal. Refer to the Example 1.20 in [9]. This example justifies Remarks 1.2.2.2, 1.2.2.4 and 1.2.2.6.

(1.3) There is a Hausdorff compact space that has a semi-C_2 property, but it cannot have a quasi (α_1)-diagonal. Let X be a set such that $|X| > \aleph_0$. Let $x_0 \in X$ and $\tau = \{G \subset X | x_0 \notin G \text{ or } |X - G| < \aleph_0\}$. Then τ is a Hausdorff compact topology on X, the Example 1.1.8 in [6]. Let ξ be a semi-open cover of X. Let $\eta = \{\{x\} | x \in X - \{x_0\}\} \cup \{\{x, x_0\} | \{x, x_0\} \subset$ some $G \in \xi\}$. Then $St(x, \eta) \subset \{x, x_0\}$ for each $x \in X - \{x_0\}$. Therefore $\overline{St(x, \eta)} = St(x, \eta) \subset St(x, \xi)$ for each $x \in X - \{x_0\}$. Also, $\overline{St(x_0, \eta)} = St(x_0, \eta) \subset St(x, \xi)$. Therefore X has a semi-C_2 property. Since this space is not first countable, by Theorem 1.1.7 the space X cannot have a quasi (α_1)-diagonal. However, if $|X| \leq \aleph_0$, then X is first countable and by Corollary 1.2.4.3, X is metrizable.

(1.4) Let X be the set of real numbers with the usual open interval topology. Let N be the set of natural numbers. Let $Y = X/N$ be the quotient space obtained from X by identifying the natural numbers. Refer to the Example 1.4.7 in [6]. Then Y is a continuous, closed image of X and hence it is a T_4 semi-stratifiable space. The space Y is not first countable, but it has a G_δ^* diagonal. Hence, by Theorem 1.1.7, this space cannot be a q-space (and it cannot be a semi-$w\Delta$ space).

(1.5) A semi-$w\Delta$ property is not a perfect invariant property. Refer to Example 9.11 on page 486 in [19]. Let R be the set of real numbers. Let X be the subspace $[0, 1] \times [0, 1]$ of the space $R \times R$ with the 'bowtie' topology. Let $K = [0, 1] \times \{0\}$. Then K is a compact subset of X. Let $Y = X/K$ be the quotient space obtained by identifying the points of K. Then Y is a perfect image of X. By Theorem 1.2.7, Y has an (α_1)-diagonal. Since Y is not first countable, by Theorem 1.1.7, Y cannot be a q-space and hence it cannot be a semi-$w\Delta$ space. This example shows that both q property and semi-$w\Delta$ are not perfect invariants.

References

[1] C. C. Alexander, Semi-developable spaces and quotient images of metric spaces, Pacific J. Math. 37 (1971), 277-293.

[2] A. V. Arhangel'skii, Mappings and spaces, Russian Math. Surveys 21(4) (1966), 115-162.

[3] P. Bacon, The compactness of countably compact spaces, Pacific J. Math. 32 (1970), 587-592.

[4] C. J. R. Borges, On metrizability of topological spaces, Canad. J. Math. 20 (1968), 795-804.

[5] J. G. Ceder, Some generalizations of metric spaces, Pacific J. Math. 11 (1961), 105-125.

[6] R. Engelking, General Topology, PWN, Warsaw, 1977.

[7] G. R. Hiremath, On generalization of G_δ diagonal and metrization, Proc. Japan Acad. 64(8) (1988), 288-291.

[8] G. R. Hiremath, Metrization in isocompact wM spaces I, Far East J. Math. Sci. 3(2) (1995), 135-149.

[9] G. R. Hiremath, Some characterizations of semimetrizability, metrizability, and generalized diagonal properties in terms of point-star-open covers, Far East J. Math. Sci. 4(1) (1996), 39-57.

[10] G. R. Hiremath, An (α_2)-property and metrization, Far East J. Math. Sci. Special Vol. Part II (1996), 123-142.

[11] G. R. Hiremath, Semi-open covers and metrization, Far East J. Math. Sci. Special Vol. Part II (1996), 243-252.

[12] G. R. Hiremath, On semi-developable spaces and a class of spaces generalizing both semi-developable spaces and $w\Delta$ concepts, AMS Abstract #930-54-1017, 1998.

[13] G. R. Hiremath, A note on metrization of wM spaces, 2005, preprint.

[14] R. E. Hodel, Moore spaces and $w\Delta$ spaces, Pacific J. Math. 38 (1971), 641-651.

[15] R. E. Hodel, Spaces defined by sequences of open covers which guarantee certain sequences have cluster points, Duke Math. J. 39 (1972), 253-263.

[16] R. E. Hodel, Metrizability of topological spaces, Pacific J. Math. 55(2) (1974), 441-459.

[17] T. Ishii and T. Shiraki, Some properties of wM spaces, Proc. Japan Acad. 47 (1971), 167-172.

[18] H. J. K. Junnila, On submetacompactness, Topology Proc. 3 (1978), 375-405.

[19] K. Kunen and J. Vaughan, Handbook of Set-theoretic Topology, North-Holland Publ., 1984.

[20] E. Michael, A note on closed maps and compact sets, Israel J. Math. 2 (1964), 173-176.