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SYMMETRIC GENERATING SET OF THE GROUPS A_{kn+1} AND S_{kn+1} USING THE WREATH PRODUCT A_m wr A_a

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Abstract

In this paper, we show how to generate A_{kn+1} and S_{kn+1} using a copy of the wreath product $A_m \text{wr} A_a$ and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all odd positive integers $n = am \geq 2$ and all positive integers $k \geq 2$. We also show how to generate A_{kn+1} and S_{kn+1} symmetrically using n elements each of order $k+1$.

1. Introduction

Al-Amri [1] showed that A_{kn+1} and S_{kn+1} can be generated using a copy of the wreath product S_m wr S_a and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all $n = am \geq 2$ and all positive integers $k \geq 2$. Moreover A_{kn+1} and S_{kn+1} can be symmetrically generated by n permutations each of order $k+1$. Further, Shafee [5] showed that A_{kn+1}

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and S_{kn+1} can be generated using a copy of the wreath product S_m wr A_a and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all $n = am \geq 2$ and all positive integers $k \geq 2$. Moreover A_{kn+1} and S_{kn+1} can be symmetrically generated by using n elements each of order $k+1$. In this paper, we give permutations to show that the group $G = \langle X, Y, Z, T \mid \langle X, Y, Z \rangle = A_m \text{ wr } A_a, T^{k+1} = [T, A_m] = 1 \rangle$ is the alternating group A_{kn+1} when k is an even integer and S_{kn+1} when k is odd for all $n = am \geq 2$, $k \geq 2$. Further, we prove that G can be symmetrically generated by n permutations each of order $k+1$ of the form T_0, $T_1, \ldots, T_{n-1}$, where $T_i = T^{x^i}$ satisfying the condition that T_0 commutes with the generators of A_m.

2. Preliminary Results

Theorem 2.1 [4]. *Let* $1 < a \neq 2a < n = am$ *and* G *be the group generated by the* n*-cycle* $(1, 2, \ldots, n)$, *the* 3*-cycle* $(n, a, 2a)$ *and the permutation* $(1, 2, 3)(a+1, a+2, a+3)(2a+1, 2a+2, 2a+3)\cdots((m-1)a+1, (m-1)a+2, (m-1)a+3)$. *If* n *is an odd integer*, *then* $G = A_m$ wr A_a.

Theorem 2.2 [4]. *Let* G *be the group generated by the* n*-cycle* $(1, 2, \ldots, n)$ *and* k*-cycle* $(1, 2, \ldots, k)$. *If* $1 < k < n$ *is an even integer*, *then* $G = S_n$.

Theorem 2.3 [4]. *Let* n *be an odd integer and* G *be the group generated by the* n*-cycle* $(1, 2, \ldots, n)$ *and* k*-cycle* $(1, 2, \ldots, k)$. *If* $1 < k < n$ *is an odd integer*, *then* $G = A_n$.

Definition 2.1. Let A be a group of permutations of a finite set Ω_1 and B be a group of permutations of a finite set Ω_2. Assume that neither of Ω_1 nor Ω_2 is empty and they are disjoint. The *wreath product* (sometimes called the *complete* or the *unrestricted wreath product*) of A

and B defined by $A \text{ wr } B = A^{\Omega_2} \times_\theta B$ which is the direct product of $|\Omega_2|$ copies of A and the mapping θ, where $\theta : B \to Aut(A^{\Omega_2})$ is defined by: $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that $|A \text{ wr } B| = (|A|)^{|\Omega_2|}|B|$.

Definition 2.2. Let G be a group and $\Gamma = \{T_0, T_1, ..., T_{n-1}\}$ be a subset of G, where each $T_i = T^{x^i}$ for all $i = 0, 1, ..., n-1$. Let S_n be the normalizer in G of the set Γ. We define Γ to be a symmetric generating set of G if and only if $G = \langle \Gamma \rangle$ and S_n permutes Γ doubly transitive by conjugation, i.e., Γ is realizable as an inner automorphism.

3. Permutational Generating Set of A_{kn+1} and S_{kn+1}

Theorem 3.1. *A_{kn+1} and S_{kn+1} can be generated using a copy of the wreath product A_m wr A_a and an element of order $k+1$ in A_{kn+1} and S_{kn+1} for all $n = am \geq 2$, n is odd and all $k \geq 2$.*

Proof. Let

$$X = (1, 2, ..., n)(n+1, n+2, ..., 2n)\cdots((k-1)n+1, (k-1)n+2, ..., kn),$$

$$Y = (a \cdot 2a, n)(n+a, n+2a, 2n)\cdots((k-1)n+a, (k-1)n+2a, kn),$$

$$\begin{aligned}Z = {} & (1, 2, 3)(a+1, a+2, a+3) \\ & \cdots((m-1)a+1, (m-1)a+2, (m-1)a+3) \\ & (n+1, n+2, n+3)(n+a+1, n+a+2, n+a+3) \\ & \cdots(n+(m-1)a+1, n+(m-1)a+2, n+(m-1)a+3) \\ & \cdots((k-1)n+1, (k-1)n+2, (k-1)n+3) \\ & \cdots((k-1)n+a+1, (k-1)n+a+2, (k-1)n+a+3) \\ & \cdots((k-1)n+(m-1)a+1, (k-1)n+(m-1)a+2, \\ & (k-1)n+(m-1)a+3)\end{aligned}$$

and $T = (n, 2n, 3n, \ldots, kn, kn+1)$ be four permutations; the first is of order n, the second of order 3, the third of order 3 and the fourth of order $k+1$. Let the H be the group generated by X, Y and Z. By Theorem 2.1, (see Al-Amri [4]) the group H is the wreath product A_m wr A_a. Let $\overline{G}$ be the group generated by X, Y, Z and T. We claim that $\overline{G}$ is either A_{kn+1} or S_{kn+1}. To show this, let $\beta = TX$. It is clear that $\beta = (1, 2, \ldots, kn+1)$, which is a cycle of length $kn+1$. Let $\alpha = T^{\beta}$. Since $\alpha = (n+1, 2n+1, \ldots, (k-1)n+1, kn+1, 1)$, conjugating α. By β, we get the cycle $\eta = (n+2, 2n+2, \ldots, (k-1)n+2, 1, 2)$. Hence the commutator $[\alpha, \eta] = (1, 2, n+1)$. Let $G = \langle \beta, [\alpha, \eta] \rangle$. It clear that $G \cong S_{kn+1}$ or A_{kn+1} depending on k either odd or even respectively, but if k is on odd in tiger, then X is on odd permutation and there for $G = \overline{G} = S_{kn+1}$. While if k is on even permutation, then $\overline{G}$ is generated by even elements. Hence $G = \overline{G} = A_{kn+1}$. □

4. Symmetric Permutational Generating Set of A_{kn+1} and S_{kn+1}

Theorem 4.1. *The groups A_{kn+1} and S_{kn+1} can be generated symmetrically using n elements each of order $k+1$.*

Proof. Let X, Y and T be the elements considered in Theorem 3.1 above. Let $\Gamma = \{T_1, T_2, \ldots, T_n\}$ for all $n = am \geq 2$, where $T_i = T^{x^i}$. Since

$$T_1 = (1, n+1, 2n+1, \ldots, (k-1)n+1, kn+1),$$

$$T_2 = (2, n+2, \ldots, (k-1)n+2, kn+1), \ldots,$$

$$T_n = T^{x^n} = T = (n, 2n, 3n, \ldots, kn, kn+1).$$

Let $H = \langle \Gamma \rangle$. We claim that $H \cong A_{kn+1}$ or S_{kn+1}. To show this, consider the element

$$\alpha = \prod_{i=1}^{n} T^{x^i}.$$

It is not difficult to show that

$$\alpha = (1,\ n+1,\ 2n+1,\ \ldots,\ (k-1)n+1,\ 2,\ n+2,\ 2n+2,\ \ldots,$$
$$(k-1)n+2,\ \ldots,\ n,\ 2n,\ \ldots,\ kn,\ kn+1),$$

which is an element of order $kn+1$.

Let $H_1 = \langle \alpha, T_1 \rangle$. We claim that $H_1 \cong A_{kn+1}$ or S_{kn+1}. To prove this θ is the mapping which takes the element in the position i of the cycle α into the element i of the cycle $(1, 2, \ldots, kn+1)$. Under this mapping the group H_1 will be mapped into the group $\theta(H_1) = \langle (1, 2, \ldots, kn+1), (1, 2, 3, \ldots, k, kn+1) \rangle$.

Therefore by Theorems 2.2 and 2.3, $\theta(H_1) \cong H_1$ is A_{kn+1} or S_{kn+1} depending on whether k is an even or odd integer respectively. Since $H_1 \leq H$, if k is an odd integer $H_1 \cong H \cong S_{kn+1}$. While if K is an even integer, then Γ contains an even permutation. Hence $H = \langle \Gamma \rangle$ is generated by even permutations. Hence $H_1 \cong H \cong A_{kn+1}$. □

In order to generate A_{kn+1} or S_{kn+1}, the set $\Gamma = \{T_1, T_2, \ldots, T_n\}$ has to have at least n elements each of order $k+1$.

If we remove m elements from the set Γ, then above results can be modified by according to the following remarks.

Remarks 4.2. Let T and X be the permutations which have been described above, where $T^{k+1} = 1$. Let $\Gamma = \{T_1, T_2, \ldots, T_n\}$ for all $n > 2$, where $T_i = T^{x^i}$.

4.2.1. Let k be an even integer. If we remove m-elements from the set Γ for all $1 \leq m \leq n-2$, then the resulting set generates $A_{k(n-m)+1}$.

4.2.2. Let k be an odd integer. If we remove m-elements from the set Γ for all $1 \leq m \leq n-2$, then the resulting set generates $S_{k(n-m)+1}$.

4.2.3. If we remove $(n-1)$-elements from the set Γ, then the resulting set generates C_{k+1}.

The proofs of above remarks are similar to those of the proof of Theorem 4.2 in [5]. □

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