JP Jour. Geometry & Topology **5**(3) (2005), 187-250

NORMAL FRAMES AND LINEAR TRANSPORTS ALONG PATHS IN VECTOR BUNDLES

BOZHIDAR Z. ILIEV

Laboratory of Mathematical Modeling in Physics
Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
Boul. Tzarigradsko chaussée 72, 1784 *Sofia*, *Bulgaria*
e-mail: bozho@inrne.bas.bg
URL: http://theo.inrne.bas.bg/~bozho/

Abstract

The theory of linear transports along paths in vector bundles, generalizing the parallel transports generated by linear connections, is developed. The normal frames for them are defined as ones in which their matrices are the identity matrix or their coefficients vanish. A number of results, including theorems of existence and uniqueness, concerning normal frames are derived. Special attention is paid to the important case when the bundle's base is a manifold. The normal frames are defined and investigated also for derivations along paths and along tangent vector fields in the last case. It is proved that normal frames always exist at a single point or along a given (smooth) path. On other subsets normal frames exist only as an exception if (and only if) certain additional conditions, derived here, are satisfied. Gravity physics and gauge theories are pointed out as possible fields for application of the results obtained.

2000 Mathematics Subject Classification: 53B99, 53C99, 53Z05, 55R25, 57R55, 83D05.

Keywords and phrases: normal frames, normal coordinates, frame fields, linear transports along paths, derivations, parallel transport.

This research was partially supported by the National Science Fund of Bulgaria under contract Grant Reg. No. 427/2005.

Communicated by Yasuo Matsushita

Received October 26, 2004; Revised February 17, 2005

1. Introduction

Conventionally, local coordinates or frames (or frame fields), which can be holonomic or not, are called *normal* if in them the coefficients of a linear connection vanish on some subset, usually a submanifold, of a differentiable manifold. Until recently the existence of normal frames was known (proved) only for symmetric linear connections on submanifolds of a (pseudo-) Riemannian manifold [7, 45, 46, 47, 52]. New light on these problems was thrown in a series of papers [15-17], where a comprehensive analysis of the normal frames for derivations of the tensor algebra over a differentiable manifold is given; in particular, they completely cover the exploration of normal frames for arbitrary linear connections on a manifold. These strict results are applied in [18] for rigorous analysis of the equivalence principle. This results in two main conclusions: the (strong) equivalence principle (in its 'conventional' formulations) is a provable theorem and the normal frames are the mathematical realization of the physical concept of 'inertial' frames. Another physical application the normal frames find is in the bundle formulation of quantum mechanics [19]. In this approach the normal frames realize the (shift to the) bundle Heisenberg picture of motion [20].

The present investigation is a completely revised and expanded version of [21]. It can also be considered as a continuation of the series of works [15-17] which are its special cases and, at the same time, its supplement. Here we study a wide range of problems concerning frames normal for linear transports and derivations along paths in vector bundles and for derivations along tangent vector fields in case when the bundle's base is a differentiable manifold. In the last case, the only general result known to the author concerning normal frames is [42, p. 102, Theorem 2.106].

The structure of this work is as follows.

Section 2 is devoted to the general theory of linear transports along paths in vector fibre bundles which is a far-reaching generalization of the theory of parallel transports generated by linear connections.[1] The

[1] This result is not explicitly proved here. The interested reader is referred to [22] for details and the proof of this assertion.

general form and other properties of these transports are studied. A bijective correspondence between them and derivations along paths is established. In Section 3, the normal frames are defined as ones in which the matrix of a linear transport along paths is the unit (identity) one or, equivalently, in which its coefficients, as defined in Section 2, vanish 'locally'. A number of properties of normal frames are found. In Section 4 is explored the problem of existence of normal frames. Several necessary and sufficient conditions for such existence are proved and the explicit construction of normal frames, if any, is presented.

Section 5 concentrates on, possibly, the most important special case of frames normal for linear transports or derivations along smooth paths in vector bundles with a differentiable manifold as a base. A specific necessary and sufficient condition for existence of normal frames in this case is proved in Subsection 5.1. In particular, normal frames may exist only for those linear transports or derivations along paths whose (2-index) coefficients linearly depend on the vector tangent to the path along which they act. Obviously, this is a generalization of the derivative along curves assigned to a linear connection. Subsection 5.2 is devoted to problems concerning frames normal for derivations along tangent vector fields in a bundle with a manifold as a base. Necessary and sufficient conditions for the existence of these frames are derived. The conclusion is made that there is a one-to-one onto correspondence between the sets of linear transports along paths, derivations along paths, and derivations along tangent vector fields all of which admit normal frames.

Section 6 concerns a special type of normal frames in which the 3-index coefficients, if any, of a linear transport along paths vanish.

In Section 7 are presented some general remarks. It is shown that the results of [15-17] remain valid, practically without changes, for (strong) normal frames in vector bundles with a manifold as a base.

All fibre bundles in this work are vectorial ones. The base and total bundle space of such bundles can be general topological spaces. However, if some kind of differentiation in one/both of these spaces is required, it/they should possess a smooth structure; if this is the case, we require it/they to be smooth, of class C^1, differentiable manifold(s). Starting from Section 5, the base and total bundle space are supposed to be C^1

manifolds. Sections 2-4 do not depend on the existence of a smoothness structure in the bundle's base. Smoothness of the bundle space is partially required in Sections 2-4.[2]

2. Linear Transports along Paths in Vector Bundles

From different view-points, the connection theory can be found in many works, like [9, 14, 33, 34, 38, 49, 56]. As pointed in these and many other references, the concept of a parallel transport is defined on the base of one of a connection. The opposite approach, i.e., the definition of a connection on the ground of an axiomatically defined concept of a parallel transport, is also known and considered in [4, 6, 32, 35-37, 39, 40, 41, 42, 51].

The purpose of the present section is an introduction and partial study of an axiomatic definition (and generalization) of parallel transport in vector bundles, called *transport along paths* which in the particular case is required to be linear.

Our basic definition of a (linear) transport along paths is Definition 2.1 below. Comparing it with [23, Definition 2.1] and taking into account [23, Proposition 4.1], we conclude that special types of general linear transports along paths are: the parallel transport assigned to a linear connection (covariant derivative) of the tensor algebra of a manifold [34, 47], Fermi-Walker transport [12, 50], Fermi transport [50], Truesdell transport [54, 55], Jaumann transport [44], Lie transport [12, 47], the modified Fermi-Walker and Frenet-Serret transports [3] etc. Consequently, Definition 2.1 is general enough to cover a list of important transports used in theoretical physics and mathematics. Thus studying the properties of the linear transports along paths, we can make corresponding conclusions for any one of the transports mentioned.[3]

---

[2] The bundle space is required to be a C^1 manifold in Section 2 (starting from Definition 2.2), in Definition 3.1′, in Propositions 3.1-3.1, if (3.1c) and (3.1d) are taken into account, in Theorem 4.2, and in Proposition 4.6.

[3] The concept of linear transport along paths in vector bundles can be generalized to the transports along paths in arbitrary bundles [39] and to transports along maps in bundles is considered in [25]. An interesting consideration of the concept of (parallel) ‘transport’ (along closed paths) in connection with homotopy theory and the classification problem of bundles can be found in [48]. These generalizations are out of the scope of the present work.

As we said above, Definition 2.1 below realizes, an axiomatic approach to the concept of a parallel transport [4, 35, 37, 40, 42, 51].[4] However, a detailed discussion of this topic is out of the scope of the present work and will be presented elsewhere.

2.1. Definition and general form

Let (E, π, B) be a complex[5] vector bundle [9, 42] with bundle (total) space E, base B, projection $\pi : E \to B$, and homeomorphic fibres $\pi^{-1}(x)$, $x \in B$.[6] Whenever some kind of differentiation in E is considered, the bundle space E will be required to be a C^1 differentiable manifold. The base B is supposed to be a general topological space in Sections 2-4 and from Section 5 onwards is required to be a C^1 differentiable manifold. By J and $\gamma : J \to B$ are denoted real interval and path in B, respectively. The paths considered are generally *not* supposed to be continuous or differentiable unless their differentiability class is stated explicitly.

Definition 2.1. A *linear transport along paths* in the bundle (E, π, B) is a map L assigning to every path γ a map L^γ, *transport along* γ, such that $L^\gamma : (s, t) \mapsto L^\gamma_{s\to t}$, where the map

$$L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t)), \quad s, t \in J, \tag{2.1}$$

called *transport along* γ *from* s *to* t, has the properties:

$$L^\gamma_{s\to t} \circ L^\gamma_{r\to s} = L^\gamma_{r\to t}, \qquad r, s, t \in J, \tag{2.2}$$

---

[4] The author of [4] states that his paper is based on unpublished lectures of Prof. Willi Rinow in 1949. See also [42, p. 46], where the author claims that the first axiomatical definition of a parallel transport in the tangent bundle case is given by Prof. W. Rinow in his lectures at the Humboldt University in 1949. Some heuristic comments on the axiomatic approach to parallel transport theory can be found in [10, Section 2.1] too.

[5] All of our definitions and results hold also for real vector bundles. Most of them are valid for vector bundles over more general fields too but this is inessential for the following.

[6] When writing $x \in X$, X being a set, we mean "for all x in X" if the point x is not specified (fixed, given) and is considered as an argument or a variable.

$$L^{\gamma}_{s\to s} = id_{\pi^{-1}(\gamma(s))}, \quad s \in J, \tag{2.3}$$

$$L^{\gamma}_{s\to t}(\lambda u + \mu v) = \lambda L^{\gamma}_{s\to t}u + \mu L^{\gamma}_{s\to t}v, \quad \lambda,\, u \in \mathbb{C}, \quad u,\, v \in \pi^{-1}(\gamma(s)), \tag{2.4}$$

where $\circ$ denotes composition of maps and id_X is the identity map of a set X.

Remark 2.1. Equations (2.2) and (2.3) mean that L is a *transport along paths* in the bundle (E, π, B), which may be an arbitrary topological bundle, not only a vector one in the general case [24, Definition 2.1],[7] while (2.4) specifies that it is *linear* [24, equation (2.8)]. In the present work only linear transports will be explored.

Remark 2.2. Definition 2.1 is a generalization of the concept of 'linear connection' given, e.g., in [4, Section 1.2] (see especially [4, p. 138, Axiom (L_1)]) which practically defines the covariant derivative in terms of linear transports along paths (see (2.34) below which is equivalent to [4, p. 138, Axiom (L_3)]). Our definition is much weaker; e.g., we completely drop [4, p. 138, Axiom (L_3)] and use, if required, weaker smoothness conditions. An excellent introduction to the theory of vector bundles and the parallel transports in them can be found in the book [42]. In particular, in this reference is proved the equivalence of the concepts parallel transport, connection and covariant derivative operator in vector bundles (as defined there). Analogous results concerning linear

[7] The definition of a connection in a *topological* bundle (E, π, B) in [51, Chapter IV, Section B.3] is, in fact, an axiomatic definition of a parallel transport. If we neglect the continuity condition in this definition, it defines a connection in (E, π, B) as a mapping $C : (\gamma, q) \mapsto C(\gamma, q)$ assigning to any continuous path $\gamma : [0, 1] \to B$ and a point $q \in \pi^{-1}(\gamma(0))$ a path $C(\gamma, q) : [0, 1] \to E$ such that $C(\gamma, q)|_0 = q$ and $\pi \circ C(\gamma, q) = \gamma$. If I is a transport along paths in (E, π, B), then $C : (\gamma, q) \mapsto C(\gamma, q) : t \mapsto C(\gamma, q)|_t = I^{\gamma}_{0\to t}(q)$ defines a connection C in (E, π, B) in the sense mentioned. Moreover, if this definition is broadened by replacing $[0, 1]$ with an arbitrary and not fixed closed interval $[a, b]$, with $a, b \in \mathbb{R}$ and $a \le b$, then the converse is also true, i.e., $C(\gamma, q)|_t = I^{\gamma}_{a\to t}(q), t \in [a, b]$, for some transport I. However, the proof of this statement is not trivial.

transports along paths will be presented below. The detailed comparison of Definition 2.1 with analogous ones in the literature is not a subject of this work and will be given elsewhere (see, e.g., [22]).

From (2.2) and (2.3), we get that $L^{\gamma}_{s\to t}$ are invertible mappings and

$$(L^{\gamma}_{s\to t})^{-1} = L^{\gamma}_{t\to s}, \qquad s, t \in J. \tag{2.5}$$

Hence the linear transports along paths are in fact linear isomorphisms of the fibres over the path along which they act.

The following two propositions establish the general structure of linear transports along paths.[8]

Proposition 2.1. *A map* (2.1) *is a linear transport along* γ *from* s *to* t *for every* $s, t \in J$ *if and only if there exist a vector space* V, *isomorphic with* $\pi^{-1}(x)$ *for all* $x \in B$, *and a family* $\{F(s;\gamma) : \pi^{-1}(\gamma(s)) \to V,\ s \in J\}$ *of linear isomorphisms such that*

$$L^{\gamma}_{s\to t} = F^{-1}(t;\gamma) \circ F(s;\gamma), \qquad s, t \in J. \tag{2.6}$$

Proof. If (2.1) is a linear transport along γ from s to t, then fixing some $s_0 \in J$ and using (2.3) and (2.5), we get $L^{\gamma}_{s\to t} = L^{\gamma}_{s_0\to t} \circ L^{\gamma}_{s\to s_0} = (L^{\gamma}_{t\to s_0})^{-1} \circ L^{\gamma}_{s\to s_0}$. So (2.6) holds for $V = \pi^{-1}(\gamma(s_0))$ and $F(s;\gamma) = L^{\gamma}_{s\to s_0}$. Conversely, if (2.6) is valid for some linear isomorphisms $F(s;\gamma)$, then a straightforward calculation shows that it converts (2.2) and (2.3) into identities and (2.4) holds due to the linearity of $F(s;\gamma)$. □

Proposition 2.2. *Let a representation* (2.6) *for a vector space* V *and some linear isomorphisms* $F(s;\gamma) : \pi^{-1}(\gamma(s)) \to V,\ s \in J,$ *be given for a linear transport along paths in the vector bundle* (E, π, B). *For a vector*

---

[8] Particular examples of Proposition 2.1 are known for parallel transports in vector bundles. For instance, Proposition 1 in [43, p. 240] realizes it for parallel transport in a bundle associated to a principal one and induced by a connection in the latter case; see also the proof of the lemma in the proof of Proposition 1.1 in [34, Chapter III, Section 1], where a similar result is obtained implicitly.

space ${}^{\star}V$, *there exist linear isomorphisms* ${}^{\star}F(s;\,\gamma):\pi^{-1}(\gamma(s))\to{}^{\star}V$, $s\in J$, *for which*

$$L^{\gamma}_{s\to t} = {}^{\star}F^{-1}(t;\,\gamma)\circ{}^{\star}F(s;\,\gamma), \qquad s,\,t\in J, \tag{2.7}$$

iff there exists a linear isomorphism $D(\gamma):V\to{}^{\star}V$ *such that*

$${}^{\star}F(s;\,\gamma) = D(\gamma)\circ F(s;\,\gamma), \qquad s\in J. \tag{2.8}$$

Proof. If equation (2.8) holds, the substitution of $F(s;\,\gamma) = D^{-1}(\gamma)\circ{}^{\star}F(s;\,\gamma)$ into (2.6) yields (2.7). Vice versa, if (2.7) is valid, then from its comparison with (2.6) follows that $D(\gamma) = {}^{\star}F(t;\,\gamma)\circ(F(t;\,\gamma))^{-1} = {}^{\star}F(s;\,\gamma)\circ(F(s;\,\gamma))^{-1}$ is the required (independent of $s,\,t\in J$) isomorphism. □

Starting from this point, we shall investigate further only the finite-dimensional case, $\dim\pi^{-1}(x) = \dim\pi^{-1}(y) < \infty$ for all $x,\,y\in B$. In this way we shall avoid a great number of specific problems arising when the fibres have infinite dimension (see, e.g., [2] for details). A lot of our results are valid, possibly *mutatis mutandis*, in the infinite-dimensional treatment too. One way for transferring results from finite to infinite dimensional spaces is the direct limit from the first to the second ones. Then, for instance, if the bundle's dimension is countably or uncountably infinite, the corresponding sums must be replaced by series or integrals whose convergence, however, requires special exploration [2]. Linear transports along paths in infinite-dimensional vector bundles naturally arise, e.g., in the fibre bundle formulation of quantum mechanics [19, 20, 26-28]. Generally, there are many difficulties with the infinite-dimensional problem which deserves a separate investigation.

2.2. Representations in frames along paths

Now we shall look locally at linear transports along paths.

Let $\{e_i(s;\,\gamma)\}$ be a basis in $\pi^{-1}(\gamma(s))$, $s\in J$.[9] So, along $\gamma:J\to B$ we

[9] Here and henceforth the Latin indices run from 1 to $\dim\pi^{-1}(x)$, $x\in B$. We also assume the usual summation rule on indices repeated on different levels.

have a set $\{e_i\}$ of bases on $\pi^{-1}(\gamma(J))$. The dependence of $e_i(s;\gamma)$ on s is inessential if we are interested only in the *algebraic* properties of the linear transports along a path; this will be the case through the proof of Proposition 2.5. Starting with two paragraphs before Definition 2.2, the mapping $s \mapsto e_i(s;\gamma)$ will be required to be of class C^1 as some kind of differentiation of liftings of paths will be considered.[10]

The *matrix* $\boldsymbol{L}(t, s; \gamma) := [L^i_j(t, s; \gamma)]$ *(along* γ *at* (s, t) *in* $\{e_i\}$*) of a linear transport* L *along* γ from s to t is defined via the expansion[11]

$$L^{\gamma}_{s\to t}(e_i(s;\gamma)) =: L^j_i(t, s; \gamma)e_j(t;\gamma), \qquad s, t \in J. \tag{2.9}$$

We call $\boldsymbol{L} : (t, s; \gamma) \to \boldsymbol{L}(t, s; \gamma)$ the *matrix* (*function*) of L; respectively L^j_i are its *matrix elements* or *components* in the given field of bases.

It is almost evident that

$$L^j_i(t, s; \gamma)e_j(t;\gamma) \otimes e^i(s;\gamma) \in \pi^{-1}(\gamma(t)) \otimes (\pi^{-1}(\gamma(s)))^*, \tag{2.10}$$

where $\otimes$ is the tensor product sign, the asterisk $(*)$ denotes dual object, and $e^i(s;\gamma) := (e_i(s;\gamma))^*$. Hence the change of the bases $\{e_i(s;\gamma)\} \mapsto \{e'_i(s;\gamma) := A^j_i(s;\gamma)e_j(s;\gamma)\}$ by means of a non-degenerate matrix $A(s;\gamma) := [A^j_i(s;\gamma)]$ implies

$$\boldsymbol{L}(t, s; \gamma) \mapsto \boldsymbol{L}'(t, s; \gamma) = A^{-1}(t;\gamma)\boldsymbol{L}(t, s; \gamma)A(s;\gamma) \tag{2.11}$$

or in component form

$$L'^j_i(t, s; \gamma) = (A^{-1}(t;\gamma))^j_k L^k_l(t, s; \gamma)A^l_i(s;\gamma). \tag{2.11$'$}$$

Evidently, for $u = u^i(s;\gamma)e_i(s;\gamma) \in \pi^{-1}(\gamma(s))$, due to (2.4), we have

$$L^{\gamma}_{s\to t}u = (L^j_i(t, s; \gamma)u^i(s;\gamma))e_j(t;\gamma). \tag{2.12}$$

---

[10] The mapping $\gamma \mapsto e_i(\cdot, \gamma)$ is, obviously, a lifting of paths.

[11] Notice the different positions of the arguments s and t in $L^{\gamma}_{s\to t}$ and in $\boldsymbol{L}(t, s; \gamma)$.

In terms of the matrix $\boldsymbol{L}$ of L, the basic equations (2.2) and (2.3) read respectively

$$\boldsymbol{L}(t, s; \gamma)\boldsymbol{L}(s, r; \gamma) = \boldsymbol{L}(t, r; \gamma), \quad r, s, t \in \boldsymbol{J}, \tag{2.13}$$

$$\boldsymbol{L}(s, s; \gamma) = \mathbb{1}, \quad s \in \boldsymbol{J} \tag{2.14}$$

with $\mathbb{1}$ being the identity (unit) matrix of corresponding size. From these equalities immediately follows that $\boldsymbol{L}$ is always non-degenerate.

Proposition 2.3. *A linear map* (2.1) *is a linear transport along* γ *from* s *to* t *iff its matrix*, *defined via* (2.9), *satisfies* (2.13) *and* (2.14).

Proof. The necessity was already proved. The sufficiency is trivial: a simple checking proves that (2.13) and (2.14) convert respectively (2.2) and (2.3) into identities. □

Proposition 2.4. *A non-degenerate matrix-valued function* $\boldsymbol{L} : (t, s; \gamma) \mapsto \boldsymbol{L}(t, s; \gamma)$ *is a matrix of some linear transport along paths* L (*in a given field* $\{e_i\}$ *of bases along* γ) *iff*

$$\boldsymbol{L}(t, s; \gamma) = \boldsymbol{F}^{-1}(t; \gamma)\boldsymbol{F}(s; \gamma), \tag{2.15}$$

where $\boldsymbol{F} : (t; \gamma) \mapsto \boldsymbol{F}(t; \gamma)$ *is a non-degenerate matrix-valued function.*

Proof. This proposition is simply a matrix form of Proposition 2.1. If $\{f_i\}$ is a basis in V and $F(s; \gamma)e_i(s; \gamma) = F_i^j(s; \gamma)f_j$, then (2.15) with $\boldsymbol{F}(s; \gamma) = [F_i^j(s; \gamma)]$ is equivalent to (2.6). □

Proposition 2.5. *If the matrix* $\boldsymbol{L}$ *of a linear transport* L *along paths has a representation*

$$\boldsymbol{L}(t, s; \gamma) = {}^{\star}\boldsymbol{F}^{-1}(t; \gamma)\,{}^{\star}\boldsymbol{F}(s; \gamma) \tag{2.16}$$

for some matrix-valued function ${}^{\star}\boldsymbol{F}(s; \gamma)$, *then all matrix-valued functions* $\boldsymbol{F}$ *representing* $\boldsymbol{L}$ *via* (2.15) *are given by*

$$\boldsymbol{F}(s; \gamma) = \boldsymbol{D}^{-1}(\gamma)\,{}^{\star}\boldsymbol{F}(s; \gamma), \tag{2.17}$$

where $\boldsymbol{D}(\gamma)$ *is a non-degenerate matrix depending only on* γ.

Proof. In fact, this propositions is a matrix variant of Proposition 2.2; $\boldsymbol{D}(\gamma)$ is simply the matrix of the map $D(\gamma)$ in some bases. □

If $\boldsymbol{F}(s;\,\gamma)$ and $\boldsymbol{F}'(s;\,\gamma)$ are two matrix-valued functions, representing the matrix of L via (2.15) in two bases $\{e_i\}$ and $\{e'_i\}$ respectively, then, as a consequence of (2.11), the relation

$$\boldsymbol{F}'(s;\,\gamma) = C(\gamma)\boldsymbol{F}(s;\,\gamma)A(s;\,\gamma) \tag{2.18}$$

holds for some non-degenerate matrix-valued function C of γ.

2.3. Linear transports and derivations along paths

Below we want to consider some properties of the linear transports along paths connected with their ‘differentiability’; in particular, we shall establish a bijective correspondence between them and the derivations along paths. For the purpose is required a smooth, of class at least C^1, transition from fibre to fibre when moving along a path in the base. Rigorously this is achieved by exploring transports in bundles whose *bundle space is a* C^1 *differentiable manifold* which will be supposed from now on.

Let $(E,\,\pi,\,B)$ be a vector bundle whose bundle space E is a C^1 differentiable manifold. A linear transport L^γ along $\gamma : J \to B$ is called *differentiable of class* C^k, $k = 0, 1,$ or simply C^k transport, if for arbitrary $s \in J$ and $u \in \pi^{-1}(\gamma(s))$, the path $\bar{\gamma}_{s;\,u} : J \to E$ with $\bar{\gamma}_{s;\,u}(t) := L^\gamma_{s\to t}u \in \pi^{-1}(\gamma(t)),\ t \in J,$ is a C^k mapping in the bundle space E.[12] If a C^k linear transport has a representation (2.6), the mapping $s \mapsto F(s;\,\gamma)$ is of class C^k. So, the transport L^γ is of class C^k iff $L^\gamma_{s\to t}$ has C^k dependence on s and t simultaneously. If $\{e_i(\cdot;\,\gamma)\}$ is a C^k frame along γ, i.e., $\{e_i(s;\,\gamma)\}$ is a basis in $\pi^{-1}(\gamma(s))$ and the mapping $s \mapsto e_i(s;\,\gamma)$ is of

[12] If E is of class C^r with $r = 0, 1, \ldots, \infty, \omega,$ we can define in an evident way a C^k transport for every $k \le r$.

class C^k for all i, from (2.12) follows that L^γ is of class C^k iff its matrix $\boldsymbol{L}(t, s; \gamma)$ has C^k dependence on s and t.

Let E be a C^1 manifold and S be a set of paths in B, $S \subseteq \{\gamma : J \to B\}$. A transport L along paths in (E, π, B), E being C^r manifold, is said to be of class C^k, $k = 0, 1, \dots, r$, on S if the corresponding transport L^γ along γ is of class C^k for all $\gamma \in S$. A transport along paths may turn to be of class C^k on some set S of paths in B and not to be of class C^k on other set S' of paths in B. Below, through Section 5, the set S will not be specialized and written explicitly; correspondingly, we shall speak simply of C^k transports implicitly assuming that they are such on some set S. Starting from Section 5, we shall suppose B to be a C^1 manifold and the set S to be the one of C^1 paths in B. Further we consider only C^1 linear transports along paths whose matrices will be referred to smooth frames along paths.

Now we want to define what a derivation along paths is (see Definition 2.2 below). For this end we will need some preliminary material.

A *lifting* (or lift)[13] (in (E, π, B)) of $g : X \to B$, X being a set, is a map $\bar{g} : X \to E$ such that $\pi \circ \bar{g} = g$; in particular, the liftings of the identity id_B of B are called *sections* and their set is $\mathrm{Sec}(E, \pi, B) :=$ $\{\sigma \mid \sigma : B \to E, \pi \circ \sigma = id_B\}$. Let $\mathrm{P}(A) := \{\gamma \mid \gamma : J \to A\}$ be the set of paths in a set A and $\mathrm{PLift}(E, \pi, B) := \{\lambda \mid \lambda : \mathrm{P}(B) \to \mathrm{P}(E), (\pi \circ \lambda)(\gamma) = \gamma$ for $\gamma \in \mathrm{P}(B)\}$ be the set of liftings of paths from B to E.[14] The set $\mathrm{PLift}(E, \pi, B)$ is: (i) A natural $\mathbb{C}$-vector space if we put $(a\lambda + b\mu) : \gamma \mapsto$

---

[13] For detail see, e.g., [13].

[14] Every linear transport L along paths provides a lifting of paths: for every $\gamma : J \to B$ fix some $s \in J$ and $u \in \pi^{-1}(\gamma(s))$, the mapping $\gamma \mapsto \bar{\gamma}_{s;u}$ with $\bar{\gamma}_{s;u}(t) := L^\gamma_{s \to t} u$, $t \in J$ is a lifting of paths from B to E.

$a\lambda_\gamma + b\mu_\gamma$ for $a, b \in \mathbb{C}$, $\lambda, \mu \in \mathrm{PLift}(E, \pi, B)$, and $\gamma \in \mathrm{P}(B)$, where, for brevity, we write λ_γ for $\lambda(\gamma)$, $\lambda : \gamma \mapsto \lambda_\gamma$; (ii) A natural left module with respect to complex functions on B: if $f, g : B \to \mathbb{C}$, we define $(f\lambda + g\mu) : \gamma \mapsto (f\lambda)_\gamma + (g\mu)_\gamma$ with $(f\lambda)_\gamma(s) := f(\gamma(s))\lambda_\gamma(s)$ for $\gamma : J \to B$ and $s \in J$; (iii) A left module with respect to the set $\mathrm{PF}(B) := \{\varphi \,|\, \varphi : \gamma \mapsto \varphi_\gamma, \gamma : J \to B, \varphi_\gamma : J \to \mathbb{C}\}$ of functions along paths in the base B: for $\varphi, \psi \in \mathrm{PF}(B)$, we set $(\varphi\lambda + \psi\mu) : \gamma \mapsto (\varphi\lambda)_\gamma + (\psi\mu)_\gamma$, where $(\varphi\lambda)_\gamma(s) := (\varphi_\gamma\lambda_\gamma)(s) := \varphi_\gamma(s)\lambda_\gamma(s)$.

If we consider $\mathrm{PLift}(E, \pi, B)$ as a $\mathbb{C}$-vector space, its dimension is equal to infinity. If we regard $\mathrm{PLift}(E, \pi, B)$ as a left $\mathrm{PF}(B)$-module, its rank is equal to the dimension of (E, π, B) (i.e., to the dimension of the fibre(s) of (E, π, B)). In the last case a basis in $\mathrm{PLift}(E, \pi, B)$ can be constructed as follows.

For every path $\gamma : J \to B$ and $s \in J$, choose a basis $\{e_i(s; \gamma)\}$ in the fibre $\pi^{-1}(\gamma(s))$; if the total space E is a C^1 manifold, then we suppose $e_i(s; \gamma)$ to have a C^1 dependence on s. Define liftings along paths $e_i \in \mathrm{PLift}(E, \pi, B)$ by $e_i : \gamma \mapsto e_i|_\gamma := e_i(\cdot; \gamma)$, i.e., $e_i|_\gamma : s \mapsto e_i|_\gamma(s) := e_i(s; \gamma)$. The set $\{e_i\}$ is a basis in $\mathrm{PLift}(E, \pi, B)$, i.e., for every $\lambda \in \mathrm{PLift}(E, \pi, B)$ there are $\lambda^i \in \mathrm{PF}(B)$ such that $\lambda = \lambda^i e_i$ and $\{e_i\}$ are $\mathrm{PF}(B)$-linearly independent. Actually, for any path $\gamma : J \to B$ and number $s \in J$, we have $\lambda_\gamma(s) \in \pi^{-1}(\gamma(s))$, so there exist numbers $\lambda^i_\gamma(s) \in \mathbb{C}$ such that $\lambda_\gamma(s) = \lambda^i_\gamma(s) e_i(s; \gamma)$. Defining $\lambda^i \in \mathrm{PF}(B)$ by $\lambda^i : \gamma \mapsto \lambda^i_\gamma$ with $\lambda^i_\gamma : s \mapsto \lambda^i_\gamma(s)$, we get $\lambda = \lambda^i e_i$; if $e_i(\cdot; \gamma)$ is of class C^1, so are λ^i_γ. The $\mathrm{PF}(B)$-linear independence of $\{e_i\}$ is an evident corollary of the $\mathbb{C}$-linear independence of $\{e_i(s; \gamma)\}$. As we notice above, if E is C^1 manifold, we choose e_i, i.e., $e_i|_\gamma$, to be of class C^1 and, consequently, the components λ^i, i.e., λ^i_γ, will be of class C^1 too.

Let (E, π, B) be a vector bundle whose bundle space E is C^1 manifold. Denote by $\mathrm{PLift}^k(E, \pi, B)$, $k = 0, 1,$ the set of liftings of paths from B to E such that the lifted paths are C^k paths and by $\mathrm{PF}^k(B)$, $k = 0, 1,$ the set of C^k functions along paths in B, i.e., $\varphi \in \mathrm{PF}^k(B)$ if φ_γ is of class C^k. Obviously, not every path in B has a C^k lifting in E; for instance, all liftings of a discontinuous path in B are discontinuous paths in E. The set of paths in B having C^k liftings in E is $\pi \circ \mathrm{P}^k(E) := \{\pi \circ \bar{\gamma} \,|\, \bar{\gamma} \in \mathrm{P}^k(E)\}$, with $\mathrm{P}^k(E)$ being the set of C^k paths in E. Therefore, when talking of C^k liftings in $\mathrm{PLift}^k(E, \pi, B)$, we shall implicitly assume that they are acting on paths in $\pi \circ \mathrm{P}^k(E) \subset \mathrm{P}(B)$. The discontinuous paths in B are, of course, not in $\pi \circ \mathrm{P}^k(E)$, so that they are excluded from the considerations below.

If E and B are C^1 manifolds, then we denote by $\mathrm{Sec}^k(E, \pi, B)$ the set of C^k sections of the bundle (E, π, B).

Definition 2.2. A *derivation along paths in* (E, π, B) or a *derivation of liftings of paths in* (E, π, B) is a map

$$D : \mathrm{PLift}^1(E, \pi, B) \to \mathrm{PLift}^0(E, \pi, B) \tag{2.19a}$$

which is $\mathbb{C}$-linear,

$$D(a\lambda + b\mu) = aD(\lambda) + bD(\mu) \tag{2.20a}$$

for $a, b \in \mathbb{C}$ and $\lambda, \mu \in \mathrm{PLift}^1(E, \pi, B)$, and the mapping

$$D_s^\gamma : \mathrm{PLift}^1(E, \pi, B) \to \pi^{-1}(\gamma(s)), \tag{2.19b}$$

defined via $D_s^\gamma(\lambda) := ((D(\lambda))(\gamma))(s) = (D\lambda)_\gamma(s)$ and called *derivation along* $\gamma : J \to B$ *at* $s \in J$, satisfies the 'Leibnitz rule':

$$D_s^\gamma(f\lambda) = \frac{\mathrm{d}f_\gamma(s)}{\mathrm{d}s}\lambda_\gamma(s) + f_\gamma(s)D_s^\gamma(\lambda) \tag{2.20b}$$

for every $f \in \mathrm{PF}^1(B)$. The mapping

$$D^{\gamma} : \mathrm{PLift}^1(E, \pi, B) \to \mathrm{P}(\pi^{-1}(\gamma(J))), \tag{2.19c}$$

defined by $D^{\gamma}(\lambda) := (D(\lambda))|_{\gamma} = (D\lambda)_{\gamma}$, is called a *derivation along* γ.

Before continuing with the study of linear transports along paths, we want to say a few words on the links between sections (along paths) and liftings of paths.

The set $\mathrm{PSec}(E, \pi, B)$ of sections along paths of (E, π, B) consists of mappings $\boldsymbol{\sigma} : \gamma \mapsto \boldsymbol{\sigma}_{\gamma}$ assigning to every path $\gamma : J \to B$ a section $\boldsymbol{\sigma}_{\gamma} \in \mathrm{Sec}((E, \pi, B)|_{\gamma(J)})$ of the bundle restricted to $\gamma(J)$. Every (ordinary) section $\sigma \in \mathrm{Sec}(E, \pi, B)$ generates a section $\boldsymbol{\sigma}$ along paths via $\boldsymbol{\sigma} : \gamma \mapsto \boldsymbol{\sigma}_{\gamma} := \sigma|_{\gamma(J)}$, i.e., $\boldsymbol{\sigma}_{\gamma}$ is simply the restriction of σ to $\gamma(J)$; hence $\boldsymbol{\sigma}_{\alpha} = \boldsymbol{\sigma}_{\gamma}$ for every path $\alpha : J_{\alpha} \to B$ with $\alpha(J_{\alpha}) = \gamma(J)$. Every $\boldsymbol{\sigma} \in \mathrm{PSec}(E, \pi, B)$ generates a lifting $\hat{\boldsymbol{\sigma}} \in \mathrm{PLift}(E, \pi, B)$ by $\hat{\boldsymbol{\sigma}} : \gamma \mapsto \hat{\boldsymbol{\sigma}}_{\gamma} := \boldsymbol{\sigma}_{\gamma} \circ \gamma$; in particular, the lifting $\hat{\sigma}$ associated to $\sigma \in \mathrm{Sec}(E, \pi, B)$ is given via $\hat{\sigma}_{\gamma} = \sigma|_{\gamma(J)} \circ \gamma$.

Every derivation D along paths generates a map

$$\overline{D} : \mathrm{PSec}^1(E, \pi, B) \to \mathrm{PLift}^0(E, \pi, B)$$

such that, if $\boldsymbol{\sigma} \in \mathrm{PSec}^1(E, \pi, B)$, then $\overline{D} : \boldsymbol{\sigma} \mapsto \overline{D}\boldsymbol{\sigma} = \overline{D}(\boldsymbol{\sigma})$, where $\overline{D}\boldsymbol{\sigma} : \gamma \mapsto \overline{D}^{\gamma}\boldsymbol{\sigma}$ is a lifting of paths defined by $\overline{D}^{\gamma}\boldsymbol{\sigma} : s \mapsto (\overline{D}^{\gamma}\boldsymbol{\sigma})(s) := D_s^{\gamma}\hat{\boldsymbol{\sigma}}$ with $\hat{\boldsymbol{\sigma}}$ being the lifting generated by $\boldsymbol{\sigma}$, i.e., $\gamma \mapsto \hat{\boldsymbol{\sigma}}_{\gamma} := \boldsymbol{\sigma}_{\gamma} \circ \gamma$. The mapping $\overline{D}$ may be called a derivation of C^1 sections along paths. Notice, if $\gamma : J \to B$ has intersection points and $x_0 \in \gamma(J)$ is such a point, the map $\gamma(J) \to \pi^{-1}(\gamma(J))$ given by $x \mapsto \{D_s^{\gamma}(\boldsymbol{\sigma}) | \gamma(s) = x, s \in J\}$, $x \in \gamma(J)$, is generally multiple-valued at x_0 and, consequently, is not a section of $(E, \pi, B)|_{\gamma(J)}$.

If B is a C^1 manifold and for some $\gamma : J \to B$ there exists a subinterval $J' \subseteq J$ on which the restricted path $\gamma|J : J' \to B$ is without self-intersections, i.e., $\gamma(s) \neq \gamma(t)$ for $s, t \in J'$ and $s \neq t$, we can define the derivation along γ of the sections over $\gamma(J')$ as a map

$$\mathsf{D}^{\gamma} : \mathrm{Sec}^{1}((E, \pi, B)|_{\gamma(J')}) \to \mathrm{Sec}^{0}((E, \pi, B)|_{\gamma(J')}) \tag{2.21}$$

such that

$$(\mathsf{D}^{\gamma}\sigma)(x) := D_{s}^{\gamma}\hat{\sigma} \quad \text{for } x = \gamma(s), \tag{2.22}$$

where $s \in J'$ is unique for a given x and $\hat{\sigma} \in \mathrm{PLift}((E, \pi, B)|_{\gamma(J')})$ is given by $\hat{\sigma} = \sigma|_{\gamma(J')} \circ \gamma|_{J'}$. Generally the map (2.21) defined by (2.22) is multiple-valued at the points of self-intersections of γ, if any, as $(\mathsf{D}^{\gamma}\sigma)(x)$ $:= \{D_{s}^{\gamma}\hat{\sigma} : s \in J, \gamma(s) = x\}$. The so-defined map $\mathsf{D} : \gamma \mapsto \mathsf{D}^{\gamma}$ is called a *section-derivation along paths*. As we said, it is single-valued only along paths without self-intersections.

Generally a section along paths or lifting of paths does not define a (single-valued) section of the bundle as well as to a lifting along paths there does not correspond some (single-valued) section along paths. The last case admits one important special exception, viz. if a lifting λ is such that the lifted path λ_{γ} is an 'exact topological copy' of the underlying path $\gamma : J \to B$, i.e., if there exist $s, t \in J$, $s \neq t$ for which $\gamma(s) = \gamma(t)$, then $\lambda_{\gamma}(s) = \lambda_{\gamma}(t)$, which means that if γ has intersection points, then the lifting λ_{γ} also possesses such points and they are in the fibres over the corresponding intersection points of γ. Such a lifting λ generates a section $\overline{\lambda} \in \mathrm{PSec}(E, \pi, B)$ along paths given by $\overline{\lambda} : \gamma \mapsto \overline{\lambda}_{\gamma}$ with $\overline{\lambda} : \gamma(s) \mapsto \lambda_{\gamma}(s)$. In the general case, the mapping $\gamma(s) \mapsto \lambda_{\gamma}(s)$ for a lifting λ of paths is multiple-valued at the points of self-intersection of $\gamma : J \to B$, if any; for injective path γ this map is a section of $(E, \pi, B)|_{\gamma(J)}$. Such mappings will be called *multiple-valued sections along paths*.

Definition 2.3. The *derivation D along paths generated by a C^{1} linear transport L along paths* in (E, π, B), E being a C^{1} manifold, is a map of type (2.19a) such that for every path $\gamma : J \to B$, we have $D^{\gamma} : \lambda \mapsto (D\lambda)_{\gamma}$ with $D^{\gamma}\lambda : s \mapsto D_s^{\gamma}\lambda$, $s \in J$, where D_s^{γ} is a map (2.19b) given via

$$D_s^{\gamma}(\lambda) := \lim_{\varepsilon \to 0}\left\{\frac{1}{\varepsilon}\left[L^{\gamma}_{s+\varepsilon\to s}\lambda_{\gamma}(s+\varepsilon) - \lambda_{\gamma}(s)\right]\right\} \tag{2.23}$$

for every lifting $\lambda \in \mathrm{PLift}^1(E, \pi, B)$ with $\lambda : \gamma \mapsto \lambda_{\gamma}$. The mapping D^{γ} (resp. D_s^{γ}) will be called a *derivation along* γ *generated by* L (resp. a *derivation along* γ *at s assigned to* L).

Remark 2.3. The operator D_s^{γ} is an analogue of the covariant derivative assigned to a linear connection; cf., e.g., [4, p. 139, equation (12)].

Remark 2.4. Notice, if γ has self-intersections and $x_0 \in \gamma(J)$ is such a point, the mapping $x \mapsto \pi^{-1}(x)$, $x \in \gamma(J)$, given by $x \mapsto \{D_s^{\gamma}(\lambda) \,|\, \gamma(s) = x, s \in J\}$ is, generally, multiple-valued at x_0.

Let L be a linear transport along paths in (E, π, B). For every path $\gamma : J \to B$, choose some $s_0 \in J$ and $u_0 \in \pi^{-1}(\gamma(s_0))$. The mapping

$$\overline{L} : \gamma \mapsto \overline{L}^{\gamma}_{s_0, u_0},\ \overline{L}^{\gamma}_{s_0, u_0} : J \to E,\ \overline{L}^{\gamma}_{s_0, u_0} : t \mapsto \overline{L}^{\gamma}_{s_0, u_0}(t) := L^{\gamma}_{s_0 \to t} u_0 \tag{2.24}$$

is, evidently, a lifting of paths.

Definition 2.4. The lifting of paths $\overline{L}$ from B to E in (E, π, B) defined via (2.24) is called *lifting* (*of paths*) *generated by the* (*linear*) *transport L*.

Equations (2.2) and (2.4), combined with (2.23), immediately imply

$$D_t^{\gamma}(\overline{L}) \equiv 0, \quad t \in J, \tag{2.25}$$

$$D_s^{\gamma}(a\lambda + b\mu) = aD_s^{\gamma}\lambda + bD_s^{\gamma}\mu,\ \ a, b \in \mathbb{C},\ \ \lambda, \mu \in \mathrm{PLift}^1(E, \pi, B), \tag{2.26}$$

where $s_0 \in J$ and u_0 are fixed. In other words, equation (2.25) means that the lifting $\overline{L}$ is constant along every path γ with respect to D.

Let $\{e_i(s;\, \gamma)\}$ be a smooth field of bases along $\gamma : J \to B$, $s \in J$. Combining (2.12) and (2.23), we find the explicit local action of D_s^γ :[15]

$$D_s^\gamma \lambda = \left[\frac{\mathrm{d}\lambda_\gamma^i(s)}{\mathrm{d}s} + \Gamma_j^i(s;\, \gamma)\lambda_\gamma^j(s) \right] e_i(s;\, \gamma). \tag{2.27}$$

Here the (2-*index*) *coefficients* Γ_j^i of the linear transport L are defined by

$$\Gamma_j^i(s;\, \gamma) := \left. \frac{\partial L_j^i(s,\, t;\, \gamma)}{\partial t} \right|_{t=s} = - \left. \frac{\partial L_j^i(s,\, t;\, \gamma)}{\partial s} \right|_{t=s} \tag{2.28}$$

and, evidently, uniquely determine the derivation D generated by L.

A trivial corollary of (2.26) and (2.27) is the assertion that the derivation along paths generated by a linear transport is actually a derivation along paths (see Definition 2.2).

Below, we shall prove that, freely speaking, a linear transport along path(s) can locally, in a given field of local bases, be described equivalently by the set of its local coefficients (with the transformation law (2.30) written below).

If the transport's matrix $\boldsymbol{L}$ has a representation (2.15), from (2.28) we get

$$\boldsymbol{\Gamma}(s;\, \gamma) := [\Gamma_j^i(s;\, \gamma)] = \left. \frac{\partial \boldsymbol{L}(s,\, t;\, \gamma)}{\partial t} \right|_{t=s} = \boldsymbol{F}^{-1}(s;\, \gamma) \frac{\mathrm{d}\boldsymbol{F}(s;\, \gamma)}{\mathrm{d}s}. \tag{2.29}$$

From here, (2.11) and (2.14), we see that the change $\{e_i\} \to \{e_i' = A_i^j e_i\}$ of the bases along a path γ with a non-degenerate C^1 matrix-valued function $A(s;\, \gamma) := [A_i^j(s;\, \gamma)]$ implies

$$\boldsymbol{\Gamma}(s;\, \gamma) = [\Gamma_j^i(s;\, \gamma)] \mapsto \boldsymbol{\Gamma}'(s;\, \gamma) = [\Gamma'^i_j(s;\, \gamma)]$$

---

[15] The existence of derivatives like $\mathrm{d}\lambda_\gamma^i(s)/\mathrm{d}s$, viz. that $\lambda_\gamma^i : J \to \mathbb{K}$ are C^1 mappings, follows from $\lambda \in \mathrm{PLift}^1(E,\, \pi,\, B)$.

with

$$\mathbf{\Gamma}'(s;\gamma) = A^{-1}(s;\gamma)\mathbf{\Gamma}(s;\gamma)A(s;\gamma) + A^{-1}(s;\gamma)\frac{\mathrm{d}A(s;\gamma)}{\mathrm{d}s}. \tag{2.30}$$

Proposition 2.6. *Let along every* (*resp. given*) *path* $\gamma : J \to B$ *be given a geometrical object* Γ *whose local components* Γ^i_j *in a field of bases* $\{e_i\}$ *along* γ *change according to* (2.30) *with* $\mathbf{\Gamma}(s;\gamma) = [\Gamma^i_j(s;\gamma)]$. *There exists a unique linear transport L along paths* (*resp. along* γ) *the matrix of whose coefficients is exactly* $\mathbf{\Gamma}(s;\gamma)$ *in* $\{e_i\}$ *along* γ. *Moreover, the matrix of the components of L in* $\{e_i\}$ *is*

$$\boldsymbol{L}(t, s;\gamma) = Y(t, s_0; -\mathbf{\Gamma}(\cdot;\gamma))Y^{-1}(s, s_0; -\mathbf{\Gamma}(\cdot;\gamma)), \quad s, t \in J, \tag{2.31}$$

where $s_0 \in J$ *is arbitrarily fixed and the matrix* $Y(s, s_0; Z)$, *for a* C^0 *matrix-valued function* $Z : s \mapsto Z(s)$, *is the unique solution of the initial-valued problem*

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = Z(s)Y, \quad Y = Y(s, s_0; Z), \quad s \in J, \tag{2.32a}$$

$$Y(s_0, s_0; Z) = \mathbb{1}. \tag{2.32b}$$

Proof. At the beginning, we note that the proof of existence and uniqueness of the solution of (2.32) can be found in [11, Chapter IV, Section 1].

Given a linear transport L with a matrix (2.15). Suppose its components are exactly $\Gamma^i_j(s;\gamma)$ in $\{e_i\}$. Solving (2.29) with respect to $\mathrm{d}\boldsymbol{F}^{-1}/\mathrm{d}s$, we obtain $\mathrm{d}\boldsymbol{F}^{-1}(s;\gamma)/\mathrm{d}s = -\mathbf{\Gamma}(s;\gamma)\boldsymbol{F}^{-1}(s;\gamma)$ and, consequently, $\boldsymbol{F}^{-1}(s;\gamma) = Y(s, s_0; -\mathbf{\Gamma}(\cdot;\gamma))\boldsymbol{F}^{-1}(s_0;\gamma)$. So, as a result of (2.15), the matrix of L is (2.31). Because of [11, Chapter IV, equation (1.10)], the expression

$$Y(t, s; Z) = Y(t, s_0; Z)Y(s_0, s; Z) = Y(t, s_0; Z)Y^{-1}(s, s_0; Z)$$

is independent of s_0. Besides, as a consequence of (2.30), the matrix

(2.31) transforms according to (2.11) when the local bases are changed. Hence (2.3) holds and, due to (2.12), the linear map L with a matrix (2.31) in $\{e_i\}$ is a linear transport along γ. In this way we have proved two things: On one hand, a linear map with a matrix (2.31) in $\{e_i\}$ is a linear transport with local coefficients $\Gamma^i_j(s;\gamma)$ in $\{e_i\}$ along γ and, on the other hand, any linear transport with local coefficients $\Gamma^i_j(s;\gamma)$ in $\{e_i\}$ has a matrix (2.31) in $\{e_i\}$. □

Now we are ready to prove a fundamental result: ***there exists a bijective mapping between the sets of*** C^1***-linear transports along paths and derivations along paths***. *The explicit correspondence between linear transports along paths and derivations along paths is through the equality of their local coefficients and components*, *respectively*, *in a given field of bases*. After the proof of this result, we shall illustrate it in a case of linear connections on a manifold.

Proposition 2.7. *A mapping* (2.19a) (*resp*. (2.19c)) *is a derivation along paths* (*resp*. *along* γ) *iff there exists a unique linear transport along paths* (*resp*. *along* γ) *generating it via* (2.23).

Proof. Let $\{e_i(s;\gamma)\}$ be a frame along γ and D (resp. D^γ) be a derivation along paths (resp. along γ). Define the *components*[16] $\Gamma^i_j(s;\gamma)$ of D^γ in $\{e_i\}$ by the expansion

$$D^\gamma_s \hat{e}_j =: \Gamma^i_j(s;\gamma)e_i(s;\gamma), \tag{2.33}$$

where $\hat{e}_i : \gamma \mapsto e_i(\cdot;\gamma)$ are liftings of the paths generated by e_i. They uniquely define D^γ as (2.20) implies (2.27). Besides, it is trivial to verify the transformation law (2.30) for them. So, by Proposition 2.6, there is a

[16] In connection with the theory of normal frames (see Section 3 and further), it is convenient to call $\Gamma^i_j(s;\gamma)$ also (2-*index*) *coefficients* of D^γ. This is consistent with the fact that Γ^i_j are coefficients of some linear transport along paths (*see below*).

unique linear transport along paths (resp. along γ) with the same local coefficients.

Conversely, as we already proved, to any linear transport L along paths (resp. along γ) there corresponds a derivation D^{γ} along γ given via (2.23) whose components coincide with the coefficients of L^{γ} and transform according to (2.30). $\square$

We end this section with two examples, the first of which is quite important and well known.

Let ∇ be a linear connection (covariant derivative) [34] on a C^1 differentiable manifold M and $\Gamma^i_{jk}(x)$, $i, j, k = 1, \dots, \dim M$, $x \in M$, be its local coefficients in a field $\{E_i(x)\}$ of bases in the tangent bundle over M, i.e., $\nabla_{E_i} E_j = \Gamma^k_{ji} E_k$. If γ is a C^1 path in M, then $\nabla_{\dot\gamma} = \dot\gamma^i \nabla_{E_i}$, $\dot\gamma$ being the vector field tangent to γ, is a derivation along γ (in the bundle tangent to M) with local components

$$\Gamma^i_j(s; \gamma) = \Gamma^i_{jk}(\gamma(s))\dot\gamma^k(s). \tag{2.34}$$

It is a simple exercise to verify that *the unique linear transport along paths corresponding, in accordance with* Proposition 2.7, *to the derivation with local components given by* (2.34) *is exactly the parallel transport generated via the initial connection* ∇.

As a second example, we consider a concrete kind of a linear transport L in the trivial line bundle $(B \times \mathbb{R}, \mathrm{pr}_1, B)$, where B is a topological space, which in particular can be a C^0 manifold, $\times$ is the Cartesian product sign, and $\mathrm{pr}_1 : B \times \mathbb{R} \to B$ is the projection on B. An element of $B \times \mathbb{R}$ is of the form $u = (b, y)$ for some $b \in B$ and $y \in \mathbb{R}$ and the fibre over $c \in B$ is $\mathrm{pr}_1^{-1}(c) = \{c\} \times \mathbb{R} = \{(c, z) : z \in \mathbb{R}\}$; the linear structure of $\mathrm{pr}_1^{-1}(c)$ is given by $\lambda_1(c, z_1) + \lambda_2(c, z_2) = (c, \lambda_1 z_1 + \lambda_2 z_2)$ for $\lambda_1, \lambda_2, z_1, z_2 \in \mathbb{R}$. The bundle $(B \times \mathbb{R}, \mathrm{pr}_1, B)$ admits a global frame field $\{e_1\}$ consisting of a single section $e_1 \in \mathrm{Sec}(B \times \mathbb{R}, \mathrm{pr}_1, B)$ such that $e_1 : B \ni b \mapsto e_1(b) = (b, 1) \in \mathrm{pr}_1^{-1}(b)$. For $\gamma : J \to B$ and $s, t \in J$, define $L : \gamma \mapsto L^{\gamma} : (s, t)$

$\mapsto L^{\gamma}_{s\to t} : \mathrm{pr}_1^{-1}(\gamma(s)) \to \mathrm{pr}_1^{-1}(\gamma(t))$ by

$$L^{\gamma}_{s\to t}(u) = \left(\gamma(t), \frac{f(\gamma(s))}{f(\gamma(t))}\, y\right) \quad \text{for } u = (\gamma(s), y) \in \mathrm{pr}_1^{-1}(\gamma(s)), \tag{2.35}$$

where $f : \gamma(J) \to \mathbb{R}\setminus\{0\}$ is a non-vanishing function on $\gamma(J)$. The verification of (2.2)-(2.4) is trivial and hence L is a linear transport along paths. Its matrix in the frame $\{e_1\}$ is $\boldsymbol{L}(t, s; \gamma) = L_1^1(t, s; \gamma) = \frac{f(\gamma(s))}{f(\gamma(t))}$, in conformity with (2.15). If $f \circ \gamma : J \to \mathbb{R}\setminus\{0\}$ is of class C^1, the single coefficient of L is (see (2.28)) $\Gamma_1^1(s; \gamma) = \frac{\mathrm{d}}{\mathrm{d}s}\ln(f(\gamma(s)))$; however, this coefficient is a useful quantity if $B \times \mathbb{R}$ (and hence B) is a C^1 manifold– see (2.27). Going some pages ahead (see Proposition 4.2 and Definition 3.4 below), we see that the transport L satisfies equation (4.1) below and therefore admits normal frames; in particular the frame $\{f_1\}$ such that (see (4.2) below)

$$f_1|_{\gamma(s)} = L^{\gamma}_{s_0\to s}(e_1|_{\gamma(s_0)}) = \left(\gamma(s), \frac{f(\gamma(s_0))}{f(\gamma(s))}\right)$$

for a fixed $s_0 \in J$ and any $s \in J$ is normal along γ, i.e., the matrix of L in $\{f_1\}$ is the identity matrix (the number one in the particular case).

3. Normal Frames

The parallel transport in a Euclidean space $\mathbb{E}^n$ (or in $\mathbb{R}^n$) has the property that, in Cartesian coordinates, it preserves the components of the vectors that are transported, changing only their initial points [5]. This evident observation, which can be taken even as a definition for parallel transport in $\mathbb{E}^n$, is of fundamental importance when one tries to generalize the situation.

Let a linear transport L along paths be given in a vector bundle (E, π, B), $U \subseteq B$ be an arbitrary subset in B, and $\gamma : J \to U$ be a path in U.

Definition 3.1. A *frame field* (*of bases*) in $\pi^{-1}(\gamma(J))$ is called *normal along* γ *for* L if the matrix of L in it is the identity matrix along the given path γ.

Definition 3.2. A *frame field* (*of bases*) defined on U is called *normal on* U *for* L if it is normal along every path $\gamma : J \to U$ in U. The *frame* is called *normal for* L if $U = B$.

Notice that 'normal' refers to a 'normal form' as opposed to orthogonal to tangential.

In the context of the present work, we pose the following problem. Given a linear transport along paths, is it possible to find a local basis or a field of bases (frame) in which its matrix is the identity one? Below we shall rigorously formulate and investigate this problem.[17] If frames with this property exist, we call them *normal* (for the transport given). According to (2.12), the linear transports do not change vectors' components in such a frame and, conversely, a frame with the last property is normal. Hence the normal frames are a straightforward generalization of the Cartesian coordinates in Euclidean space.[18] Because of this and following the established terminology with respect to metrics [1, 34], we call *Euclidean* a linear transport admitting normal frame(s).

Since a frame field, for instance on a set U, is actually a basis in the set $\mathrm{Sec}((E, \pi, B)|_U) = \mathrm{Sec}(\pi^{-1}(U), \pi|_U, U)$, we call such a basis *normal* if the corresponding field of bases is normal on U.

Definition 3.3. A *linear transport along paths* (or *along a path* γ) is called *Euclidean along some* (*or the given*) *path* γ if it admits a frame normal along γ.

---

[17] The problem for exploring normal frames for linear transports seems to be set in the present paper for the first time. One studies usually normal frames for some kinds of derivations which, in particular, can be linear connections [29].

[18] According to the argument presented, it is more natural to call Cartesian the special kind of local bases (or frames) we are talking about. But, in our opinion and for historical reasons, it is better to use the already established terminology for linear connections and derivations of the tensor algebra over a differentiable manifold (see below and [18, Appendix A] or [17]).

Definition 3.4. A *linear transport along paths* is called *Euclidean* on U if it admits frame(s) normal on U. It is called *Euclidean* if $U = B$.

We want to note that the name "Euclidean transport" is connected with the fact that if we put $B = \mathbb{R}^n$ and $\pi^{-1}(x) = T_x(\mathbb{R}^n)$ (the tangent space to $\mathbb{R}^n$ at x) and identify $T_x(\mathbb{R}^n)$ with $\mathbb{R}^n$, then in an orthonormal frame, i.e., in Cartesian coordinates, the Euclidean transport coincides with the standard parallel transport in $\mathbb{R}^n$ (leaving the vectors' components unchanged).

Euclidean transports exist always in a case of a trivial bundle $(B \times V, \mathrm{pr}_1, B)$, with V being a vector space and $\mathrm{pr}_1 : B \times V \to B$ being the projection on B; cf. the last example at the end of Subsection 2.3. For instance, the mapping $L^{\gamma}_{s\to t}(\gamma(s), v) = (\gamma(t), v)$, for $v \in V$, defines a Euclidean transport which is similar to the parallel one in $\mathbb{R}^n$. Indeed, if $\{f_i : i = 1, \dots, \dim V\}$ is a basis of V and $v = v^i f_i$, then $e_i : p \mapsto e_i|_p :=$ (p, f_i), $p \in B$, is a (global) frame on B if we put $v^i e_i|_p = (p, v^i f_i) = (p, v)$ and therefore $L^{\gamma}_{s\to t}(e_i|_{\gamma(s)}) = e_i|_{\gamma(t)}$, which means that $L : \gamma \mapsto L^{\gamma} : (s, t)$ $\mapsto L^{\gamma}_{s\to t}$ is a Euclidean transport and $\{e_i\}$ is a normal frame for it (see Corollary 3.1 below).

Below we present some general results concerning normal frames leaving the problem of their existence for the next section.

Proposition 3.1. *The following statements are equivalent in a given frame* $\{e_i\}$ *over* $U \subseteq B$:

(i) *The matrix of L is the identity matrix on U, i.e., along every path* γ *in U*

$$\boldsymbol{L}(t, s; \gamma) = \mathbb{1}. \tag{3.1a}$$

(ii) *The matrix of L along every* $\gamma : J \to U$ *depends only on* γ, *i.e., it is independent of the points at which it is calculated*:

$$\boldsymbol{L}(t,\ s;\ \gamma) = C(\gamma), \tag{3.1b}$$

where C is a matrix-valued function of γ.

(iii) *If E is a C^1 manifold, the coefficients $\Gamma^i_j(s;\ \gamma)$ of L vanish on U, i.e., along every path γ in U*

$$\boldsymbol{\Gamma}(s;\ \gamma) = 0. \tag{3.1c}$$

(iv) *The explicit local action of the derivation D along paths generated by L reduces on U to differentiation of the components of the liftings with respect to the path's parameter if the path lies entirely in U:*

$$D^{\gamma}_{s}\lambda = \frac{\mathrm{d}\lambda^{i}_{\gamma}(s)}{\mathrm{d}s}\, e_i(s;\ \gamma), \tag{3.1d}$$

where $\lambda = \lambda^i e_i \in \mathrm{PLift}^1((E,\ \pi,\ B)|_U)$, with E being a C^1 manifold, and $\lambda : \gamma \mapsto \lambda_\gamma$.

(v) *The transport L leaves the vectors' components unchanged along any path in U:*

$$L^{\gamma}_{s\to t}(u^i e_i(s;\ \gamma)) = u^i e_i(t;\ \gamma), \tag{3.1e}$$

where $u^i \in \mathbb{C}$.

(vi) *The basic vector fields are L-transported along any path $\gamma : J \to U$:*

$$L^{\gamma}_{s\to t}(e_i(s;\ \gamma)) = e_i(t;\ \gamma). \tag{3.1f}$$

Proof. We have to prove the equivalences

$$\boldsymbol{L}(t,\ s;\ \gamma) = C(\gamma) \Leftrightarrow \boldsymbol{L}(t,\ s;\ \gamma) = \mathbb{1} \Leftrightarrow \boldsymbol{\Gamma}(s;\ \gamma) = 0$$

$$\Leftrightarrow D^{\gamma}_{s}\lambda = \frac{\mathrm{d}\lambda^{i}_{\gamma}(s)}{\mathrm{d}s}\, e_i(s;\ \gamma) \Leftrightarrow L^{\gamma}_{s\to t}(u^i e_i(s;\ \gamma)) = u^i e_i(t;\ \gamma)$$

$$\Leftrightarrow L^{\gamma}_{s\to t}(e_i(s;\ \gamma)) = e_i(t;\ \gamma). \tag{3.2}$$

If $\boldsymbol{L}(t, s; \gamma) = C(\gamma)$, then, using the representation (2.15), we get $\boldsymbol{F}(t; \gamma) = \boldsymbol{F}(s; \gamma)C(\gamma) = \boldsymbol{F}(s_0; \gamma)$ for some fixed $s_0 \in J$ as s and t are arbitrary, so $\boldsymbol{L}(t, s_0; \gamma) = \boldsymbol{F}^{-1}(s_0; \gamma)\boldsymbol{F}(s_0; \gamma) = \mathbb{1}$. The inverse implication is trivial. The second equivalence is a consequence of (2.29) and (2.15) since $\boldsymbol{\Gamma} = 0$ implies $\boldsymbol{F}(s; \gamma) = \boldsymbol{F}(\gamma)$, while the third one is a corollary of (2.27). The validity of the last but one equivalence is a consequence of $\boldsymbol{L}(t, s; \gamma) = \mathbb{1} \Leftrightarrow L^{\gamma}_{s\to t}(u^i e_i(s; \gamma)) = u^i e_i(t; \gamma)$ which follows from (2.12). The last equivalence is a corollary of the linearity of L and the arbitrariness of u^i. □

Remark 3.1. An evident corollary of the last proof is

$$\boldsymbol{L}(t, s; \gamma) = \mathbb{1} \Leftrightarrow \boldsymbol{F}(s; \gamma) = \boldsymbol{B}(\gamma) \tag{3.3}$$

with $\boldsymbol{B}$ being a matrix-valued function of the path γ only. According to Proposition 2.5, this dependence is inessential and, consequently, in a normal frame, we can always choose representation (2.15) with

$$\boldsymbol{F}(s; \gamma) = \mathbb{1}. \tag{3.4}$$

Corollary 3.1. *The equalities* (3.1a)-(3.1f) *are equivalent and any one of them expresses a necessary and sufficient condition for a frame to be normal for L in U. In particular, for* $U = \gamma(J)$ *they express such a condition along a fixed path* γ.

Proof. This result is a direct consequence of Definition 3.2 and Proposition 3.1. □

A lifting of paths $\lambda \in \mathrm{PLift}(E, \pi, B)$ is called *L-transported* along $\gamma : J \to B$, if for every $s, t \in J$ is fulfilled $\lambda_\gamma(t) = L^{\gamma}_{s\to t}\lambda_\gamma(s)$ with $\lambda : \gamma \mapsto \lambda_\gamma$. Hence a frame $\{e_i(s, \gamma)\}$ along γ is L-transported along γ if the basic vectors $\hat{e}_1, \ldots, \hat{e}_{\dim B}$, considered as liftings of paths, i.e., $\hat{e}_i : \gamma \mapsto e_i(\cdot; \gamma)$, are L-transported along γ.

Therefore a frame is normal for L along γ iff it is L-transported along γ, i.e., if, by definition, its basic vectors $e_i(s; \gamma)$ satisfy (3.1f). As we shall see below (see Proposition 3.3), this allows a convenient and useful way for constructing normal frames, if any.

For the above reasons, sometimes, it is convenient for the Definition 3.1 to be replaced, equivalently, by the next ones.

Definition 3.1′. If E is a C^1 manifold, a *frame* (*or frame field*) over $\gamma(J)$ is called *normal along* $\gamma : J \to B$ for a linear transport L along paths if the coefficients of L along γ vanish in it.

Definition 3.1″. A *frame* over $\gamma(J)$ is called *normal along* $\gamma : J \to B$ for a linear transport L along paths if it is L-transported along γ.

The last definition of a normal frame is, in a sense, the 'most invariant (basis-free)' one.

The next proposition describes the class of normal frames, if any, along a given path.

Proposition 3.2. *All frames normal for some linear transport along paths which is Euclidean along a certain* (*fixed*) *path are connected by linear transformations whose matrices may depend only on the given path but not on the point at which the bases are defined.*

Proof. Let $\{e_i\}$ and $\{e'_i := A_i^j e_j\}$ be frames normal along $\gamma : J \to B$ for a linear transport L along paths and $\boldsymbol{L}$ and $\boldsymbol{L}'$ be the matrices of L in them respectively. As, by definition $\boldsymbol{L} = \boldsymbol{L}' = \mathbb{1}$, from (2.11), we get $A(s; \gamma) = A(t; \gamma)$ for any $s, t \in J$, i.e., $A(s; \gamma)$ depends only on γ and not on s.

If E is a C^1 manifold and $\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}'$ are the matrices of the coefficients of L in $\{e_i\}$ and $\{e'_i\}$, respectively, by Proposition 3.1 we have $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}' = 0$, so the transformation law (2.30) implies $\mathrm{d}A(s; \gamma)/\mathrm{d}s = 0$, $A(s; \gamma) := [A_i^j(s; \gamma)]$. □

Corollary 3.2. *All frames normal for a Euclidean transport along a given path are obtained from one of them via linear transformations whose matrices may depend only on the path given but not on the point at which the bases are defined.*

Proof. See Proposition 3.2 or its proof. □

The following two results describe the class of all frames normal on an arbitrary set U, if such frames exist.

Corollary 3.3. *If a linear transport along paths admits frames normal on a set* U, *then all of them are connected via linear transformations with constant* (*on* U) *matrices.*

Proof. Let $\{e_i\}$ and $\{e'_i := A_i^j e_j\}$ be frames normal on U and $x \in U$. By Proposition 3.2 (see also Definition 3.2), for any paths β and γ in U passing through x, we have $A(x) := [A_i^j] = \boldsymbol{B}(\beta) = \boldsymbol{B}(\gamma)$ for some matrix-valued function $\boldsymbol{B}$ on the set of the paths in U. Hence $A(x) = \text{const}$ on U, due to the arbitrariness of β and γ. □

Corollary 3.4. *If a linear transport along paths admits a frame normal on a set* U, *then all such frames on* U *for it are obtained from that frame by linear transformations with constant* (*on* U) *coefficients.*

Proof. The result immediately follows from Corollary 3.3. □

We end this section with a simple but important result which shows how the normal frames, if any, can be constructed along a given path.

Proposition 3.3. *If* L *is Euclidean transport along* $\gamma : J \to B$ *and* $\{e_i^0\}$ *is a basis in* $\pi^{-1}(\gamma(s_0))$ *for some* $s_0 \in J$, *then the frame* $\{e_i\}$ *along* γ *defined by*

$$e_i(s; \gamma) = L^{\gamma}_{s_0 \to s}(e_i^0), \quad s \in J \tag{3.5}$$

is normal for L *along* γ.

Proof. Due to (2.2) and (3.5), the frame $\{e_i\}$ satisfies (3.11) along γ. Hence, by Corollary 3.1, it is normal for L along γ. □

An analogous result on a set $U \subseteq B$ will be presented in the next section (see below - Proposition 4.5).

4. On the Existence of Normal Frames

In the previous section there were derived a number of properties of the normal frames, but the problem of their existence was neglected. This is the subject of the present section.

At a given point $x \in B$ the following result is valid.

Proposition 4.1. *A linear transport L^{γ} along $\gamma : J \to B$ such that $\gamma(J) = \{x\}$ for a given point $x \in B$ admits normal frame(s) iff it is the identity mapping of the fibre over x, i.e., $L^{\gamma}_{s \to t} = id_{\pi^{-1}(x)}$ for every $s, t \in J$.*

Proof. The sufficiency is trivial (see Definition 2.1). If $\{e_i\}$ is normal for L^{γ} (at x), then $L^{\gamma}_{s \to t}(u^i e_i|_x) = u^i L^{\gamma}_{s \to t} e_i|_x = u^i e_i|_x$, $u^i \in \mathbb{C}$ due to $\gamma(s) = \gamma(t) = x$ and Proposition 3.1, point (iv). Therefore $L^{\gamma}_{s \to t} = id_{\pi^{-1}(x)}$. □

Thus, for a degenerate path $\gamma : J \to \{x\} \subset B$ for some $x \in B$, the identity mapping of the fibre over x is the only realization of a Euclidean transport along paths. Evidently, for such a transport every basis of that fibre is a frame normal at x for it.

Proposition 4.2. *A linear transport L along paths admits frame(s) normal along a given path $\gamma : J \to B$ iff*

$$L^{\gamma}_{s \to t} = id_{\pi^{-1}(\gamma(s))} \quad \text{for every } s, t \in J \text{ such that } \gamma(s) = \gamma(t), \tag{4.1}$$

i.e., if γ contains loops, the L-transport along each of them reduces to the identity mapping of the fibre over the initial/final point of the transportation.

Remark 4.1. For $s = t$ the equation (4.1) is identically satisfied due to (2.3). But for $s \neq t$, if such s and t exist, this is highly non-trivial restriction: it means that the result of L-transportation along γ of a vector $u \in \pi^{-1}(x_0)$ for some $x_0 \in \gamma(J)$ from x_0 to a point $x \in \gamma(J)$ is independent of how long the vector has 'traveled' along γ or, more precisely, if $x_0, x \in \gamma(J)$ are fixed and, for each $y \in \gamma(J)$, $J_y := \{r \in J : \gamma(r) = y\}$, then the vector $L^{\gamma}_{s_0 \to s}(u)$ is independent of the choice of the points $s_0 \in J_{x_0}$ and $s \in J_x$ (if some of the sets J_{x_0} and/or J_x contain more than one point). This is trivial if γ is without self-intersections (see (2.2)). If γ has self-intersections, e.g., if γ intersects itself one time at $\gamma(s)$, i.e., if $\gamma(s) = \gamma(t)$ for some $s, t \in J$, $s \neq t$, then the result of L-transportation of $u \in \pi^{-1}(\gamma(s_0))$ from $x_0 = \gamma(s_0)$ to $x = \gamma(s) = \gamma(t)$ along γ is $u_s = L^{\gamma}_{s_0 \to s} u$ or $u_t = L^{\gamma}_{s_0 \to t} u$. We have $u_s = u_t$ iff (4.1) holds. Rewording, if we fix some $u_0 \in \pi^{-1}(\gamma(s_0))$, the bundle-valued function $u : \gamma(J) \to E$ given by $u : \gamma(s) \to u_s = L^{\gamma}_{s_0 \to s} u_0 \in \pi^{-1}(\gamma(s))$ for $s \in J$ is single-valued iff (4.1) is valid.[19] Notice, since $\pi \circ u_s \equiv \gamma(s)$ (see (2.1)), the map u is (a single-valued) lifting of γ in E through u_0 irrespectively of the validity of (4.1).

Prima facie the above may be reformulated in terms of the concept of holonomy in vector bundles [42, pp. 51-54]. But a rigorous analysis reveals that this is impossible in the general case without imposing further restrictions, like equation (4.4) below, on the transports involved. For instance, without requiring equation (4.4) below to be valid, one cannot introduce the concept of a holonomy group.

Proof. If L is Euclidean along γ, then (4.1) follows from equation (3.1e) as it holds for every $u^i \in \mathbb{C}$ in some normal frame $\{e_i\}$. Conversely, let (4.1) be valid. Put

$$e_i|_{\gamma(s)} := L^{\gamma}_{s_0 \to s}(e_i^0), \tag{4.2}$$

where $\{e_i^0\}$ is a fixed basis in $\pi^{-1}(\gamma(s_0))$ for a fixed $s_0 \in J$. Due to the nondegeneracy of L, $\{e_i\}$ is a basis at $\gamma(s)$ for every s. According to (4.1),

[19] The so-defined map u is a section along γ of (E, π, B) [26]. Generally it is a multiple-valued map (see Section 2).

the so-defined field of bases $\{e_i\}$ along γ is single-valued. By means of (2.2), we easily verify that (3.1f) holds for $\{e_i\}$. Hence $\{e_i\}$ is normal for L along γ. □

Remark 4.2. Regardless of the validity of (4.1), equation (4.2) defines a field of, generally multiple-valued, normal frames in the set of sections along γ of (E, π, B). (For details on sections along paths, see Section 2.)

Such a multi-valued property can be avoided if γ is supposed to be injective ($\Leftrightarrow$ without self-intersections). *Prima facie* one may think that this solves the multi-valued problem in the general case by decomposing γ into a union of injective paths. However, this is not the most general situation because a transport along a composition of paths, in general, is not equal to the composition of the transports along its constituent sub-paths (see equation (4.4) below); besides, since equation (4.8) below does not hold, in general, the absence of a natural/canonical definition of composition (product) of paths introduces an additional indefiniteness.

Corollary 4.1. *Every linear transport along paths is Euclidean along every fixed path without self-intersections.*

Proof. For a path $\gamma : J \to B$ without self-intersections, the equality $\gamma(s) = \gamma(t)$, $s, t \in J$ is equivalent to $s = t$. So, according to (2.3), the condition (4.1) is identically satisfied. □

Now we shall establish an important necessary and sufficient condition for the existence of frames normal on an arbitrary subset $U \subseteq B$.

Theorem 4.1. *A linear transport along paths admits frames normal on some set* (*resp. along a given path*) *if and only if its action along every path in this set* (*resp. along the given path*) *depends only on the initial and final point of the transportation but not on the particular path connecting these points.* In other words, *a transport is Euclidean on* $U \subseteq B$ *iff it is path-independent on* U.

Proof. Let a linear transport L admit a frame $\{e_i\}$ normal in $U \subseteq B$. By Definitions 3.1 and 3.2 and equation (2.12), this implies $L^{\gamma}_{s\to t}u^i(\gamma(s))(e_i|_{\gamma(s)}) = u^i(\gamma(s))e_i|_{\gamma(t)}$ for $\gamma : J \to U$ and $u(x) \in \pi^{-1}(x)$,

$x \in B$. Conversely, let $L^{\gamma}_{s\to t} u(\gamma(s))$ depend only on $\gamma(s)$ and $\gamma(t)$ but not on γ and $\{e_i\}$ be a field of bases on U (resp. on $\gamma(J)$). Then, due to (2.12), the matrix $\boldsymbol{L}$ of L in $\{e_i\}$ has the form $\boldsymbol{L}(t, s; \gamma) = \boldsymbol{B}(\gamma(t), \gamma(s))$ for some matrix-valued function $\boldsymbol{B}$ on $U \times U$. Combining this result with Propositions 2.4 and 2.5, we see that $\boldsymbol{L}$ admits a representation

$$\boldsymbol{L}(t, s; \gamma) = \boldsymbol{F}_0^{-1}(\gamma(t))\boldsymbol{F}_0(\gamma(s)), \quad s, t \in J \tag{4.3}$$

for a non-degenerate matrix-valued function $\boldsymbol{F}_0$ on U. At last, putting $e'_i|_x = (\boldsymbol{F}_0^{-1}(x))_i^j e_j|_x$, $x \in U$, from (2.11) we obtain that the matrix of L in $\{e'_i\}$ is $\boldsymbol{L}'(t, s; \gamma) = \mathbb{1}$, i.e., the frame $\{e'_i\}$ is normal for L on U. □

An evident corollary of Theorem 4.1 is the following assertion. Let a linear transport L be Euclidean on $U \subseteq B$ and $h_a : J \to U$, $a \in [0, 1]$, be a homotopy of paths passing through two fixed points $x, y \in U$, i.e., $h_a(s_0) = x$ and $h_a(t_0) = y$ for some $s_0, t_0 \in J$ and any $a \in [0, 1]$. Then $L^{h_a}_{s_0\to t_0}$ is independent of $a \in [0, 1]$. In particular, we have $L^{h_a}_{s_0\to t_0}|_{y=x} = id_{\pi^{-1}(x)}$ owing to Proposition 4.2.

Equation (4.3) and the part of the proof of Theorem 4.1 after it are a hint for the formulation of the following result.

Theorem 4.2. *A linear transport L along paths in a vector bundle, with C^1 manifold as a bundle space, is Euclidean on U (resp. along γ) iff for some, and hence for every, frame $\{e_i\}$ on U (resp. on $\gamma(J)$) there exists a non-degenerate matrix-valued function $\boldsymbol{F}_0$ on U such that the matrix $\boldsymbol{L}$ of L in $\{e_i\}$ is given by* (4.3) *for every $\gamma : J \to U$ (resp. for the given γ) or, equivalently, iff the matrix $\boldsymbol{\Gamma}$ of the coefficients of L in $\{e_i\}$ is*

$$\boldsymbol{\Gamma}(s; \gamma) = \boldsymbol{F}_0^{-1}(\gamma(s))\frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s}. \tag{4.3'}$$

Proof. Suppose L is Euclidean. There is a frame $\{e_i^0\}$ normal for L on U (resp. along γ). Define a matrix $\boldsymbol{F}_0(x)$ via the expansion $e_i|_x =$

$(\boldsymbol{F}_0(x))_i^j e_j^0|_x$, $x \in U$. Since, by definition, the matrix of L in $\{e_i^0\}$ is the unit (identity) matrix on U, the matrix of L in $\{e_i\}$ is given via (4.3) due to (2.11). Conversely, if (4.3) holds in $\{e_i\}$ on U, then the frame $\{e_i'|_x = (\boldsymbol{F}_0^{-1}(x))_i^j e_j^0|_x\}$ is normal for L on U (resp. along γ), as we saw at the end of the proof of Theorem 4.1. The equivalence of (4.3′) and (4.3) is a consequence of (2.28) (cf. (2.29), (2.30), and (3.2)). □

The proof of Theorem 4.2 suggests a way for generating Euclidean transports along paths by 'inverting' the definition of normal frames: take a given field of bases over $U \subseteq B$ and define a linear transport by requiring its matrix to be unit in the given field of bases. We call this Euclidean transport *generated by* (*or assigned to*) the given initial frame, which is normal for it.

Proposition 4.3. *All frames normal for a Euclidean transport along paths in U generate one and the same Euclidean transport along paths in U coinciding with the initial one.*

Proof. The result is an almost evident consequence of the last definition and Corollary 3.4. □

Proposition 4.4. *Two or more frames on U generate one and the same Euclidean transport along paths iff they are connected via linear transformations with constant* (*on U*) *coefficients.*

Proof. If $\{e_i\}$ and $\{e_i'\}$ generate L, then they are normal for it (Proposition 4.3) and, by Corollary 3.3, they are connected in the way pointed. The converse is a trivial corollary of the last definition. □

In this way we have established a bijective correspondence between the set of Euclidean linear transports along paths in U and the class of sets of frames on U connected by linear transformations with constant coefficients.

The comparison of Proposition 4.2 with Theorem 4.1 suggests that a transport is Euclidean in $U \subseteq B$ iff (4.1) holds for every $\gamma : J \to U$. But this is not exactly the case. The right result is the following one.

Theorem 4.3. *A linear transport L along paths is Euclidean on a path-connected set* $U \subseteq B$ *iff the next three conditions are valid*: (i) *Equation* (4.1) *holds for every continuous path* $\gamma : J \to U$; (ii) *The transport along a product of paths is equal to the composition of the transports along the paths of the product, i.e.*,

$$L^{\gamma_1\gamma_2} = L^{\gamma_2} \circ L^{\gamma_1}, \tag{4.4}$$

where γ_1 *and* γ_2 *are paths in U such that the end of* γ_1 *coincides with the beginning of* γ_2 *and* $\gamma_1\gamma_2$ *is the product of these paths*; (iii) *For any subinterval* $J' \subseteq J$ *the locality condition*

$$L^{\gamma|J'}_{s\to t} = L^{\gamma}_{s\to t}, \;\; s, t \in J' \subseteq J, \tag{4.5}$$

with $\gamma|J'$ *being the restriction of* $\gamma : J \to U$ *to* J', *is valid.*

Remark 4.3. Here and below we do not present and use a particular definition of the product of paths. There are slightly different versions of that definition; for details see [13, 53] or [24, Section 3]. Our results are independent of any concrete such definition because the transports, we are considering here, are independent of the particular path they are acting along (see Theorem 4.1).

Proof. If L is Euclidean, then, by Definition 3.4, it admits normal frame(s) along every $\gamma : J \to U$ and, consequently, according to Proposition 4.2, the condition (4.1) is valid along every $\gamma : J \to U$. By Theorem 4.1, the transport $L^{\gamma}_{s\to t}$, $s, t \in J$ depends only on the points $x = \gamma(s)$ and $y = \gamma(t)$ but not on the particular path γ connecting $x, y \in U$. Equations (4.4) and (4.5) follow from here.

Conversely, let (4.1), (4.4), and (4.5) be true for all paths γ, γ_1, and γ_2 in U, the end of γ_1 coinciding with the beginning of γ_2, and subinterval $J' \subseteq J$. Meanwhile, we notice the equality

$$L^{\gamma^{-1}} = (L^{\gamma})^{-1}, \tag{4.6}$$

γ^{-1} being the path inverse to γ,[20] which is a consequence of (4.1) and (4.4).

Let x_0 be arbitrarily chosen fixed point in U and $\{e_i^0\}$ be an arbitrarily fixed basis in the fibre $\pi^{-1}(x_0)$ over it. In the fibre $\pi^{-1}(x)$ over $x \in U$ we define a basis $\{e_i|_x\}$ via (cf. (4.2))

$$e_i|_x := L^{\gamma_{x_0,x}}_{s_0 \to s}(e_i^0), \tag{4.7}$$

where $\gamma_{x_0,x} : J \to U$ is an arbitrary continuous path through x_0 and x, i.e., for some $s_0, s \in J$, we have $\gamma_{x_0,x}(s_0) = x_0$ and $\gamma_{x_0,x}(s) = x$. Below we shall prove that the field $\{e_i\}$ of bases over U is normal for L on U.

At first, we shall prove the independence of $e_i|_x$ from the particular continuous path $\gamma_{x_0,x}$. Let $\beta_a : J_a \to U$, $a = 1, 2$ and $\beta_a(s_a) = x_0$ and $\beta_a(t_a) = x$ for some $s_a, t_a \in J_a$, $a = 1, 2$. For definiteness, we assume $s_a \le t_a$. (The other combinations of ordering between s_1, t_1, s_2, and t_2 can be considered analogously.) Defining $\beta'_a := \beta_a|[s_a, t_a]$, $a = 1, 2$ and using (4.5), (4.6), (4.4), and (4.1), we get

$$L^{\beta_2}_{s_2\to t_2} \circ L^{\beta_1}_{t_1\to s_1} = L^{\beta'_2}_{s_2\to t_2} \circ L^{\beta'_1}_{t_1\to s_1} = L^{\beta'_2}_{s_2\to t_2} \circ L^{(\beta'_1)^{-1}}_{s_1\to t_1}$$

$$= L^{(\beta'_1)^{-1}\beta'_2}_{s_0\to t_0} = id_{\pi^{-1}(x)},$$

where $(\beta'_1)^{-1}\beta'_2 : [s_0, t_0] \to U$ is the product of $(\beta'_1)^{-1}$ and β'_2 and we have used that, from the definition of $(\beta'_1)^{-1}$ and β'_2, it is clear that $((\beta'_1)^{-1}\beta'_2)(s_0) = ((\beta'_1)^{-1}\beta'_2)(t_0) = x$, i.e., $(\beta'_1)^{-1}\beta'_2$ is a closed path passing through x. Applying the last result, (2.2), and (2.3), we obtain

---

[20] If $\gamma : [p, q] \to U$, and $\gamma^{-1} : [p', q'] \to U$, for $p, q, p', q' \in \mathbb{R}$, $p < q$, $p' < q'$, and $\gamma^{-1}(p') = \gamma(q)$, we shall apply (4.6) in the form $L^{\gamma^{-1}}_{p'\to q'} = (L^{\gamma}_{p\to q})^{-1} = L^{\gamma}_{q\to p}$.

$$L^{\beta_2}_{s_2\to t_2}e^0_i = (L^{\beta_2}_{s_2\to t_2} \circ L^{\beta_1}_{t_1\to s_1}) \circ (L^{\beta_1}_{s_1\to t_1})e^0_i = L^{\beta_1}_{s_1\to t_1}e^0_i.$$

Since β_1 and β_2 are arbitrary, from here we conclude that the frame $\{e_i\}$, defined via (4.7) on U, is independent from the particular path used in (4.7).

Now we shall prove that $\{e_i\}$ is normal for L on U, which will complete this proof.

From the proof of Proposition 4.2 (compare (4.7) and (4.2)) follows that $\{e_i\}$ is normal for L along any path in U passing through x_0. Let $\gamma : J \to U$ be such a path, $s_0 \in J$ be fixed, and $\beta : [0, 1] \to U$ be such that $\beta(0) = x$ and $\beta(1) = \gamma(s_0) =: x_0$. Defining $\gamma_\pm := \gamma | J_\pm$ for $J_\pm := \{s \in J, \pm s \geq \pm s_0\}$, we conclude that $\{e_i\}$ is normal for L along $\beta\gamma_+$ and $\beta\gamma_-^{-1}$. Take, for example, the path $\beta\gamma_+$. If for some $s'_0, s', s^* \in \mathbb{R}$ is fulfilled $(\beta\gamma_+)(s'_0) = x, (\beta\gamma_+)(s') = \gamma(s)$, and $(\beta\gamma_+)(s^*) = x_0$, then, applying (4.7), (4.4), and (4.5), we find for $s \geq s_0$:

$$e_i|_{\gamma(s)} = L^{\beta\gamma_+}_{s'_0\to s'}(e_i|_x) = L^{\beta\gamma_+}_{s'_0\to s'} \circ L^{\beta\gamma_+}_{s^*\to s'_0}(e_i|_{x_0}) = L^{\beta\gamma_+}_{s^*\to s'}(e_i|_{x_0})$$

$$= L^{\gamma_+}_{s_0\to s} \circ L^{\beta}_{0\to 1}(e_i|_{x_0}) = L^{\gamma_+}_{s_0\to s}(e_i|_x) = L^{\gamma}_{s_0\to s}(e_i|_x).$$

Analogously one can prove that $e_i|_{\gamma(s)} = L^{\gamma}_{s_0\to s}(e_i|_x)$ for $s \leq s_0$ by using $\beta\gamma_-^{-1}$ instead of $\beta\gamma_+$. So, due to (2.2), the frame $\{e_i\}$ satisfies (3.1f) along γ. Consequently, by Corollary 3.1, the frame so-constructed is normal for L along γ. □

Remark 4.4. According to [21, Proposition 3.4], the equality (4.4) is a consequence of (4.5) and the reparametrization condition

$$L^{\gamma\circ\tau}_{s\to t} = L^{\gamma}_{\tau(s)\to\tau(t)}, \quad s, t \in J'', \tag{4.8}$$

where J'' is an $\mathbb{R}$-interval and $\tau : J'' \to J$ is bijection. Hence in the formulation of Theorem 4.3 we can (equivalently) replace the condition (4.4) with (4.8). So, we have

Theorem 4.3′. *A transport L is Euclidean on a path-connected set $U \subseteq B$ iff* (4.1), (4.5), *and* (4.8) *are valid for every continuous path $\gamma : J \to U$.*

The next result is analogous to Proposition 3.3. According to it, a frame normal for L on $U \subseteq B$, if any, can be obtained by L-transportation of a fixed basis over some point in U to the other points of U.

Proposition 4.5. *If L is a Euclidean transport on a path-connected set $U \subseteq B$ and $\{e_i^0\}$ is a given basis in $\pi^{-1}(x_0)$ for a fixed $x_0 \in U$, then the frame $\{e_i\}$ over U defined via*

$$e_i|_x = L^{\gamma}_{s_0 \to s}(e_i^0), \tag{4.9}$$

where $\gamma : J \to U$ is such that $\gamma(s_0) = x_0$ and $\gamma(s) = x$ for some $s_0, s \in J$, is normal for L on U.

Proof. By Theorem 4.1, the basis $\{e_i|_x\}$ is independent of the particular path γ used in (4.9). According to Theorem 4.3, the conditions (4.1), (4.4), and (4.5) hold for L. Further, repeating step-by-step the last paragraph of the proof of Theorem 4.3, we verify that $\{e_i\}$ is normal for L on U.

Alternatively, the assertion is a consequence of (2.25) and Proposition 4.6 presented a few lines below. □

A simple way to check whether a given frame is normal along some path is provided by the following proposition.

Proposition 4.6. *A frame $\{e_i\}$ along $\gamma : J \to B$ is normal for a linear transport L in (E, π, B), E being a C^1 manifold, along paths if and only if the liftings $\hat{e}_i : \gamma \mapsto e_i(\cdot, \gamma)$ are constant* (*along* γ) *with respect to the derivation D generated by L*:

$$D^{\gamma}\hat{e}_i = 0. \tag{4.10}$$

Proof. If $\{e_i\}$ is normal for L along γ, equation (3.1f) is valid (see

Corollary 3.1), so (4.10) follows from (2.25). If (4.10) holds, by virtue of (2.25), its solution is[21] $e_i|_{\gamma(s)} = L^{\gamma}_{s_0 \to s}(e_i|_{\gamma(s_0)})$ and consequently, by Proposition 3.3, the frame $\{e_i\}$ is normal along γ. □

Recall (see the remark preceding Definition 2.2), the path γ in Proposition 4.6 cannot be an arbitrary continuous path in B as it must be in the set $\pi \circ \mathrm{P}^k(E)$, with $\mathrm{P}^k(E)$, $k = 0, 1$, being the set of C^k paths in E. Notice, the derivative in (4.10) does not require B to be a manifold.

Of course, it is true that if (4.10) holds in a frame $\{e_i\}$ along every path γ in U, the frame $\{e_i\}$ is normal for L on U. But it is more natural to find a 'global' version of (4.10) concerning the whole set U, not the paths in it. Since it happens that such a result cannot be formulated solely in terms of transports along paths, it will be presented elsewhere.[22]

5. The Case of a Manifold as a Base

Starting from this section, we consider some peculiarities of frames normal for linear transports along paths in a vector bundle (E, π, M) whose base M is a C^1 differentiable manifold. Besides, the bundle space E will be required to be a C^1 manifold. This will allow links to be made with the general results of [17] concerning frames normal for derivations of the tensor algebra of the vector space of vector fields over a manifold which, in particular, can be linear connections.

The local coordinates of $x \in M$ will be denoted by x^{μ}. Here and below the Greek indices $\alpha, \beta, \ldots, \mu, \nu, \ldots$ run from 1 to dim M and, as usual, a summation from 1 to dim M on such indices repeated on different levels will be assumed. The below-considered paths, like

---

[21] Equation (4.10) is an ordinary differential equation of first order with respect to the local components of e_i (see (2.27)).

[22] For this purpose is required the concept of (linear) transports along maps (see [25]). Alternatively, the concept of a curvature of a linear transport along paths can be used [30, 31].

$\gamma : J \to M$, are supposed to be of class C^1 and by $\dot{\gamma}(s)$ is denoted the vector tangent to γ at $\gamma(s)$, $s \in J$, (more precisely at s), i.e., $\dot{\gamma}$ is the vector field tangent to γ provided γ is injective. By $\{E_\mu\}$ will be denoted a frame along γ in the bundle space tangent to M, i.e., for every $s \in J$ the vectors $E_1|_{\gamma(s)}, \ldots, E_{\dim M}|_{\gamma(s)}$ form a basis in the space $T_{\gamma(s)}(M)$ tangent to M at $\gamma(s)$. In particular, the frame $\{E_\mu\}$ can be a coordinate one, $E_\mu|_x = \left.\frac{\partial}{\partial x^\mu}\right|_x$, in some neighborhood of $x \in \gamma(J)$. Notice, if we say that U is a neighborhood of a set $V \subseteq M$, we mean that U is an open set in M containing V. Otherwise by a neighborhood we understand any open set in M (which set is a neighborhood of any its point in the just pointed sense). The transports along paths investigated below are supposed to be of class C^1 on the set of C^1 paths in M.

5.1. Normal frames for linear transports

Proposition 5.1. *Let L be a linear transport along paths in (E, π, M), E and M being C^1 manifolds, and L be Euclidean on $U \subseteq M$ (resp. along a C^1 path $\gamma : J \to M$). Then the matrix $\mathbf{\Gamma}$ of its coefficients has the representation*

$$\mathbf{\Gamma}(s; \gamma) = \sum_{\mu=1}^{\dim M} \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s) \equiv \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s) \tag{5.1}$$

in any frame $\{e_i\}$ along every (resp. the given) C^1 path $\gamma : J \to U$, where $\Gamma_\mu = [\Gamma^i_{j\mu}]_{i,j=1}^{\dim \pi^{-1}(x)}$ are some matrix-valued functions, defined on an open set V containing U (resp. $\gamma(J)$) or equal to it, and $\dot{\gamma}^\mu$ are the components of $\dot{\gamma}$ in some frame $\{E_\mu\}$ along γ in the bundle space tangent to M, $\dot{\gamma} = \dot{\gamma}^\mu E_\mu$.

Proof. By Theorem 4.2, the representation (4.3′) is valid in $\{e_i\}$ for some matrix-valued function $\boldsymbol{F}_0$ on U. Hence, if U is a neighborhood, equation (5.1) holds for

$$\Gamma_\mu(x) = \boldsymbol{F}_0^{-1}(x)(E_\mu(\boldsymbol{F}_0)|_x) \tag{5.2}$$

with $x \in U$. In the general case, e.g., if U is a submanifold of M of dimension less than the one of M, the terms $E_\mu(\boldsymbol{F}_0)|_U$, $\mu = 1, \dots, \dim M$, in the last equality may turn to be undefined as the matrix-valued function $\boldsymbol{F}_0$ is defined only on U. To overcome this possible problem, let us take some C^1 matrix-valued function $\boldsymbol{F}$, defined on an open set V containing U (resp. $\gamma(J)$) or equal to it, such that $\boldsymbol{F}|_U = \boldsymbol{F}_0$. Since (4.3) and (4.3′) depend only on the values of $\boldsymbol{F}_0$, i.e., on the ones of $\boldsymbol{F}$ on U, these equations hold also if we replace $\boldsymbol{F}_0$ in them with $\boldsymbol{F}$. From the so-modified equality (4.3′), with $\boldsymbol{F}$ for $\boldsymbol{F}_0$, we see that (5.1) is valid for

$$\Gamma_\mu(x) = \boldsymbol{F}^{-1}(x)(E_\mu(\boldsymbol{F}))|_x \tag{5.3}$$

with $x \in V$. □

Consider now the transformation properties of the matrices Γ_μ in (5.1). Let U be an open set, e.g., $U = M$. If we change the frame $\{E_\mu\}$ in the bundle space tangent to M, $\{E_\mu\} \mapsto \{E'_\mu = B^\nu_\mu E_\nu\}$ with $B = [B^\nu_\mu]$ being non-degenerate matrix-valued function, and simultaneously the bases in the fibres $\pi^{-1}(x)$, $x \in M$, $\{e_i|_x\} \mapsto \{e'_i|_x = A^j_i(x) e_j|_x\}$, then, from (2.30) and (5.1), we see that Γ_μ transforms into Γ'_μ such that

$$\Gamma'_\mu = B^\nu_\mu A^{-1}\Gamma_\nu A + A^{-1}E'_\mu(A) = B^\nu_\mu A^{-1}(\Gamma_\nu A + E_\nu(A)), \tag{5.4}$$

where $A := [A^j_i]^{\dim \pi^{-1}(x)}_{i, j=1}$ is non-degenerate and of class C^1.

Note 5.1. While deriving (5.4), we supposed (5.1) to be valid on M, i.e., for $U = M$. If $U \neq M$, equation (5.1) holds only on U, i.e., for $\gamma : J \to U$. Therefore the result (5.4) is true only on U, but in this case the frames $\{e_i\}$ and $\{e'_i\}$ *must* be defined on an open set containing or equal to U. This follows from (2.30) in which the derivative $\frac{\mathrm{d}A(s;\, \gamma)}{\mathrm{d}s} = \frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}$ enters. To derive (5.4), we have expressed $\frac{\mathrm{d}A(\gamma(s))}{\mathrm{d}s}$ as $(E_\mu(A))|_{\gamma(s)}\dot{\gamma}^\mu(s)$ which is meaningful iff A is defined on a neighborhood of each point in U. Consequently A, as well as $\{e_i\}$ and $\{e'_i\}$, must be defined on an open set $V \supseteq U$. For this reason, below, when derivatives like $E_\mu(A)$ appear, we admit the employed frames in the bundle space E to be defined always on some neighborhood in M containing or equal to the set U on which some normal frames are investigated.

Denoting by $\Gamma^i_{j\mu}$ the components of Γ_μ, we can rewrite (5.4) as

$$\Gamma'^i_{j\mu} = \sum_{\nu=1}^{\dim M} \sum_{k,\, l=1}^{\dim \pi^{-1}(x)} B^\nu_\mu (A^{-1})^i_k A^l_j \Gamma^k_{l\nu} + \sum_{\nu=1}^{\dim M} \sum_{k=1}^{\dim \pi^{-1}(x)} B^\nu_\mu (A^{-1})^i_k E_\nu(A^k_j). \quad (5.5)$$

Thus, we observe that the functions $\Gamma^i_{j\mu}$ are very similar to the coefficients of a linear connection [34, Chapter III, Section 7]. Below, in Section 7, we shall see that this is not accidental (compare (5.1) with (2.34)). These functions are also called coefficients of the transport L. To make a distinction between Γ^i_j and $\Gamma^i_{j\mu}$, we call the former ones 2-*index coefficients* of L and the latter ones 3-*index coefficients* of L when there is a risk of ambiguities. Besides, if (5.1) holds for every $\gamma : J \to U$ for a transport L, then, in the general case, there are (infinitely) many such representations unless U is an open set. For instance, if (5.1) is valid for some Γ_μ, it is also true if we replace in it Γ_μ with $\Gamma_\mu + G_\mu$, where the matrix-valued functions G_μ are such that $G_\mu \dot{\gamma}^\mu = 0$ for every $\gamma : J \to U$; the 3-index coefficients $\Gamma^i_{j\mu}$ of a given linear transport L admitting them are defined *uniquely* on $U \subseteq M$ by (5.3) or (5.2) if (and only if) U is an open subset of M, e.g., if $U = M$.

Note that any linear transport has 2-index coefficients while 3-index ones exist only for some of them; in particular such are the Euclidean transports (see Proposition 5.1 and Theorem 5.2 below).

The equation (5.1) is generally only a necessary, but not sufficient condition for a frame to be normal as it is stated by the following theorem.

Theorem 5.1. *A C^2 linear transport L along paths is Euclidean on a neighborhood $U \subseteq M$ if and only if in every frame the matrix $\mathbf{\Gamma}$ of its coefficients has a representation* (5.1) *along every C^1 path γ in U in which the matrix-valued functions Γ_μ, defined on an open set containing U or equal to it, satisfy the equalities*

$$(R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M}))(x) = 0, \tag{5.6}$$

where $x \in U$ and

$$R_{\mu\nu}(-\Gamma_1, \dots, -\Gamma_{\dim M}) := -\frac{\partial \Gamma_\mu}{\partial x^\nu} + \frac{\partial \Gamma_\nu}{\partial x^\mu} + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu \tag{5.7}$$

in a coordinate frame $\left\{ E_\mu = \frac{\partial}{\partial x^\mu} \right\}$ in a neighborhood of x.

Remark 5.1. This result is a direct analogue of [15, Proposition 3.1] in the theory considered here.

Proof. NECESSITY. For a transport L Euclidean on U is valid (5.1) due to Proposition 5.1. Moreover, we know from the proof of this proposition that Γ_μ admit representation (5.3) for some C^1 non-degenerate matrix-valued function $\boldsymbol{F}$. The proof of the necessity is completed by the following lemma.

Lemma 5.1. *A set of matrix-valued functions $\{\Gamma_\mu : \mu = 1, \dots, \dim M\}$, of class C^1 and defined on a neighborhood V, admits a representation* (5.3) *iff the conditions* (5.6) *are fulfilled for $x \in V$.*

Proof of Lemma 5.1. A representation (5.3) exists iff it, considered as a matrix linear partial differential equation of first order, has a solution with respect to $\boldsymbol{F}$. Rewriting (5.3) as

$$\left. \frac{\partial \boldsymbol{F}^{-1}}{\partial x^\mu} \right|_x = -\Gamma_\mu(x) \boldsymbol{F}^{-1}(x), \quad x \in V,$$

from [17, Lemma 3.1] we conclude that the solutions of this equation with respect to $\boldsymbol{F}^{-1}$ exist iff (5.6) holds. In fact, fixing some initial value $\boldsymbol{F}^{-1}(x_0) = f_0$, we see that

$$\boldsymbol{F}(x) = f_0^{-1} Y^{-1}(x, x_0; -\Gamma_1, \dots, -\Gamma_{\dim M}), \tag{5.8}$$

where $Y(x, x_0; Z_1, \dots, Z_{\dim M})$ is the solution of the initial-value problem

$$\left.\frac{\partial Y}{\partial x^\mu}\right|_x = Z_\mu(x) Y|_x, \quad Y|_{x=x_0} = \mathbb{1}. \tag{5.9}$$

Here $Z_1, \dots, Z_{\dim M}$ are continuous matrix-valued functions and $\mathbb{1}$ is the identity (unit) matrix of the corresponding size. According to [17, Lemma 3.1], the problem (5.9) with $Z_\mu = -\Gamma_\mu$ has (a unique) solution (of class C^2) iff the (integrability) conditions (5.6) are valid. □

SUFFICIENCY. Let (5.1) and (5.6) be valid. As a consequence of Lemma 5.1, there is a representation (5.3) for Γ_μ with some $\boldsymbol{F}$. Substituting (5.3) into (5.1), we get

$$\boldsymbol{\Gamma}(s; \gamma) = \boldsymbol{F}^{-1}(\gamma(s)) \left.\frac{\partial \boldsymbol{F}(x)}{\partial x^\mu}\right|_{x=\gamma(s)} \dot{\gamma}^\mu(s) = \boldsymbol{F}^{-1}(\gamma(s)) \frac{\mathrm{d}\boldsymbol{F}(\gamma(s))}{\mathrm{d}s}.$$

So, by Theorem 4.2 (see (4.3′) for $\boldsymbol{F}_0 = F|_U$), the considered transport L along paths is Euclidean. □

The just-proved Theorem 5.1 expresses a very important practical necessary and sufficient condition for existence of frames normal on neighborhoods because the conditions (5.1) and (5.6) are easy to check for a given linear transport along paths in bundles with a differentiable manifold as a base.

Now, combining (3.1c) and (5.1), applying Corollary 3.1, and using the arbitrariness of γ, we can formulate the following essential result.

Proposition 5.2. *A necessary and sufficient condition for a frame to be normal on a neighborhood $U \subseteq M$ for a Euclidean linear transport on U along paths in (E, π, M) is the vanishment of its 3-index coefficients,* *i.e.*,

$$\Gamma_\mu(x) := [\Gamma^i_{j\mu}]^{\dim \pi^{-1}(x)}_{i,j=1} = 0 \tag{5.10}$$

for every $x \in U$, *where* $\Gamma_\mu(x)$ *define the* (2-*index*) *coefficients of the transport via* (5.1).

Now we are going to find an analogue of Theorem 5.1 when the neighborhood $U \subseteq M$ in it is replaced with a submanifold of the base M.

Let N be a submanifold of M and L be a linear transport along paths in (E, π, M) which is Euclidean on N. Let the C^1 matrix-valued function $\boldsymbol{F}_0$ determine the coefficients' matrix of L via (4.3′). Suppose $p_0 \in N$ and (V, x) is a chart of M such that $V \ni p_0$ and the local coordinates of every $p \in N \cap V$ are $x(p) = (x^1(p), \ldots, x^{\dim N}(p), t_0^{\dim N+1}, \ldots, t_0^{\dim M})$, where t_0^ρ, $\rho = \dim N + 1, \ldots, \dim M$, are constant numbers.[23]

In the chart (V, x), we have $\dfrac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s} = \sum_{\alpha=1}^{\dim N} \dfrac{\partial \boldsymbol{F}_0}{\partial x^\alpha}\bigg|_{\gamma(s)} \dot{\gamma}^\alpha(s)$, with $\gamma^\mu := x^\mu \circ \gamma$, for every C^1 path $\gamma : J \to N$ and $s \in J$. From here and (4.3′), it follows that (5.1) holds for

$$\Gamma_\alpha(p) = \boldsymbol{F}_0^{-1}(p) \frac{\partial \boldsymbol{F}_0}{\partial x^\alpha}\bigg|_p, \quad \alpha = 1, \ldots, \dim N \tag{5.11}$$

and *arbitrary* $\Gamma_{\dim N+1}, \ldots, \Gamma_{\dim M}$ since in the coordinates $\{x^\mu\}$ is fulfilled $\gamma^\rho(s) = t_0^\rho = \text{const}$ and hence

$$\dot{\gamma}^{\dim N+1} = \cdots = \dot{\gamma}^{\dim M} \equiv 0. \tag{5.12}$$

Comparing (5.11) with (5.2) for $E_\mu = \dfrac{\partial}{\partial x^\mu}$, we conclude that Γ_α, given via (5.11), are exactly the first dim N of the matrices $\Gamma_\mu = [\Gamma^i_{j\mu}]$ of the

---

[23] We are using the definition of a submanifold presented in [1, p. 227].

3-index coefficients of the transport L in the pair of frames $\left(\{e_i\}, \left\{\frac{\partial}{\partial x^\mu}\right\}\right)$.

As we said, the rest of the 3-index coefficients of L (on N) are completely arbitrary. In particular, one can choose them according to (5.3),

$$\Gamma_\rho(p) = \boldsymbol{F}^{-1}(p)\frac{\partial \boldsymbol{F}}{\partial x^\rho}, \quad \rho = \dim N + 1, \ldots, \dim M, \quad \boldsymbol{F}|_N = \boldsymbol{F}_0, \tag{5.13}$$

which leads to the validity of (5.3) in every frame, or, if the representation (5.1) holds for every $\gamma : J \to M$ (this does not mean that L is Euclidean on M!), the matrices Γ_ρ can be identified with the ones appearing in (5.1) in the frame $\left\{\frac{\partial}{\partial x^\mu}\right\}$.

If $\{x'^\mu\}$ are other coordinates on V like $\{x^\mu\}$, i.e., $x'^\rho(p) = \text{const}$ for $p \in N \cap V$ and $\rho = \dim N + 1, \ldots, \dim M$, the change $\{x^\mu\} \mapsto \{x'^\mu\}$, combined with $\{e_i\} \mapsto \{e'_i = A^j_i e_j\}$ leads to

$$\Gamma_\alpha \mapsto \Gamma'_\alpha = B^\beta_\alpha A^{-1}\Gamma_\beta A + A^{-1}\frac{\partial A}{\partial x'^\alpha}, \quad B^\beta_\alpha := \frac{\partial x^\beta}{\partial x'^\alpha}, \ \alpha, \beta = 1, \ldots, \dim N \tag{5.14}$$

on $N \cap V$. So, equation (5.4) remains valid only for frames $\{E_\mu\}$ normal on N. But using the arbitrariness of Γ_ρ, we can force (5.4) to hold on N for arbitrary frames defined on a neighborhood of N.

The above discussion implies that the conditions (5.6) in Theorem 5.1, when applied on a submanifold N, imposes restrictions on the transport L as well as ones on the 'inessential' 3-index coefficients of L, like Γ_ρ above, or on the matrix-valued function $\boldsymbol{F}$ entering in (5.3) or in (5.13). Since the restrictions of the last type are not connected with the transport L, below we shall 'repair' Theorem 5.1 on submanifolds in such a way as to exclude them from the final results.

Theorem 5.2. *A linear transport L along paths is Euclidean on a submanifold N of M if and only if in every frame $\{e_i\}$, in the bundle space over N, the matrix of its coefficients has a representation* (5.1) *along every*

C^1 path in N and, for every $p_0 \in N$ and a chart (V, x) of M such that $V \ni p_0$ and $x(p) = (x^1(p), \dots, x^{\dim N}(p), t_0^{\dim N+1}, \dots, t_0^{\dim M})$ for every $p \in N \cap V$ and constant numbers $t_0^{\dim N+1}, \dots, t_0^{\dim M}$, the equalities

$$(R^N_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim N}))(p) = 0, \ \ \alpha, \beta = 1, \dots, \dim N \tag{5.15}$$

hold for all $p \in N \cap V$ and

$$R^N_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim N}) := R_{\alpha\beta}(-\Gamma_1, \dots, -\Gamma_{\dim M})$$
$$= -\frac{\partial \Gamma_\alpha}{\partial x^\beta} - \frac{\partial \Gamma_\beta}{\partial x^\alpha} + \Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha. \tag{5.16}$$

Here $\Gamma_1, \dots, \Gamma_{\dim N}$ are first $\dim N$ of the matrices of the 3-index coefficients of L in the coordinate frame $\left\{\frac{\partial}{\partial x^\mu}\right\}$ in the tangent bundle space over $N \cap V$. They are uniquely defined via (5.11).

Remark 5.2. In the theory considered here, this result is a direct analogue of [17, Theorem 3.1].

Remark 5.3. This theorem is, in fact, a special case of Theorem 5.1: if in the latter theorem we put $U = N$, restrict the transport L to the bundle $(\pi^{-1}(N), \pi|_{\pi^{-1}(N)}, N)$, replace M with N, and notice that $\{x^1, \dots, x^{\dim N}\}$ provide an internal coordinate system on N, we get the former one. Because of the importance of the result obtained, we call it 'theorem' and present below its independent proof.

Proof. If L is Euclidean on N, equation (5.1) holds in every frame on N (Proposition 5.1); in particular it is valid in the frame $\left\{\frac{\partial}{\partial x^\mu}\right\}$, induced by the chart (V, x), in which, as was proved above, equation (5.11) is satisfied. The substitution of (5.11) into (5.16) results in (5.15). Conversely, let (5.1) for $\gamma : J \to N$ and (5.15) be valid. By Lemma 5.1 with N for M, from (5.15) follows the existence of a representation (5.11) for some matrix-valued function $\boldsymbol{F}_0$ on N. Substituting (5.11) into (5.1) and using that γ is a path in N and (5.12) is valid, in the frame $\left\{\frac{\partial}{\partial x^{\mu}}\right\}$, we obtain:

$$\boldsymbol{\Gamma}(s;\ \gamma) = \Gamma_{\mu}(\gamma(s))\dot{\gamma}^{\mu}(s) = \sum_{\alpha=1}^{\dim N}\Gamma_{\alpha}(\gamma(s))\dot{\gamma}^{\alpha}(s)$$

$$= \boldsymbol{F}_0^{-1}(\gamma(s))\sum_{\alpha=1}^{\dim N}\frac{\partial \boldsymbol{F}_0}{\partial x^{\alpha}}\bigg|_{\gamma(s)}\dot{\gamma}^{\alpha}(s)$$

$$= \boldsymbol{F}_0^{-1}(\gamma(s))\frac{\partial \boldsymbol{F}}{\partial x^{\mu}}\bigg|_{\gamma(s)}\dot{\gamma}^{\mu}(s) = \boldsymbol{F}_0^{-1}(\gamma(s))\frac{\mathrm{d}\boldsymbol{F}(\gamma(s))}{\mathrm{d}s}$$

$$= \boldsymbol{F}_0^{-1}(\gamma(s))\frac{\mathrm{d}\boldsymbol{F}_0(\gamma(s))}{\mathrm{d}s},$$

where $\boldsymbol{F}$ is a C^1 matrix-valued function defined on an open set containing N or equal to it and such that $\boldsymbol{F}|_N = \boldsymbol{F}_0$. Thus, by Theorem 4.2, the transport L is Euclidean on N. □

Corollary 5.1. *Every linear transport along paths in a vector bundle whose base and bundle spaces are C^1 manifolds, is Euclidean at every single point or along every path without self-intersections.*

Proof. See Theorem 5.2 for $\dim N = 0, 1,$ in which cases $R^{N}_{\alpha\beta} \equiv 0.$ □

It should be noted, the last result agrees completely with Proposition 4.1 and Corollary 4.1.

5.2. Normal frames for derivations

For a general bundle (E, π, B) whose bundle space E is C^1 manifold, we call a frame $\{e_i\}$ *normal on* $U \subseteq B$ (resp. along $\gamma : J \to M$) for a derivation D along paths (resp. D^{γ} along γ) (see Definition 2.2) if $\{e_i\}$ is normal on U (resp. along γ) for the linear transport L along paths generating it by (2.23) (see Proposition 2.7). We can also equivalently define a frame normal for D (resp. D^{γ}) as one in which the components of D (resp. D^{γ}) vanish (see the proof of Proposition 2.7, and Corollary 3.1). A derivation admitting normal frame(s) is called *Euclidean*.

In connection with concrete physical applications, far more interesting case is the case of a bundle (E, π, M) with a differentiable manifold M as a base. The cause for this is the existence of natural structures over M, e.g., the different tensor bundles and the tensor algebra over it. Below we concentrate on this particular case.

Definition 5.1. A *derivation* over an open set $V \subseteq M$ or in $(E, \pi, M)|_V$ *along tangent vector fields* is a map $\mathcal{D}$ assigning to every tangent vector field X over V a linear map

$$\mathcal{D}_X : \mathrm{Sec}^1((E, \pi, M)|_V) \to \mathrm{Sec}^0((E, \pi, M)|_V), \tag{5.17}$$

called a *derivation along* X, such that

$$\mathcal{D}_X(f \cdot \sigma) = X(f) \cdot \sigma + f \cdot \mathcal{D}_X(\sigma) \tag{5.18}$$

for every C^1 section σ over V and every C^1 function $f : V \to \mathbb{C}$.

Obviously (see Definition 2.2), if $\gamma : J \to V$ is a C^1 path, the map $\overline{D} : \hat{\sigma} \mapsto \overline{D}\hat{\sigma}$, with $\overline{D}\hat{\sigma} : \gamma \mapsto \overline{D}^{\gamma}\hat{\sigma}$, where $\overline{D}^{\gamma}\hat{\sigma} : s \mapsto \overline{D}^{\gamma}_s\hat{\sigma}$ is defined via

$$\overline{D}^{\gamma}_s(\hat{\sigma}) = ((\mathcal{D}_X\sigma)|_{X=\dot{\gamma}})(\gamma(s)), \;\; \hat{\sigma} : \gamma \mapsto \sigma \circ \gamma, \tag{5.19}$$

is a derivation along paths on the set of C^1 liftings generated by sections of $(E, \pi, M)|_V$. From Section 2, we know that along paths without self-intersections every derivation along these paths generates a derivation of the sections of (E, π, M) (see (2.21) and (2.22)). Thus to any derivation $\mathcal{D}$ along (tangent) vector fields on V there corresponds, via (5.19), a natural derivation D along the paths in V on the set of liftings generated by sections. These facts are a hint for the possibility to introduce 'normal' frames for $\mathcal{D}$. This can be done as follows.

Let $\{e_i\}$ be a C^1 frame in $\pi^{-1}(V)$. We define the *components* or (2-*index*) *coefficients* $\Gamma_{X}{}^{i}_{j} : V \to \mathbb{C}$ of $\mathcal{D}_X$ by the expansion (cf. (2.33))

$$\mathcal{D}_X e_i = \Gamma_X{}_i^j e_j. \tag{5.20}$$

So $\Gamma_X := [\Gamma_X{}_i^j]$ is the *matrix* of $\mathcal{D}_X$ in $\{e_i\}$.

Applying (5.18) to $\sigma = \sigma^i e_i$ and using the linearity of $\mathcal{D}_X$, we get the explicit expression (cf. (2.27))

$$\mathcal{D}_X(\sigma) = (X(\sigma^i) + \Gamma_X{}_j^i \sigma^j) e_i. \tag{5.21}$$

A simple verification proves that the change $\{e_i\} \mapsto \{e'_i = A_i^j e_j\}$, with a non-degenerate C^1 matrix-valued function $A = [A_i^j]$, leads to (cf. (2.30))

$$\Gamma_X := [\Gamma_X{}_j^i] \mapsto \Gamma'_X := [\Gamma'_X{}_j^i] = A^{-1}\Gamma_X A + A^{-1}X(A), \tag{5.22}$$

where $X(A) := [X(A_i^j)]$. Conversely, if a geometrical object with components $\Gamma_X{}_j^i$ is given in a frame $\{e_i\}$ and a change $\{e_i\} \mapsto \{e'_i = A_i^j e_j\}$ implies the transformation (5.22), then there exists a unique derivation along X, defined via (5.21), whose components in $\{e_i\}$ are exactly $\Gamma_X{}_j^i$ (cf. Proposition 2.6).

Below, for the sake of simplicity, we take $V = M$, i.e., the derivations are over the whole base M.

Definition 5.2. A *frame* $\{e_i\}$, defined on an open set containing U or equal to it, is called *normal for a derivation* $\mathcal{D}$ *along tangent vector fields* (*resp. for* $\mathcal{D}_X$ *along a given tangent vector field* X) *on* U if in $\{e_i\}$ the components of $\mathcal{D}$ (resp. $\mathcal{D}_X$) vanish on U for every (resp. the given) tangent vector field X.

If $\mathcal{D}$ (resp. $\mathcal{D}_X$) admits frames normal on $U \subseteq M$, we call it *Euclidean* on U. A number of results, analogous to those of Sections 3-5.1, can be proved for such derivations. Here we shall mention only a few of them.

Proposition 5.3 (cf. Theorem 4.2)**.** *A derivation $\mathcal{D}$ along vector fields admits frame(s) normal on $U \subseteq M$ iff in every frame its matrix on U has the form*

$$\Gamma_X|_U = (F^{-1}X(F))_U, \tag{5.23}$$

where F is a C^1 non-degenerate matrix-valued function defined on an open set containing U.

Proof. If $\{e'_i\}$ is normal on U for $\mathcal{D}$, then (5.23) with $F = A^{-1}$ follows from (5.22) with $\Gamma'_X|_U = 0$. Conversely, if (5.23) holds, then (5.22) with $A = F^{-1}$ yields $\Gamma'_X|_U = 0$. □

Proposition 5.4 (cf. Corollary 3.4)**.** *The frames normal on a set $U \subseteq M$ for a Euclidean derivation along vector fields (resp. given vector field X) are connected by linear transformations whose matrices A are constant (resp. $X(A) = 0$) on U.*

Proof. The result is a consequence of (5.22) for $\Gamma_X = \Gamma'_X = 0$. □

Definition 5.3. A *derivation $\mathcal{D}$ along (tangent) vector fields* is called *linear on U* if in one (and hence in any) frame its components admit the representation

$$\Gamma_X{}^i_j(x) = \Gamma^i_{j\mu}(x)X^\mu(x) \quad \text{or} \quad \Gamma_X = \Gamma_\mu X^\mu, \tag{5.24}$$

where $x \in U$, $\Gamma_\mu = [\Gamma^i_{j\mu}(x)]^{\dim \pi^{-1}(x)}_{i,j=1}$ are matrix-valued functions on U, and X^μ are the local components of a vector field X in some frame $\{E_\mu\}$ of tangent vector fields, $X = X^\mu E_\mu$.

Remark 5.4. The invariant definition of a derivation linear on U is via the equation

$$\mathcal{D}_{fX+gY} = f\mathcal{D}_X + g\mathcal{D}_Y, \tag{5.25}$$

where $f, g : U \to \mathbb{C}$, and X and Y are tangent vector fields over U. But for the purposes of this work the above definition is more suitable.

Comparing Definitions 5.1 and 5.3 (see also (5.25)) with [42, p. 74, Definition 2.51], we see that a derivation along tangent vector fields is linear iff it is a covariant derivative operator in (E, π, B). Therefore the concepts of a linear derivation along tangent vector fields and that of a covariant derivative operator coincide.

We call $\Gamma^i_{j\mu}$ 3-*index coefficients* of $\mathcal{D}$ or simply coefficients if there is no risk of misunderstanding. It is trivial to check that under changes of the frames they transform according to (5.5). It is easy to verify that to every linear derivation $\mathcal{D}$ there corresponds a unique derivation along paths or linear transport along paths whose 2-index coefficients are given via (5.1) with $\Gamma_\mu := [\Gamma^i_{j\mu}]$ being the matrices of the 3-index coefficients of $\mathcal{D}$.[24] Conversely, to any such transport or derivation along paths there corresponds a unique linear derivation along tangent vector fields with components ((2-index) coefficients) given by (5.24), i.e., with the same 3-index coefficients. So, *there is a bijective correspondence between the sets of linear derivations along tangent vector fields and derivations* (*or linear transports*) *along paths whose* (2-*index*) *coefficients admit the representation* (5.1). It should be emphasized, if the above discussion is restricted to a subset U, i.e., only for paths lying entirely in U, it remains valid iff U is an open set in M.

Proposition 5.5. *A derivation along tangent vector fields is Euclidean on U iff it is linear on U and, in every frame* $\{e_i\}$ *over U in the bundle space and every local coordinate frame* $\left\{E_\mu = \frac{\partial}{\partial x^\mu}\right\}$ *over U in the tangent bundle space over U, the matrices* Γ_μ *of its* 2-*index coefficients have the form* (5.3) *for some non-degenerate* C^1 *matrix-valued function* $\boldsymbol{F}$ *on U.*

Proof. The result is a corollary from Proposition 5.17 as

[24] One can verify that the action of the derivation along paths induced by $\mathcal{D}$ on the liftings generated by sections is given by (5.19).

$X = X^{\mu} \dfrac{\partial}{\partial x^{\mu}}$ and (5.23) imply (5.24) with $\Gamma_{\mu} := [\Gamma^{i}_{j\mu}] = F^{-1} \dfrac{\partial F}{\partial x^{\mu}}$. □

Theorem 5.3 (cf. Theorem 5.1)**.** *Frames normal on a neighborhood U for a derivation* $\mathcal{D}$ *along vector fields exist iff it is linear on U and its* 3-*index coefficients satisfy the conditions* (5.6) *on U.*

Proof. By Proposition 5.5, a derivation $\mathcal{D}$ along vector fields is Euclidean iff (5.3) holds for some $\boldsymbol{F}$ which, according to Lemma 5.1, is equivalent to (5.6). □

Proposition 5.6 (cf. Proposition 5.2)**.** *A frame is normal on a set U for some linear derivation along tangent vector fields iff the derivation's* 3-*index coefficients vanish on U.*

Proof. This result is a corollary of Definition 5.2, equation (5.24) and the arbitrariness of X in it. □

In this way we have proved the *existence of a bijective mapping between the sets of Euclidean derivations along paths and Euclidean linear transports along paths*. It is given via the (local) coincidence of their 3-index coefficients in some (local) frame. Moreover, the normal frames for the corresponding objects of these sets coincide. What concerns the frames normal for Euclidean derivations along tangent vector fields, in them, by Proposition 5.6, vanish not only their 2-index coefficients, but also the 3-index ones. Hence the set of these frames is, generally, a subset of the one of frames normal for derivations or linear transports along paths.

6. Strong Normal Frames

Let M be a manifold and $(T(M), \pi, M)$ be the tangent bundle over it. Let ∇ and P be, respectively, a linear connection on M and the parallel transport along paths in $(T(M), \pi, M)$ generated by ∇ (see (2.34) and the statement after it). Suppose ∇ and P admit frames normal on a set $U \subseteq M$. Here a natural question arises: what are the links between both types of normal frames, the ones normal for ∇ on U and the ones for P on U?

Recall, if Γ^i_{jk} are the coefficients of ∇ in a frame $\{E_i\}$, then the frame $\{E_i\}$ is normal on $U \subseteq M$ for ∇ or P iff respectively

$$\Gamma^i_{jk}(p) = 0 \tag{6.1}$$

$$\Gamma^i_j(s;\,\gamma) = \Gamma^i_{jk}(\gamma(s))\dot{\gamma}^k(s) = 0, \tag{6.2}$$

for every $p \in U$, $\gamma : J \to U$, and $s \in J$. Two simple but quite important conclusions can be made from these equalities: (i) The frames normal for ∇ are normal for P, the converse being generally not valid, and (ii) in a frame normal for ∇ vanish the 2-index as well as the 3-index coefficients of P.

Definition 6.1. Let P be a parallel transport in $(T(M),\, \pi,\, M)$ and $U \subseteq M$. A *frame* $\{E_i\}$, defined on an open set containing U, is called *strong normal on U for* P if the 3-index coefficients of P in $\{E_i\}$ vanish on U. Respectively, $\{E_i\}$ is *strong normal along* $g : Q \to M$ if it is strong normal on $g(Q)$.

Obviously, the set of frames strong normal on U for a parallel transport P coincides with the set of frames normal for the linear connection ∇ generating P.

The above considerations can be generalized directly to linear transports for which 3-index coefficients exist and are fixed.

Definition 6.2. Let E and M be C^1 manifolds, $U \subseteq M$, and $(E,\, \pi,\, M)$ be a vector bundle over M. Let L (resp. D) be a linear transport (resp. derivation) along paths in $(E,\, \pi,\, M)$ admitting 3-index coefficients on U which are supposed to be fixed, i.e., its coefficient matrix is of the form

$$\mathbf{\Gamma}(s;\,\gamma) = \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s) \tag{6.3}$$

in every pair of frames $\{e_i\}$ in E and $\{E_\mu\}$ in $T(M)$ defined on an open set containing U or equal to it, where $\gamma : J \to U$ is of class C^1 and

$\Gamma_\mu := [\Gamma^i_{j\mu}]$ are the (fixed) matrices of the 3-index coefficients of L. A *frame* $\{e_i\}$, defined on an open set containing U or equal to it, is called *strong normal on U for L* (resp. D), if in the pair $(\{e_i\}, \{E_\mu\})$ for some (and hence any) $\{E_\mu\}$ the 3-index coefficients of L vanish on U. Respectively, $\{e_i\}$ is *strong normal along* $g : Q \to M$ if it is strong normal on $g(Q)$.

So, a frame $\{e_i\}$ is strong normal or normal on U if (cf. (6.1) and (6.2)) respectively

$$\Gamma_\mu(x) = 0 \tag{6.4}$$

$$\mathbf{\Gamma}(s;\, \gamma) = \Gamma_\mu(\gamma(s))\dot{\gamma}^\mu(s) = 0 \tag{6.5}$$

for every $x \in U$, $\gamma : J \to U$, and $s \in J$. From these equations, it is evident that a strong normal frame is a normal one, the opposite being valid as an exception, e.g., if U is a neighborhood. This situation is identical with the one for parallel transports in $(T(M), \pi, M)$ which is a consequence of the fact that Definition 6.2 incorporates Definition 6.1 as its obvious special case.

The main difference between the cases of parallel transports and arbitrary linear transports along paths is that for the former the condition (6.3) holds globally, i.e., for every path $\gamma : J \to M$, for some uniquely fixed Γ_μ, while for the latter (6.3) is valid, generally, locally, i.e., for $\gamma : J \to U$ with $U \subseteq M$, and in it Γ_μ are fixed but are not uniquely defined by the transport and may depend on U (see Section 5). The cause for this is that for a parallel transport, equation (6.3) on M with uniquely defined Γ_μ follows from its definition, while if for a given linear transport L this equation holds on U for some Γ_μ, it is also true if we replace Γ_μ with $\Gamma_\mu + G_\mu$, where the matrix-valued functions G_μ are subjected to the condition $G_\mu\dot{\gamma}^\mu = 0$ for every path γ in U. If U is an open set, then $\dot{\gamma}(s)$ is an arbitrary vector in $T_{\gamma(s)}(M)$, which implies $G_\mu|_U = 0$, i.e., in this case the 3-index coefficients of L are unique; just this is the case with a parallel transport when $U = M$ and its 3-index coefficients are fixed and, by definition, are equal to the coefficients of the linear connection generating it.

If in Definition 6.2 one replaces D with a derivation $\mathcal{D}$ along tangent vector fields and (6.4) with (5.24), the definition of a frame strong normal on U for $\mathcal{D}$ will be obtained. But, by Proposition 5.6, every frame normal on U for $\mathcal{D}$ is strong normal on U for $\mathcal{D}$ and *vice versa*. Therefore the concepts of a 'normal frame' and 'strong normal frame', when applied to derivations along tangent vector fields, are identical. Returning to the considerations in Subsection 5.2, we see that frames (strong) normal for a derivation along tangent vector fields are *strong* normal for some derivation or linear transport along paths and *vice versa*. For this reason, below only strong normal frames for the latter objects will be investigated.

To make the situation easier and clearer, below the following problem will be studied. Let (E, π, M) be a vector bundle over a C^1 manifold M, $V \subseteq M$ be an *open* subset, $U \subseteq V$, and L be a linear transport along paths in (E, π, M) whose coefficient matrix has the form (6.3) on V, i.e., for every C^1 path $\gamma : J \to V$.[25] The problem is to be investigated frames strong normal for L on U.

Let $\{e_i\}$ be a frame over V in E and $\{E_\mu\}$ be a frame over V in $T(M)$. A frame $\{e'_i = A^j_i e_j\}$ over V in E is strong normal on $U \subseteq V$ if for some frame $\{E'_\mu\}$ over V in $T(M)$ is fulfilled $\Gamma'_\mu|_U = 0$ with Γ'_μ given by (5.4). Hence $\{e'_i\}$ is strong normal on U iff the matrix-valued function $A = [A^j_i]$ satisfies the (*strong*) *normal frame equation*

$$(\Gamma_\mu A + E_\mu(A))|_U = 0, \tag{6.6}$$

where Γ_μ are the 3-index coefficients' matrices of L in $(\{e_i\}, \{E_\mu\})$.

---

[25] From here follows the existence of unique 3-index coefficients of L on V which, under a change of frames, transform into (5.5). We suppose the 3-index coefficients of L on U to be fixed and equal to the ones on V when restricted to U.

If on U exists a frame $\{e_i\}$ strong normal for L, then all frames $\{e'_i = A_i^j e_j\}$ which are normal or strong normal on U can easily be described: for the normal frames, the matrix $A = [A_i^j]$ must be constant on U (Corollary 3.4), $A|_U = 0$, while for the strong normal frames it must be such that $E_\mu(A)|_U = 0$ for some (every) frame $\{E_\mu\}$ over U in $T(M)$ (see (6.6) with $\Gamma_\mu|_U = 0$).

Comparing equation (6.6) with analogous ones in [15-17], we see that they are identical with the only difference that the size of the square matrices $\Gamma_1, \ldots, \Gamma_{\dim M}$, and A in [15-17] is $\dim M \times \dim M$ while in (6.6) it is $\upsilon \times \upsilon$, where υ is the dimension of the vector bundle (E, π, M), i.e., $\upsilon = \dim \pi^{-1}(x)$, $x \in M$, which is generally not equal to dim M. But this difference is completely insignificant from the view-point of solving these equations (in a matrix form) or with respect to the integrability conditions for them. Therefore all of the results of [15-17], concerning the solution of the matrix differential equation (6.6), are (*mutatis mutandis*) applicable to the investigation of the frames strong normal on a set $U \subseteq M$.

The transferring of results from [15-17] is so trivial that their explicit reformulation makes sense only if one really needs the corresponding rigorous assertions for some concrete purpose. For this reason, we describe below briefly the general situation and one of its corollary.

The only peculiarity one must have in mind, when such transferring is carried out, consist in the observation that in this way can be obtained, generally, only *part* of the frames normal for some linear transport, viz. the frames strong normal for it. But such a state of affairs is not a trouble as we need a single normal frame to construct all of them by means of Corollary 3.4.

If $\gamma_n : J^n \to M$, J^n a neighborhood in $\mathbb{R}^n$, $n \in \mathbb{N}$, is a C^1 injective map, then [17, Theorem 3.1] a necessary and sufficient condition for the existence of frame(s) strong normal on $\gamma_n(J^n)$ for some linear transport along paths or derivation along paths or along vector fields tangent to M, is in some neighborhood (in $\mathbb{R}^n$) of every $s \in J^n$ their (3-index) coefficients to satisfy the equations

$$(R_{\mu\nu}(-\Gamma_1 \circ \gamma_n, \ldots, -\Gamma_{\dim M} \circ \gamma_n))(s) = 0, \quad \mu, \nu = 1, \ldots, n, \tag{6.7}$$

where $R_{\mu\nu}$ are given via (5.7) for $x^\mu = s^\mu$, $\mu, \nu = 1, \ldots, n$ with $\{s^\mu\}$ being Cartesian coordinates in $\mathbb{R}^n$.

From (6.7) an immediate observation follows [17, Section 6]: strong normal frames always exist at every point $(n = 0)$ or/and along every C^1 injective path $(n = 1)$. Besides, these are the *only cases* when normal frames *always exist* because for them (6.7) is identically valid. On submanifolds with dimension greater than or equal to two normal frames exist only as an exception if (and only if) (6.7) holds. For $n = \dim M$ equations (6.7) express the flatness of the corresponding linear transport [30] or derivation [16, Section 2] to which we shall return to elsewhere.

It is almost evident, in the coordinates used, equation (6.7) is identical with (5.15) for $N = \gamma_n(J^n)$ and $p = \gamma_n(s)$. Thus, on a submanifold or along injective mappings, the existence of normal frames (for linear transports of the considered type) implies the existence of strong normal frames.

7. Conclusion

In the preceding sections we have developed the generic theory of linear transports along paths in vector bundles and of frames normal for them and for derivations along paths and/or along tangent vector fields (if the bundle's base is a manifold in the last case). Below we make some conclusions from the material presented and point out links with other results in this field.

From Proposition 5.1 and Theorem 5.2, we know that only linear transports/derivations along paths with (2-index) coefficients given by (5.1) admit normal frames. Besides, from equations (5.1) and (5.4), it follows that frames normal on a subset $U \subseteq M$ for such transports/

derivations along paths exist if and only if the matrix differential equation

$$\left[\dot{\gamma}^{\mu}\left(\Gamma_{\mu}A + \frac{\partial A}{\partial x^{\mu}}\right)\right]\Big|_{U} = 0 \tag{7.1}$$

has a solution for every $\gamma : J \to U$ with respect to A.[26] In fact, the equations (5.15) are the integrability conditions for (7.1).[27] Evidently, the same is the situation with derivations along tangent vector fields (see Subsection 5.2) when, due to (5.22), such a derivation admits frames normal on U iff the equation

$$(\Gamma_X A + X(A))|_U = 0, \tag{7.2}$$

Γ_X being the derivation's matrix along a vector field X, has a solution with respect to A. As we proved in Subsection 5.2, if X is arbitrary and tangent to the paths in U, this equation is equivalent to (7.1) with Γ_{μ} being the matrices of the 3-index coefficients of the derivation; if X is completely arbitrary, (7.2) is equivalent to equation (7.3) below.

Now it is time to recall that, from a mathematical view-point, the series of papers [15-17] is actually devoted precisely to the solution of the equation[28]

$$\left(\Gamma_{\mu}A + \frac{\partial A}{\partial x^{\mu}}\right)\Big|_{U} = 0 \tag{7.3}$$

which is equivalent to (7.1) if U is a neighborhood. The general case is explored in [17], while [16] investigates the case $U = \gamma(J)$ for $\gamma : J \to M$ and [15] is concentrated on the one in which U is a single point or a neighborhood in M. The fact that in the works mentioned are studied frames normal for derivations of the tensor algebra over a manifold M is inessential because the *equations describing the matrices*

---

[26] If such A exists in a frame $\{e_i\}$, then the frame $\{e'_i = A^j_i e_j\}$ is normal on U and vice versa; see (5.4) and Proposition 5.3.

[27] If (5.6) holds and U is a neighborhood, then $A = Y(p, p_0; -\Gamma_1, \ldots, -\Gamma_{\dim M})A_0$, A_0 being non-degenerate matrix.

[28] In [15-17] the notation W_X instead of Γ_X is used.

by means of which is performed the transformation from an arbitrary frame to a (strong) normal one are the same in these papers and in the present investigation. The only difference is *what objects* are transformed by means of the matrices satisfying (7.2): in the present work these are the frames in the restricted bundle space $\pi^{-1}(U) \subseteq E$, while in the above series of works they are the tensor bases over U, in particular the ones in the bundle tangent to M. In [15-17] the only explicit use of the derivations of the tensor algebra over M was to define their components (2-index coefficients) and the transformation law for the latter. Since this law [8, equation (2.2)] is identical with (5.22),[29] all results concerning the 2- and 3-index coefficients of derivations of the tensor algebra over M and the ones of derivations along tangent vectors in vector bundle (E, π, M) *coincide*.

Thus, we have come to a very important conclusion: *all of the results of* [15-17] *concerning S-derivations*, *their components*, *and frames normal for them are mutatis mutandis valid* (*as investigated in the present work*) *for linear transports along paths*, *derivations along paths or along tangent vector fields*, *their coefficients* (*or components*), *and the frames* (*strong*) *normal for them in vector bundles with a differentiable manifold as a base*. The only change, if required, to transfer the results is to replace the term 'S-derivation' with 'derivation along tangent vector fields', or 'derivation along paths', or 'linear transport along paths' and, possibly, the term 'normal frame' with 'strong normal frame'.

Because of the widespread usage of covariant derivatives (linear connections), we want to mention them separately regardless of the fact that this case was completely covered in [15-17]. As a consequence of (2.34), the covariant derivatives are linear derivations on the whole base M (as well as on any its subset). Thus for them the condition (5.1) is identically satisfied. Therefore, by Theorem 5.2, a covariant derivative (or the corresponding parallel transport) admits normal frames on a submanifold $U \subseteq M$ iff (5.15) holds on U. Consequently, every covariant

[29] The transformation laws (2.30) and (5.4) can be considered, under certain conditions, as special cases of (5.22).

derivative admits normal frames at every point or along any given smooth injective path. However, only the flat covariant derivatives on U admit frames normal on U if U is a neighborhood $(\dim U = \dim M)$. Further general details concerning this important case can be found in [17, Section 5].

In theoretical physics, we find applications of a number of linear transports along paths [23]: parallel [34, 47], Fermi-Walker [12, 50], Fermi [50], Truesdell [54, 55], Jaumann [44], Lie [12, 47], modified Fermi-Walker and Frenet-Serret [3] etc. Our results are fully applicable to all of them (see [23, Proposition 4.1]), in particular for all of them there exist frames normal at a given point or/and along smooth injective paths.

We end with a few words about gravity. A comprehensive analysis, based on [15-17], of the connections between gravity and normal frames is given in [18]. The importance of the concept of 'normal frame' for physics comes from the fact that it is the mathematical object representing the physical concept of an 'inertial frame'. Moreover, in [18] we proved that the (strong) equivalence principle is a theorem according to which these two types of frames coincide. Thus, we hope, the present investigation may find applications in the further exploration of gravity.

The formalism developed in the present work can find natural application in the theory of gauge fields [29], which mathematically are linear connections, to whose coefficients our results are applicable.

Acknowledgements

The author expresses his gratitude to Professor James D. Stasheff (Math-UNC, Chapel Hill, NC, USA) for his interest in the present paper and help for correcting the English of the manuscript. The current paper has benefited from comments by the referee of an earlier version.

References

[1] Y. Choquet-Bruhat et al., Analysis, Manifolds and Physics, North-Holland Publ. Co., Amsterdam, 1982.

[2] Richard G. Cooke, Infinite Matrices and Sequence Spaces, London, 1950. Reprinted: Dover Publications, New York, 1966.

[3] R. Dandoloff and W. J. Zakrzewski, Parallel transport along a space curve and related phases, J. Phys. A: Math. Gen. 22 (1989), L461-L466.

[4] Peter Dombrowski, Krümmungsgrößen gleichungsdefinierter Untermannigfaltigkeiten Riemannscher Mannigfaltigkeiten, Mathematische Nachrichten 38(3/4) (1968), 133-180.

[5] B. A. Dubrovin, S. P. Novikov and A. T. Fomenko, Modern geometry: methods and applications, Pt. 1, Vol. 93 of Graduate Text in Mathematics: The geometry of surfaces, transformation groups, and fields, 2 ed., Springer, New York, 1992. Translation from the original Russian ed., Moscow, Nauka, 1979.

[6] B. Durhuus and J. M. Leinaas, On the loop space formulation of gauge theories, Preprint TH 3110, CERN, Geneva, 1981.

[7] E. Fermi, Sopra i fenomeni che avvengono in vicinonza di una linear oraria (On phenomena near a world line), Atti R. Accad. Lincei Rend., Cl. Sci. Fis. Mat. Nat. 31(1) (1922), 21-23, 51-52. In Italian. Russian translation in [56, pp. 64-71].

[8] Enrico Fermi, Scientific Papers, 1921-1938, Italy, Vol. I, Nauka, Moscow, 1971 (in Russian).

[9] W. Greub, S. Halperin and R. Vanstone, De Rham cohomology of manifolds and vector bundles, Connections, Curvature, and Cohomology, Vol. 1, Academic Press, New York and London, 1972.

[10] D. Gromoll, W. Klingenberg and W. Meyer, Riemannian Geometry At Large, Mir, Moscow, 1971. Russian translation from the German original: Riemannsche Geometrie im Großen, Springer, Berlin-Heidelberg-New York, 1968.

[11] Ph. Hartman, Ordinary Differential Equations, John Wiley & Sons, New York-London-Sydney, 1964.

[12] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-time, Cambridge Univ. Press, Cambridge, 1973.

[13] Sze-Tsen Hu, Homotopy Theory, Academic Press, New York-London, 1959.

[14] D. Husemoller, Fibre Bundles, McGraw-Hill Book Co., New York-St. Louis-San Francisco-Toronto-London-Sydney, 1966. Russian translation: Mir, Moscow, 1970.

[15] Bozhidar Z. Iliev, Normal frames and the validity of the equivalence principle: I, Cases in a neighborhood and at a point, J. Phys. A: Math. Gen. 29(21) (1996), 6895-6901.

http://www.arXiv.org e-Print archive, E-print No. gr-qc/9608019, August 1998.

[16] Bozhidar Z. Iliev, Normal frames and the validity of the equivalence principle: II, The case along paths, J. Phys. A: Math. Gen. 30(12) (1997), 4327-4336.

http://www.arXiv.org e-Print archive, E-print No. gr-qc/9709053, September 1997.

[17] Bozhidar Z. Iliev, Normal frames and the validity of the equivalence principle: III, The case along smooth maps with separable points of self-intersection, J. Phys. A: Math. Gen. 31(4) (1998), 1287-1296.

http://www.arXiv.org e-Print archive, E-print No. gr-qc/9805088, May 1998.

[18] Bozhidar Z. Iliev, Is the principle of equivalence a principle? J. Geom. Phys. 24(3) (1998), 209-222.
http://www.arXiv.org e-Print archive, E-print No. gr-qc/9806062, June 1998.

[19] Bozhidar Z. Iliev, Fibre bundle formulation of nonrelativistic quantum mechanics, I, Introduction, The evolution transport, J. Phys. A: Math. Gen. 34(23) (2001), 4887-4918.
http://www.arXiv.org e-Print archivel, E-print No. quant-ph/9803084, March 1998.

[20] Bozhidar Z. Iliev, Flibre bundle formulation of nonrelativistic quantum mechanics, III. Pictures and integrals of motion, J. Phys. A: Math. Gen. 34(23) (2001), 4935-4950.
http://www.arXiv.org e-Print archive, E-print No. quant-ph/9806046, June 1998.

[21] Bozhidar Z. Iliev, Linear transports along paths in vector bundles, I, General theory, JINR Communication E5-93-239, Dubna, 1993, p. 22.
http://www.arXiv.org e-Print archive, E-print No. math.DG/0411023, November 1, 2004.

[22] Bozhidar Z. Iliev, Transports along paths in fibre bundles, II. Ties with the theory of connections and parallel transports, JINR Communication E5-94-16, Dubna, 1994, p. 24.
http://www.arXiv.org e-Print archive, E-print No. math.DG/0503006, March 1, 2005.

[23] Bozhidar Z. Iliev, Parallel transports in tensor spaces generated by derivations of tensor algebras, JINR Communication E5-93-1, Dubna, 1993, p. 19.
http://www.arXiv.org e-Print archive, E-print No. math.DG/0502008, February 1, 2005.

[24] Bozhidar Z. Iliev, Transports along paths in fibre bundles, General theory, JINR Communication E5-93-299, Dubna, 1993, p. 26.
http://www.arXiv.org e-Print archive, E-print No. math.DG/0503005, March 1, 2005.

[25] Bozhidar Z. Iliev, Transports along maps in fibre bundles, JINR Communication E5-97-2, Dubna, 1997, p. 19.
http://www.arXiv.org e-Print archive, E-print No. dg-ga/9709016, September 1997.

[26] Bozhidar Z. Iliev, Fibre bundle formulation of nonrelativistic quantum mechanics, II. Equations of motion and observables, J. Phys. A: Math. Gen. 34(23) (2001), 4919-4934.
http://www.arXiv.org e-Print archive, E-print No. quant-ph/9804062, Apri1 1998.

[27] Bozhidar Z. Iliev, Fibre bundle formulation of nonrelativistic quantum mechanics, IV. Mixed states and evolution transport's curvature, Internat. J. Modern Phys. A 17(2) (2002), 229-243.
http://www.arXiv.org e-Print archive, E-print No. quant-ph/9901039, January 1999.

[28] Bozhidar Z. Iliev, Fibre bundle formulation of nonrelativistic quantum mechanics, V, Interpretation, summary, and discussion, Internat. J. Modern Physics A 17(2) (2002), 245-258.
http://www.arXiv.org e-Print archive, E-print No. quant-ph/9902068, February 1999.

[29] Bozhidar Z. Iliev, Normal frames for derivations and linear connections and the equivalence principle, J. Geom. Phys. 45(1-2) (2003), 24-53.

http://www.arXiv.org e-Print archive, E-print No. hep-th/0110194, October 2001.

[30] Bozhidar Z. Iliev, Linear transports along paths in vector bundles, III, Curvature and torsion, JINR Communication E5-93-261, Dubna, 1993, p. 12.

http://www.arXiv.org e-Print archive, E-print No. math. DG/0502008, February 1, 2005.

[31] Bozhidar Z. Iliev, Linear transports along paths in vector bundles, V, Properties of curvature and torsion, JINR Communication E5-97-1, Dubna, 1997, p. 11.

http://www.arXiv.org e-Print archive, E-print No. dg-ga/9709017, September 1997.

[32] O. M. Khudaverdian and A. S. Schwarz, A new comments on the string representation of gauge fields, Phys. Lett. B 91(1) (1980), 107-110.

[33] S. Kobayashi, Theory of connections, Ann. Mat. Pura Appl. 43 (1957), 119-194.

[34] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I, Interscience Publishers, New York-London, 1963. Russian translation: Nauka, Moscow, 1981.

[35] Ülo G. Lumiste, To the foundations of the global connection theory, Scientific Writings of the Tartu State University, Number 150 in Works on Mathematics and Mechanics, Vol. IV, pp. 69-107, University of Tartu, Tartu, 1964 (in Russian).

[36] Ülo G. Lumiste, Homogeneous fibre bundles with connection and their immersions, Works of the Geometrical Seminar, Vol. 1, pp. 191-237, VINITI, Moscow, 1966 (in Russian).

[37] Ülo G. Lumiste, Connection theory in fibre bundles, Science Review, Mathematics: Algebra, Topology, Geometry, 1969, pp. 123-168, VINITI, Moscow, 1971 (in Russian).

[38] L. Mangiarotti and G. Sardanashvily, Connections in Classical and Quantum Field Theory, World Scientific, Singapore-New Jersey-London-Hong Kong, 2000.

[39] Mathematical Encyclopedia, Chief Editor Vinogradov, I. M., Vol. 4, Moscow, 1984, Sovetskaya Entsiklopediya (Soviet Encyclopedia) (in Russian). English translation in: Encyclopedia of Mathematics, Vol. I-VI, Kluwer, 1995.

[40] P. Nikolov, On the correspondence between infinitesimal and integral description of connections, Internal Report, IC/81/196, ICTP, Trieste, 1981.

[41] N. M. Ostianu, V. V. Rizhov and P. I. Shveikin, Article on the scientific investigations of German Fedorovich Laptev, Works of the Geometrical Seminar, Vol. 4, pp. 7-70, VINITI, Moscow, 1973 (in Russian).

[42] Walter A. Poor, Differential Geometric Structures, McGraw-Hill Book Company, Inc., New York, 1981.

[43] Pham Mau Quan, Introduction a la Géométrie des Variétés Différentiables, Dunod, Paris, 1969.

[44] L. Radhakrishna, L. N. Katkar and T. H. Date, Jaumann transport in relativistic continuum mechanics, Gen. Relativity Gravitation 13(10) (1981), 939-946.

[45] L. Ó Raifeartaigh, Fermi coordinates, Proc. Roy. Irish Acad. Sect. A 59(2) (1958), 15-24.

[46] J. A. Schouten, Tensor Analysis for Physicists, Clarendon Press, Oxford, 1951. Russian translation: Nauka, Moscow, 1965.

[47] J. A. Schouten, Ricci-Calculus: An Introduction to Tensor Analysis and its Geometrical Applications, 2nd ed., Springer-Verlag, Berlin-Gottingen-Heidelberg, 1954.

[48] James D. Stasheff, Parallel transport in fibre spaces, Reimpreso del boletin de la sociedad matematica mexicana, Sociedad Matematica Mexicana, Printed in USA, 1966, pp. 68-84.

[49] N. Steenrod, The Topology of Fibre Bundles, 9th ed., Princeton Univ. Press, Princeton, 1974.

[50] J. L. Synge, Relativity: The General Theory, North-Holland Publ. Co., Amsterdam, 1960. Russian translation: IL (Foreign Literature), Moscow, 1963.

[51] C. Teleman, Elements of Topology and Differentiable Manifolds, Mir, Moscow, 1967 (in Russian). Translation from the Rumanian original: Elemente de Topologie si Varietati Differentiabile, Bucuresti, 1964, German translation: Grundzüge der Topologie und Differenzierbare Mannigfaltigkeiten (Autorisierte und vom Autor erg. Übersetzung aus dem Rumänischen: Horst Antel-mann.) Mit 22 Abbildungen, Berlin, Deutscher Verlag der Wissenschaften, 1968, p. 411.

[52] O. Veblen, Normal coordinates for the geometry of paths, Proc. Nat. Acad. 8 (1922), 192-197.

[53] O. Ya. Viro and D. B. Fuks, Introduction to homotopy theory, Reviews of Science and Technique, Vol. 24, Modern Problems in Mathematics, pp. 6-121, VINITI, Moscow, 1988 (in Russian).

[54] B. B. Walwadkar, Truesdell transport in general relativity, Gen. Relativity Gravitation 15(12) (1983), 1107-1116.

[55] B. B. Walwadkar and K. V. Virkar, Truesdell invariance in relativistic electromagnetic fields, Gen. Relativity Gravitation 16(1) (1984), 1-7.

[56] F. W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1983. Russian translation: Mir, Moscow, 1987.

■