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ISOMETRIC CYLINDER METHOD FOR STRICT MIXED-INTEGER LINEAR PROGRAMMING

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Abstract

In this paper, we solve strict mixed-integer linear programming not including integer linear programming by isometric cylinder method under the MATLAB environment. The algorithm considers the optimization problems for both whole and real variables of the strict mixed-integer linear programming, and can quickly obtain the optimal mixed-integer point simultaneously using isometric planes and cutting planes derived from polyhedral-cones and rounded-minimal-cylinders at the highest vertex. Only a few linear programming problems need to be solved. Numerical tests show the conclusions.

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1. Introduction

We consider the following mixed-integer linear programming (MILP):

$$\max z = c^T x + c_0,$$

$$\text{such that } Ax \geq b, \; x = (x_1, x_2, ..., x_n)^T \text{ : mixed-integer vector,} \quad (1.1)$$

where $A = (a_{ij})$, $c = (c_1, c_2, ..., c_n)^T$, $b = (b_1, b_2, ..., b_m)^T$ are given $m \times n$ matrix and vectors in R^n and R^m respectively, c_0 is a parameter independent of x, and $n = n_i + n_r$ $(n > n_i > 1)$, $x_1, ..., x_{n_i}$ are integer variables, $x_{n_i+1}, ..., x_n$ real variables. The MILP (1.1) becomes an integer linear programming (ILP) when $n_r = 0$, which has been solved by the isometric surface method, see [6] and [10]. We suppose $n_r > 0$ here. The point where the first n_i components are integers is referred to as *mixed-integer point*. The relaxation linear programming (LP) of (1.1) is

$$\max z = c^T x,$$

$$\text{such that } Ax \geq b. \quad (1.2)$$

There is no equality in the constraint conditions of problems (1.1) and (1.2). Otherwise, for instance, an equality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \quad (1.3)$$

can be replaced by two inequalities

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i - \varepsilon_0,$$

$$-a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \geq -b_i - \varepsilon_0, \quad (1.4)$$

where ε_0 is a positive number near to zero, for example, $\varepsilon_0 = 10^{-12}$ (ε_0^{-1} is understood as infinite). Clearly, the replacement (1.4) does not decrease mixed-integer points of the hyperplane (1.3) in R^n; however, it is possible to increase mixed-integer points. When the distance (L_2 norm) from some mixed-integer point to the hyperplane (1.3) is not bigger than a multiplier of ε_0, the mixed-integer point is included in the *thin space* defined by (1.4). If this phenomenon happens to the replacement (1.4), then we approximately think of the mixed-integer point is located on the hyperplane (1.3).

Generally, the following hypothesis is adopted in this paper:

Thick-whole-point hypothesis: Assume $x, y \in R^n$ and y is a rounded mixed-integer point of x, where each one of the first n_i components of x is rounded off to the nearest whole number. If

$$\| x - y \|_2 = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \le \varepsilon_0,$$

then the point x is considered a mixed-integer point.

If there are m_s equalities which can eliminate m_s real variables $(m_s \le n_r)$, then the dimension of problems (1.1) and (1.2), and the number of constraint is reduced to $n - m_s$ and $m - m_s$ respectively.

The algorithms for ILP and MILP have been studied for many years. Most algorithms are LP-based branch-and-cut, branch-and-bound and their improvements, e.g., [1-4, 8]. However, branch-and-cut and branch-and-bound are essentially ill-implied enumeration [5]. Therefore, they are essentially NP-hard method for general ILP and MILP. For ILP, isometric surface method is a well-implied enumeration [5, 6], which has been successfully applied to travelling salesman problem [9]. The isometric surface method with some additional techniques is also well for the MILP (1.1), [10]. In this paper, we present an isometric cylinder method for the strict MILP (1.1) not including ILP. In order to get a solution of the problem (1.1), the proposed method needs not any additional technique of chance, and needs to solve relaxation problem only one time in most cases. Using the isometric plane method [7] or using the MATLAB function *linprog*$(\cdots)$, a simplex algorithm, we can quickly get the solution of relaxation LP (1.2). Numerical tests show that the isometric cylinder method, if it gives a solution, is more accurate than the isometric surface method for strict MILP.

2. Polyhedral-Cone and Rounded-Minimal-Cylinder

Assume now, we have obtained, using the isometric plane method or simplex algorithm, the highest vertex x^* for relaxation LP (1.2) whatever the nonempty constraint polyhedron Ω^m is bounded or unbounded. x^* is usually an intersection point of n hyperplanes. If there are more than n hyperplanes intersecting at x^*, then test n hyperplanes by x^*, which are linearly independent of each other as intersection hyperplanes. These n hyperplanes with vertex x^* and normal vectors $A_{l_k}^T = (a_{l_k,1}, \ldots, a_{l_k,n})^T$ $(k = 1, 2, \ldots, n)$ form a polyhedral-cone $C_{x^*}^n$ in R^n, that is,

$$C_{x^*}^n = \{x \in R^n \mid A_{l_k}(x - x^*) \geq 0,\ k = 1, 2, \ldots, n\}. \tag{2.1}$$

There are n one-dimensional edges on the cone $C_{x^*}^n$. Each 1D-edge is a ray passing through x^* and satisfying

$$A_{l_i}(x - x^*) = 0,\ i = 1, \ldots, k-1, k+1, \ldots, n;\ A_{l_k}(x - x^*) > 0,\ 1 \leq k \leq n.$$

Let $y^{l_k} = x - x^*$ be a 1D-edge vector such that y^{l_k} can be found by

$$A_{l_i} y^{l_k} = 0,\ i = 1, \ldots, k-1, k+1, \ldots, n;\ A_{l_k} y^{l_k} > 0,\ (1 \leq k \leq n). \tag{2.2}$$

We have known that

$$y^{l_k} = -d_k = \sum_{i \neq k} p_i A_{l_i}^T - c,$$

$$\sum_{i \neq k} p_i A_{l_j} A_{l_i}^T = A_{l_j} c\ \ (j = 1, \ldots, k-1, k+1, \ldots, n), \tag{2.3}$$

satisfies (2.2) [7]. Thus, the 1D-edge is the ray

$$L_{x^*}^{y^{l_k}} = \{x \in R^n \mid x - x^* = t y^{l_k},\ t \geq 0\},\ (1 \leq k \leq n). \tag{2.4}$$

In order to find n 1D-edge vectors y^{l_k} $(k = n, \dots, 1)$, we notice the symmetric positive matrix $\overline{A} = (A_{l_i} A_{l_j}^T)$ and its inverse matrix $\overline{B} = (A_{l_i} A_{l_j}^T)^{-1}$. Let E_{kn} be the matrix commuting k-n row/column. Then $(E_{kn}\overline{A}E_{kn})^{-1} = E_{kn}\overline{B}E_{kn}$, namely

$$\begin{pmatrix} \overline{A}_k & \overline{a}_k \\ \overline{a}_k^T & A_{l_k} A_{l_k}^T \end{pmatrix}^{-1} = \begin{pmatrix} \overline{B}_k & \overline{b}_k \\ \overline{b}_k^T & \overline{b}_{kk} \end{pmatrix},$$

where $\overline{a}_k$ and $\overline{b}_k$ are known column vectors. We have

$$\overline{A}_k^{-1} = \overline{B}_k - \frac{1}{\overline{b}_{kk}} \overline{b}_k \overline{b}_k^T.$$

The coefficients p_i of equation (2.3) can be got by $\overline{A}_k^{-1}$.

The highest vertex x^* of (1.2) is not a mixed-integer point usually; otherwise the mixed integer programming (1.1) has been solved. We can round each one of the first n_i components of x^* off to the nearest whole number, thus obtain the rounded-mixed-integer point x^N nearest to x^*. Since $x_i^N = x_i^*$ $(i = n_i + 1, \dots, n)$,

$$r_{x^*} = \| x^N - x^* \|_2 = \left(\sum_{i=1}^{n_i} (x_i^N - x_i^*)^2 \right)^{1/2}.$$

If $r_{x^*} \le \varepsilon_0$, then x^* is considered a mixed-integer point in thick-whole-point hypothesis, so that suppose $r_{x^*} > \varepsilon_0$. The cylinder

$$O_{x^*} = \left\{ x \in R^n \mid \left(\sum_{i=1}^{n_i} (x_i - x_i^*)^2 \right)^{1/2} \le r_{x^*} \right\} \tag{2.5}$$

is referred to as *rounded-minimal-cylinder*. Clearly, there is no mixed-integer point inside the cylinder O_{x^*}, and there is one *mixed-integer*

n_r*D-manifold*

$$M_{x^N} = \{x \in R^n \mid x_i = x_i^N,\ i = 1, \dots, n_i\}$$

at least on the surface of O_{x^*}, and the radius $r_{x^*} \le \sqrt{n_i}/2$ [6].

It is possible that there are 2^k mixed-integer manifolds on the surface of O_{x^*}, where fractional part of k components of x^* is near to 0.5, namely

$$E_k : \lceil x_j^* \rceil - x_j^* = 0.5 \pm eps,\ j = j_1, \dots, j_k, 1 \le k \le n_i, \tag{2.6}$$

$\lceil x_j^* \rceil$ denotes the least integer larger than or equal to x_j^*, $eps = 2^{-52} < 10^{-15.6}$ is a MATLAB constant near to zero.

However, the probability of event (2.6) is very small. Let fractional part of independent components, which drops equally likely into an interval with length $2eps$ in [0, 1], be a sample. Then the sample space consists of $(1/(2eps))^{n_i}$ samples. And the probability for the event (2.6) to happen is

$$p(E_k) = C_{n_1}^k (2eps)^k (1 - 2eps)^{n_i - k},\ 0 \le k \le n_i,$$

which belongs to binomial distribution. Specially,

$$p(E_0) = (1 - 2eps)^{n_i} \ge 1 - 2n_i eps,$$

$$p(E_1) = 2n_i eps(1 - 2eps)^{n_i - 1} \le 2n_i eps,$$

namely on the surface of rounded-minimal-cylinder O_{x^*} there is only one mixed-integer manifold with probability larger than or equal to $1 - 2n_i eps$, and there are two manifolds with probability less than or equal to $2n_i eps$.

We have shown

Theorem 1. *Let* x^* *be the highest vertex of relaxation linear programming* (1.2), $C_{x^*}^n$ *the polyhedral-cone defined by* (2.1) *and* O_{x^*} *the* *rounded-minimal-cylinder with radius* $r_{x^*} > \varepsilon_0$ *defined by* (2.5). *Then there is no mixed-integer point inside the* O_{x^*} *and* $r_{x^*} \leq \sqrt{n_i}/2$, *there is only one mixed-integer* $n_r D$*-manifold with probability larger than or equal to* $1 - 2n_i eps$ *on the surface of* O_{x^*}, *here,* $eps = 2^{-52}$ *is a* MATLAB *constant near to zero.*

The n 1D-edges of $C^n_{x^*}$ with formula (2.4) intersect to the surface of rounded-minimal-cylinder O_{x^*} or the boundary of constraint polyhedron Ω^m at n points. These n intersection points define an $(n-1)D$ hyperplane π_{x^*} which makes the inside of close cone $\overline{C}^n_{x^*}$ a non-whole point set because of Theorem 1. Let x^{1l_k} be the intersection point, then via (2.5)

$$x^{1l_k} = x^* + t_1^{l_k} y^{l_k},\ k = 1, 2, \ldots, n, \tag{2.7}$$

where

$$t_1^{l_k} = \min\{t_c^{l_k}, t_b^{l_k}\},\ t_c^{l_k} = r_{x^*} \Big/ \left(\sum_{i=1}^{n_i} (y_i^{l_k})^2 \right)^{1/2},$$

$$t_b^{l_k} = \min\{(b_i - A_i x^*)/A_i y^{l_k} > 0;\ i \neq l_1, \ldots, l_{k-1}, l_{k+1}, \ldots, l_n\}.$$

It is possible $t_1^{l_k} = \varepsilon_0^{-1}$ for unbounded Ω^m. Suppose that the *bottom plane* π_{x^*} of $\overline{C}^n_{x^*}$ is defined by

$$(a_1^1, a_2^1, \ldots, a_n^1)x = \pm 1,\ (a_1^1, a_2^1, \ldots, a_n^1)x^* < \pm 1.$$

We have clearly

$$(a_1^1, a_2^1, \ldots, a_n^1)x^{1l_k} = \pm 1,\ k = 1, 2, \ldots, n,\ (a_1^1, a_2^1, \ldots, a_n^1)x^* < \pm 1. \tag{2.8}$$

The linear equations (2.8) can uniquely determine the normal vector $(a_1^1, a_2^1, \ldots, a_n^1)^T$ of π_{x^*}. Cutting-plane π_{x^*} cuts out a non-whole mixed part near x^*. Therefore, $(a_1^1, \ldots, a_n^1)x \geq \pm 1$ may be appended to the constraint set of (1.1). Via (2.7), the n 1D-edges $x^* + ty^{l_k}$ $(t \geq 0,$ $k = 1, 2, \ldots, n)$ intersect to the plane π_{x^*} of new constraint polyhedron Ω_1^m at n points $x^{1l_1}, \ldots, x^{1l_n}$. Let

$$c^T x^{1l_*} = \max\{c^T x^{1l_i}, \; i = 1, 2, \ldots, n\}. \tag{2.9}$$

Theorem 2. *Suppose the highest vertex x^* for relaxation* LP (1.2) *is the intersection point of n linearly independent hyperplanes of Ω^m. Let $\pi_{x^*} : (a_1^1, \ldots, a_n^1)x = \pm 1$ be the bottom plane of $\overline{C}_{x^*}^n$ defined by* (2.7) *and* (2.8). *Append π_{x^*} to the constraints* (1.1) *and form a new constraint polyhedron Ω_1^m. Then Ω_1^m has the same* MILP *solution with Ω^m. Moreover, x^{1l_*} is the highest vertex of Ω_1^m.*

Proof. Since via Theorem 1, π_{x^*} cuts out a non-whole mixed part near x^* of Ω^m, evidently, Ω_1^m has the same MILP solution with Ω^m.

Let y^* be another vertex of the convex polyhedron Ω^m, and $(a_1^1, \ldots, a_n^1)y^* \geq \pm 1$. According to the suppositions, we know that $c^T y^* \leq c^T x^*$, and that the connected line between y^* and x^* is located on some boundary surface of Ω^m and intersects with π_{x^*} at the point z^*. Evidently, $c^T y^* \leq c^T z^* \leq c^T x^*$ and via (2.9), $c^T z^* \leq c^T x^{1l_*}$. Therefore, $c^T y^* \leq c^T x^{1l_*}$.

Notice that for some particular cases, x^{1l_*} is the intersection point of a 1D-edge with both π_{x^*} and a boundary plane of Ω^m, and therefore the polyhedral-cone $C_{x^{1l_*}}^n$ is not unique. In these cases, the highest vertex next to x^{1l_*} should be found by relaxation LP after the cutting plane $\pi_{x^{1l_*}}$ is appended to the constraints (1.1).

We need not to solve a new relaxation linear programming problem via Theorem 2. The additional cutting plane π_{x^*} may form the highest vertex of Ω_1^m. This will improve not only efficiency but also accuracy of computation.

To determine whether the rounded-mixed-integer point x^N belongs to $\overline{\Omega}^m$, we consider the constraint set of (1.2). If $Ax^N \geq b - \varepsilon_0 e$, where $e = (1, 1, ..., 1)^T$, and ε_0 is the same as right-hand side of (1.4), then x^N belongs to $\overline{\Omega}^m$. Now, suppose that a mixed-integer point x^N of rounded-minimal-cylinder surface belongs to $\overline{\Omega}^m$. Then we need only to consider the mixed-integer points x which satisfy $c^T x \geq c^T x^N$, namely, we may append the inequality $c^T x \geq c^T x^N$ to the constraint set of (1.1).

The rounded-mixed-integer point x^N is usually an external point of $\overline{\Omega}_1^m$. However; there are many mixed-integer points in the mixed-integer $n_r D$-manifold M_{x^N}, where some of them belong to $\overline{\Omega}_1^m$ possibly. In order to determine whether some mixed-integer points of M_{x^N} belong to $\overline{\Omega}_1^m$, we should verify whether the mixed-integer points of M_{x^N} satisfy the constraint conditions of $\overline{\Omega}_1^m$. If some satisfy, we should look for the highest one as well. Let $x = (x_1^N, ..., x_{n_i}^N, x_{n_i+1}, ..., x_n)^T$ be a mixed-integer point of M_{x^N}. Substituting the mixed-integer point x into (1.1) and $(a_1^1, ..., a_n^1)x \geq \pm 1$, we obtain that

$$\max z = (c_{n_i+1}, ..., c_n)(x_{n_i+1}, ..., x_n)^T,$$

$$\text{such that} \begin{pmatrix} a_{1,n_i+1} & \cdots & a_{1,n} \\ \cdot & \cdots & \cdot \\ a_{m,n_i+1} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_{n_i+1} \\ \cdot \\ x_n \end{pmatrix} \geq \begin{pmatrix} b_1 - \sum_{j=1}^{n_i} a_{1,j} x_j^N \\ \cdot \\ b_m - \sum_{j=1}^{n_i} a_{m,j} x_j^N \end{pmatrix},$$

$$a_{n_i+1}^1 x_{n_i+1} \cdots a_n^1 x_n \geq \pm 1 - \sum_{j=1}^{n_i} a_j^1 x_j^N. \tag{2.10}$$

This is an $n_r D$-LP problem. Using the isometric plane method or the MATLAB function $linprog(\cdots)$, we can quickly get the solution of (2.10) or conclude that there is no solution for (2.10). If $(x_{n_i+1}^1, \ldots, x_n^1)^T$ is the solution of (2.10), then $x^{N1} = (x_1^N, \ldots, x_{n_i}^N, x_{n_i+1}^1, \ldots, x_n^1)$ is a higher interior mixed-integer point of $\overline{\Omega}_1^m$, and we may append the inequality $c^T x \geq c^T x^{N1}$ to the constraint set of (1.1).

It is important to find quickly an interior rounded-mixed-integer point x^N of $\overline{\Omega}_1^m$ in practical computation. Therefore, the algorithm gives the following 4 selections of x^N for (2.10): to round x^* off to the nearest mixed-integer point, to round x^* off to the most distant mixed-integer point, and to round x^* off to the two mixed-integer points of alternative distant and near components.

If there is still no solution of (2.10) after many cutting-planes have been appended to Ω^m, then the computation will fail possibly.

3. Description of the Algorithm

Step 1. Record the solution $x^H = (-1/(\varepsilon_0 \| c \|_2))c$ of MILP (1.1). Let $k_c = k_n = 0$. Assign $\overline{k}_c \leq n$, $\overline{k}_n \leq n_i$.

Step 2. Check whether the constraint polyhedron Ω^m of relaxation linear programming (1.2) is empty. If it is, then go to *exit*.

Step 3. Find the highest vertex x^* of (1.2) using isometric plane method or simplex algorithm. If $\| c^T(x^* - x^H)/\| c \|_2 \|_2 \leq \varepsilon_0$, then go to *exit*.

Step 4. Make the rounded-minimal-cylinder O_{x^*} with rounded-mixed-integer points x^N and radius r_{x^*}. If $r_{x^*} \leq \varepsilon_0$, then x^* is the solution of (1.1) in thick-whole-point hypothesis, record $x^H = x^*$, $k_n = 1$, and go to *exit*; If $c^T x^* \le -\| c \|_2/\varepsilon_0$, then there is no solution for the problem (1.1), go to *exit*.

Step 5. Find the polyhedral-cone $C^n_{x^*}$ and its 1D-edges $L^{y^{l_k}}_{x^*}$ using (2.1)-(2.4).

Step 6. Look for higher mixed-integer point x^{N1} of $\overline{\Omega}^m_1$ according to (2.10) or $\overline{\Omega}^m$ according to similar to (2.10). If there is no solution for (2.10) or similar to (2.10), then go to Step 8.

Step 7. If there exists x^{N1} of $\overline{\Omega}^m$ or $\overline{\Omega}^m_1$ and $k_n = 0$, then append the inequality $c^T x \ge c^T x^{N1}$ to the constraint set of (1.1), else if there exist x^{N1} and $k_n > 0$, then append new inequality instead of old one. Record $x^H = x^{N1}$ and print x^H.

Step 8. Construct the cutting-plane π_{x^*} using (2.2), (2.7) and (2.8). Append the inequality $(a^1_1, a^1_2, \ldots, a^1_n)x \ge \pm 1$ to the constraint set of (1.1) and let $k_c = k_c + 1$. Find x^{1l_*} using (2.9).

Step 9. If $k_c \le \overline{k}_c$ or $k_n \le \overline{k}_n$, then go to Step 4 or 2 else go to *exit*.

The conclusion on the algorithm is

Theorem 3. *The algorithm will be terminated at finite steps. And the solutions of* MILP (1.1) *are under the thick-whole-point hypothesis or nearly located in the highest plane of polyhedron cut out.*

Proof. We consider the constraint polyhedron Ω^m of the relaxation linear programming (1.2). If original Ω^m is empty, then the algorithm is stopped at Step 2, and the record $x^H = (-1/(\varepsilon_0 \| c \|_2))c$ means that (1.1) has no solution. So, assume original Ω^m is nonempty.

Each time to perform Step 4-Step 8, we append a cutting-plane π_{x^*} to Ω^m which cuts out a part of Ω^m containing mixed-fractions near to x^*, and if there exists $x^{N1} \in \overline{\Omega}^m$, we also record $x^H = x^{N1}$ and append an isometric plane $P_{x^{N1}} = \{x \in R^n \mid c^T(x - x^{N1}) = 0\}$ to Ω^m. The algorithm thus can get vertex series $x^{*1}, x^{*2}, \ldots, x^{*i}, \ldots$ and mixed-whole-point series $x^{H1}, x^{H2}, \ldots, x^{Hj}, \ldots$ such that

$$c^T x^{*1} \geq c^T x^{*2} \geq \cdots \geq c^T x^{*i} \geq \cdots,$$

$$c^T x^{H1} \leq c^T x^{H2} \leq \cdots \leq c^T x^{Hj} \leq \cdots.$$

And the equality signs cannot always hold on the first sequence of inequalities. Therefore, the algorithm will be terminated at finite steps when the constraint polyhedron Ω^m of (1.2) become empty or $\| c^T(x^{*i} - x^{Hj})/\| c \|_2 \|_2 \leq \varepsilon_0$.

If the algorithm stops with $\| x^{*i} - x^{Hj} \|_2 \leq \varepsilon_0$ and the solution x^{*i} of the last relaxation programming is unique, then the problem (1.1) has a unique solution.

Generally, if the same isometric plane $c^T x = c^T x^H$ is recorded continually over $\overline{k}_n$ times or cutting-planes are made over $\overline{k}_c$ times, then the mixed-integer points are the solutions of (1.1). In fact, if once the same isometric plane is recorded continually a few times, the cutting-plane π_{x^*} is hard to construct because of rounded-error accumulation.

The arithmetical-operation quantity making cutting-planes and isometric planes is dependent on mainly Steps 5-8. In order to find inverse of symmetric positive matrix and to solve n linear equations (2.3), we need $O(n^3)$ arithmetical-operations which is the same estimation with solving linear equations (2.8). To solve the $n_r D$-LP problem (2.10) usually needs $O(mn_r^3)$ operations on Step 6. So that to perform Steps 5-8 need $O(mn^3)$ arithmetical-operations, which do not exceed the flops to perform Steps 2 and 3.

The algorithm need not solve LP (2.10) many times except original constraint polyhedron Ω^m of (1.2) is unbounded and the MILP (1.1) has no solution.

4. Numerical Examples

In order to investigate efficiency of the algorithm, we have done some numerical experiments. The program is run using MATLAB7.0 under Windows^{xp}. The CPU time required on a PC is given in seconds.

Example 1. Consider the following MILP:

$$\max z = \sum_{i=1}^{n} x_i,$$

$$\text{such that } x \leq (1.5 + (-1)^i 0.15)e,\ x \geq 0,$$

$$x \text{ is a mixed-integer vector } (n_i = \lceil 3n/5 \rceil), \tag{4.1}$$

where $e = (1, \ldots, 1)^T$. The solution of (4.1) is clearly

$$x^H = (1, \ldots, 1, 1.5 + (-1)^{n_i+1} 0.15, \ldots, 1.5 + (-1)^n 0.15)^T.$$

And the optimal objective function value $z(x^H) = 1.5n - 0.5n_i$ for even numbers n and n_i or for odd numbers n and n_i.

The numerical experiments have been done for $n = 50, 75, 100, 300, 500$. Table 1 lists comparison between the isometric cylinder method (ICM) and the isometric surface method with additional techniques (ISM) for (4.1). For LP problems, the isometric plane method (Ipm) is used in both ICM and ISM. In the table, “maximum”, “ncp” and “nip” mean the optimal objective function value of (4.1), the number of cutting-planes and the number of isometric planes, respectively. The two algorithms are comparative for (4.1). Here, ICM makes one cutting-plane more than ISM and gives an approximate optimal solution for $n = 75$.

Table 1. Comparison between ICM and ISM for (4.1)

	ICM(Ipm)					ISM(Ipm)				
n	50	75	100	300	500	50	75	100	300	500
ni	30	45	60	180	300	30	45	60	180	300
CPU time	0.33	0.83	1.64	48.67	201.53	0.28	0.67	1.19	33.97	138.70
maximum	60	88.614421	120	360	600	60	90	120	360	600
ncp	2	2	2	2	2	1	1	1	1	1
nip	1	1	1	1	1	1	1	1	1	1

Example 2. Consider the following MILP:

$$\min z = \sum_{i=1}^{n} x_i,$$

$$\text{such that } x \leq 1.75e,\ (2ww^T - I)x \geq 0.625e,$$

$$x \text{ is a mixed-integer vector } (n_i = \lceil 3n/5 \rceil), \tag{4.2}$$

where I is unit matrix, $w = (1/\sqrt{n}, \ldots, 1/\sqrt{n})^T$. The solution and the optimal objective function value of the first relaxation LP of (4.2) are $x = 0.625e$ and $z^r = 0.625n$, respectively. However, since the integer component of x^H may equal to zero or negative number, the computational solution of (4.2) is dependent on n and difference between the two algorithms. The numerical experiments have been done for $n = 10, 20, 30, 40, 50$.

Table 2 lists comparison between ICM and ISM for (4.2). For LP problems, the simplex method is used in ICM, while the isometric plane method (Ipm) is used in ISM.

Table 2. Comparison between ICM and ISM for (4.2)

	ICM(Simplex)					ISM(Ipm)				
n	10	20	30	40	50	10	20	30	40	50
ni	6	12	18	24	30	6	12	18	24	30
CPU time	0.14	0.17	0.23	0.36	0.52	0.09	0.11	0.17	0.30	0.44
minimum	8.125	16.25	24.375	32.5	40.625	8.3163	17	24.5	33	41.5
ncp	2	2	2	2	2	4	3	3	4	3
nip	1	1	1	1	1	1	1	1	1	1

Both ICM and ISM obtain the solution of (4.2) for $n = 10, ..., 50$. But the solutions obtained by them are a little different, so that the optimal objective function values are different. The result of isometric cylinder algorithm is better than that of isometric surface algorithm. Table 3 lists the solutions of (4.2) obtained by ICM and ISM for $n = 10$. It is easy to see, via Table 3, that there are optimization problems for both whole and real variables of the strict MILP.

Table 3. Solutions of (4.2) obtained by ICM and ISM for $n = 10$

ICM (Simplex)	1	1	1	1	1	1	1.0	1.0	0.125	0.0
ISM(Ipm)	1	1	1	1	1	1	0.808712	0.808712	-0.109847	0.808712

Example 3. Consider the following MILP:

$$\max z = \sum_{i=1}^{n} x_i,$$

$$\text{such that } (2ww^T - I)x \le (1.5 + (-1)^i 0.15)e, \; x \ge 0,$$

$$x \text{ is a mixed-integer vector } (n_i \ge \lceil 3n/5 \rceil). \tag{4.3}$$

The computational solution of (4.3) is dependent on n and difference between the two algorithms. The numerical experiments have been done for $n = 10, 20, 30, 40, 50$. Table 4 lists comparison between ICM and ISM for (4.3).

Table 4. Comparison between ICM and ISM for (4.3)

	ICM(Simplex)					ISM(Ipm)				
n	10	20	30	40	50	10	20	30	40	50
ni	7	13	19	27	33	7	13	19	27	33
CPU time	0.20	0.38	0.44	0.61	1.63	0.02	0.06	0.33	0.44	0.47
maximum	11.75	23.50	35.25	47.00	58.75	11.35	23.47	35.23	46.35	58.35
ncp	2	2	2	2	2	4	4	9	5	4
nip	2	1	1	2	2	1	1	1	1	1

The result of isometric cylinder algorithm is clearly better than that of isometric surface algorithm, although a little more CUP time is spent with the isometric cylinder algorithm. Table 5 lists the solutions of (4.3) obtained by ICM and ISM for $n = 10$. It is easy to see, via Table 5, that ICM and ISM give respectively different solutions for both whole and real variables of the strict MILP.

Table 5. Solutions of (4.3) obtained by ICM and ISM for $n = 10$

ICM(Simplex)	1	1	1	1	1	1	2	0.70	1.00	2.05
ISM(Ipm)	1	1	1	1	1	1	1	1.35	1.65	1.35

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