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SOME FIXED POINT THEOREMS UNDER WEAK CONDITIONS

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Abstract

In this paper, we obtain fixed point theorems without the continuity condition, and completeness of X is weakened.

Let (X, d) be a metric space. Two maps S and T are said to be *compatible* if, $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\} \subseteq X$ is such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t \in X$. Two maps S and T are said to be *weakly compatible* if they commute at coincidence points.

Let A, B, S and T be selfmaps of a metric space (X, d) such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. Define $\{x_n\}$ by $x_0 \in X$, x_1 such that $Tx_1 = Ax_0$, x_2 such that $Sx_2 = Bx_1$, and, in general, define $\{x_n\}$ so that $Tx_{2n+1} = Ax_{2n}$, $Sx_{2n+2} = Bx_{2n+1}$. Define $\{y_n\}$ by $y_{2n} = Sx_{2n}$, $y_{2n+1} = Tx_{2n+1}$.

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Theorem 1. *Let A, B, S and T be selfmaps of a metric space (X, d) satisfying $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and for each x, $y \in X$, either*

$$d(Ax, By) \leq \alpha \left\{ \frac{d(Ax, Sx)^2 + d(By, Ty)^2}{d(Ax, Sx) + d(By, Ty)} \right\} + \beta d(Sx, Ty) \tag{1}$$

if $d(Ax, Sx) + d(By, Ty) \neq 0$, α, $\beta > 0$, $\alpha + \beta < 1$, *or*

$$d(Ax, By) = 0 \text{ if } d(Ax, Sx) + d(By, Ty) = 0. \tag{2}$$

If one of $A(X)$, $B(X)$, $S(X)$ or $T(X)$ is complete subspace of X, and $\{A, S\}$, $\{B, T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. In (1) set $x = x_{2n}$, $y = x_{2n+1}$ to get

$$d(Ax_{2n}, Bx_{2n+1}) \leq \alpha \left\{ \frac{d(Ax_{2n}, Sx_{2n})^2 + d(Bx_{2n+1}, Tx_{2n+1})^2}{d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1})} \right\}$$

$$+ \beta d(Sx_n, Tx_{2n+1})$$

or

$$d(y_{2n+1}, y_{2n+2}) \leq \alpha \left\{ \frac{d(y_{2n+1}, y_{2n})^2 + d(y_{2n+2}, y_{2n+1})^2}{d(y_{2n+1}, y_{2n}) + d(y_{2n+2}, y_{2n+1})} \right\}$$

$$+ \beta d(y_{2n}, y_{2n+1}).$$

Setting $x = x_{2n}$, $y = x_{2n+1}$ in (1) gives

$$d(y_{2n+1}, y_{2n}) \leq \alpha \left\{ \frac{d(y_{2n+1}, y_{2n})^2 + d(y_{2n}, y_{2n-1})^2}{d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})} \right\} + \beta d(y_{2n}, y_{2n-1}).$$

Therefore, for all n,

$$d(y_n, y_{n+1}) \leq \alpha \left\{ \frac{d(y_{n-1}, y_n)^2 + d(y_n, y_{n+1})^2}{d(y_{n-1}, y_n) + d(y_n, y_{n+1})} \right\} + \beta d(y_{n-1}, y_n). \tag{3}$$

From the argument of Theorem 4 of [6], if $y_n = y_{n+1}$ for some n, then A, B, S and T have a common fixed point.

Suppose now that $y_n \neq y_{n+1}$ for all n. With $d_n := d(y_n, y_{n+1})$, it follows from (3) that

$$(1-\alpha)d_n^2 + (1-\beta)d_n d_{n-1} - (\alpha+\beta)d_{n-1}^2 \leq 0.$$

The corresponding quadratic equation has one positive solution k with $0 < k < 1$. Therefore the above inequality implies that $d_n \leq kd_{n-1}$. Then $d(y_n, y_{n+1}) = d_n \leq k^n d_0 = k^n d(y_0, y_1)$ and hence $\{y_n\}$ is Cauchy.

Now suppose that $S(X)$ is complete. Then the subsequence $\{y_{2n}\}$ has a limit in $S(X)$ since $\{y_{2n}\}$ is contained in $S(X)$. Let $u = \lim_{n\to\infty} y_{2n}$. Since $\lim_{n\to\infty} d_n = 0$, the subsequence $\{y_{2n-1}\}$ also converges to u. There exists a $v \in X$ such that $Sv = u$ since $u \in S(X)$. To prove that $Av = u$, let $r_1 = d(Av, u)$, and suppose that $r_1 > 0$. Setting $x = v$, $y = x_{2n-1}$ in (1) gives, since $y_n \neq y_{n+1}$ for each n,

$$\begin{aligned} d(Av, y_{2n}) &= d(Av, Bx_{2n-1}) \\ &\leq \alpha\left\{\frac{d(Av, Sv)^2 + d(Bx_{2n-1}, Tx_{2n-1})^2}{d(Av, Sv) + d(Bx_{2n-1}, Tx_{2n-1})}\right\} + \beta d(Sv, Tx_{2n-1}) \\ &\leq \alpha[d(Av, Sv) + d(y_{2n}, y_{2n-1})] + \beta d(u, y_{2n-1}). \end{aligned}$$

Taking the limit as $n \to \infty$ yields $r_1 = d(Av, u) \leq \alpha d(Av, Sv) = \alpha d(Av, u)$, a contradiction. Therefore $r_1 = 0$; i.e., $Av = u = Sv$.

Since $A(X) \subseteq T(X)$ and $Av = u$, $u \in T(X)$, there exists a $w \in X$ such that $Tw = u$. To prove that $Bw = u$, let $r_2 = d(Bw, u)$, and assume that $r_2 > 0$. Setting $x = x_{2n-2}$ and $y = w$ in (1) gives

$$\begin{aligned} d(y_{2n-1}, Bw) &= d(Ax_{2n-2}, Bw) \\ &\leq \alpha\left\{\frac{d(Ax_{2n-2}, Sx_{2n-2})^2 + d(Bw, Tw)^2}{d(Ax_{2n-2}, Sx_{2n-2}) + d(Bw, Tw)}\right\} + \beta d(Sx_{2n-2}, Tw) \\ &\leq \alpha[d(y_{2n-1}, y_{2n-2}) + d(Bw, u)] + \beta d(y_{2n-2}, u). \end{aligned}$$

Taking the limit as $n \to \infty$ yields $r_2 = d(u, Bw) \le \alpha d(Bw, u)$, a contradiction. Therefore $r_2 = 0$; i.e., $Bw = u = Tw$.

If we assume that $T(X)$ is complete, then, by the same argument, A and S have a coincidence point, and B and T also have a coincidence point.

If $B(X)$ is complete, then $u \in B(X) \subseteq S(X)$. Similarly if $A(X)$ is complete, then $u \in A(X) \subseteq T(X)$. Thus, by the previous cases, A and S have a coincidence point, and B and T also have a coincidence point.

Therefore $u = Sv = Av = Tw = Bw$.

Since A and S are weakly compatible, they commute at a coincidence point v. Thus $Au = ASv = SAv = Su$. Since B and T are weakly compatible, we get $Bu = BTw = TBw = Tu$. Since $d(Av, Sv) + d(Bu, Tu) = d(u, u) + d(Bu, Tu) = 0$, we obtain, from (2), that $d(Av, Bu) = d(u, Bu) = 0$. Hence $Bu = u$. Since $d(Au, Su) + d(Bu, Tu) = 0$, from (2), $d(Au, Bu) = d(Au, u) = 0$. Hence $Au = u$.

Therefore u is a common fixed point of A, B, S and T.

The uniqueness of u follows from (2).

Theorem 1 of Ahmad and Imdad [1] is a special case of Theorem 1, since weakly commuting implies compatibility. We have improved Theorem 4 of Jeong and Rhoades [6] by removing any assumption of continuity and by not assuming that X is complete.

Theorem 2. *Let A, B, S and T be selfmaps of a metric space (X, d) satisfying $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and, for each $x, y \in X$, either*

$$d(Ax, By) \le \alpha\left\{\frac{d(Ax, Sx)d(Sx, By) + d(By, Ty)d(Ty, Ax)}{d(Sx, By) + d(Ty, Ax)}\right\} + \beta d(Sx, Ty) \tag{4}$$

if $d(Sx, By) + d(Ty, Ax) \neq 0$, $\alpha, \beta > 0$, $\alpha + \beta < 1$, or

$$d(Ax, By) = 0 \text{ if } d(Sx, By) + d(Ty, Ax) = 0. \tag{5}$$

If one of $A(X)$, $B(X)$, $S(X)$ *or* $T(X)$ *is complete subspace of* X, *and* $\{A, S\}$, $\{B, T\}$ *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. From (4) we get, for all n,

$$d(y_n, y_{n+1}) \le \alpha\left\{\frac{d(y_{n-1}, y_n)d(y_{n-1}, y_{n+1})}{d(y_{n-1}, y_{n+1})}\right\} + \beta d(y_{n-1}, y_n). \tag{6}$$

If, as in the proof of Theorem 5 of [6], $y_n = y_{n+1}$ for some n, then A, B, S and T have a common fixed point.

If $y_n = y_{n+2}$ for some n, then $d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1}) = 0$, and, from (5), $d(y_{n+1}, y_{n+2}) = 0$; i.e., $y_{n+1} = y_{n+2}$, which we have already taken care of. Therefore, we may assume that $y_n \ne y_{n+2}$ for all n. From (6), one obtains $d(y_n, y_{n+1}) \le (\alpha + \beta) d(y_{n-1}, y_n), \alpha + \beta < 1$. Then $d(y_n, y_{n+1}) \le (\alpha + \beta)^n d(y_0, y_1)$ and hence $\{y_n\}$ is a Cauchy sequence.

Now suppose that $S(X)$ is complete. Then the subsequence $\{y_{2n}\}$ has a limit in $S(X)$ since $\{y_{2n}\}$ is contained in $S(X)$. Let $u = \lim_{n\to\infty} y_{2n}$. The subsequence $\{y_{2n-1}\}$ also converges to u. There exists a $v \in X$ such that $Sv = u$ since $u \in S(X)$. To prove that $Av = u$, let $r_1 = d(Av, u)$, and suppose that $r_1 > 0$. Setting $x = v, y = x_{2n-1}$ in (4) gives,

$$\begin{aligned}
d(Av, y_{2n}) &= d(Av, Bx_{2n-1}) \\
&\le \alpha\left\{\frac{d(Av, Sv)d(Sv, Bx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1})d(Tx_{2n-1}, Av)}{d(Sv, Bx_{2n-1}) + d(Tx_{2n-1}, Av)}\right\} \\
&\quad + \beta d(Sv, Tx_{2n-1}) \\
&= \alpha\left\{\frac{d(Av, u)d(u, y_{2n}) + d(y_{2n}, y_{2n-1})d(y_{2n-1}, Av)}{d(u, y_{2n}) + d(y_{2n-1}, Av)}\right\} \\
&\quad + \beta d(Sv, y_{2n-1}).
\end{aligned}$$

Since $r_1 > 0$, there exists an N_1 such that, for all $n > N_1$, and $r_3 :=$ $d(u, y_{2n}) + d(y_{2n-1}, Av) \ne 0$. Therefore, for all $n > N_1$,

$$d(Av,\ y_{2n}) \leq \alpha[d(Av,\ u) + d(y_{2n},\ y_{2n-1})] + \beta d(Sv,\ y_{2n-1}).$$

Taking the limit as $n \to \infty$ yields $r_1 = d(Av, u) \leq \alpha d(Av, u)$, a contradiction. Therefore $r_1 = 0$; i.e., $Av = u = Sv$.

Since $A(X) \subseteq T(X)$ and $Av = u$, $u \in T(X)$, there exists a $w \in X$ such that $Tw = u$. To prove that $Bw = u$, let $r_2 = d(Bw, u) > 0$. Setting $x = x_{2n-2}$ and $y = w$ in (4) gives

$$d(y_{2n-1},\ Bw)$$

$$= d(Ax_{2n-2},\ Bw)$$

$$\leq \alpha\left\{\frac{d(Ax_{2n-2},\ Sx_{2n-2})d(Sx_{2n-2},\ Bw) + d(Bw,\ Tw)d(Tw,\ Ax_{2n-2})}{d(Sx_{2n-2},\ Bw) + d(Tw,\ Ax_{2n-2})}\right\}$$

$$+ \beta d(Sx_{2n-2},\ Tw)$$

$$= \alpha\left\{\frac{d(y_{2n-1},\ y_{2n-2})d(y_{2n-2},\ Bw) + d(Bw,\ u)d(u,\ y_{2n-1})}{d(y_{2n-2},\ Bw) + d(u,\ y_{2n-1})}\right\} + \beta d(y_{2n-2},\ u).$$

Since $r_2 > 0$, there exists an N_2 such that $r_5 := d(y_{2n-2}, Bw) + d(u, y_{2n-1}) > 0$ for all $n > N_2$. Thus, for all $n > N_2$,

$$d(y_{2n-1},\ Bw) \leq \alpha[d(y_{2n-1},\ y_{2n-2}) + d(Bw,\ u)] + \beta d(y_{2n-2},\ u).$$

Taking the limit as $n \to \infty$ yields $0 < r_2 = d(u, Bw) \leq \alpha d(Bw, u)$, a contradiction. Therefore $r_2 = 0$; i.e., $Bw = u = Tw$.

If we assume that $T(X)$ is complete, then, by a similar argument, A and S have a coincidence point, and B and T also have a coincidence point.

If $B(X)$ is complete, then $u \in B(X) \subseteq S(X)$. Similarly if $A(X)$ is complete, then $u \in A(X) \subseteq T(X)$. Thus, by the previous cases, A and S have a coincidence point, and B and T also have a coincidence point.

Therefore $u = Sv = Av = Tw = Bw$.

Since A and S are weakly compatible, they commute at a coincidence point v. Thus $Au = ASv = SAv = Su$. Since B and T are weakly compatible, we get $Bu = BTw = TBw = Tu$.

Let $Bu \neq u$. Then $d(Sv, Bu) + d(Tu, Av) = 2d(u, Bu) \neq 0$. Setting $x = v$ and $y = u$ in (4) gives

$$\begin{aligned} d(u, Bu) &= d(Av, Bu) \\ &\leq \alpha\left\{\frac{d(Av, Sv)d(Sv, Bu) + d(Bu, Tu)d(Tu, Av)}{d(Sv, Bu) + d(Tu, Av)}\right\} + \beta d(Sv, Tu) \\ &= \beta d(u, Bu), \end{aligned}$$

a contradiction. Thus $Bu = u$.

Let $Au \neq u$. Then $d(Su, Bw) + d(Tw, Au) = 2d(u, Au) \neq 0$. Setting $x = u$ and $y = w$ in (4) gives

$$\begin{aligned} d(Au, u) &= d(Au, Bw) \\ &\leq \alpha\left\{\frac{d(Au, Su)d(Su, Bw) + d(Bw, Tw)d(Tw, Au)}{d(Su, Bw) + d(Tw, Au)}\right\} + \beta d(Su, Tw) \\ &= \beta d(Au, u), \end{aligned}$$

a contradiction. Thus $Au = u$.

Therefore u is a common fixed point of A, B, S and T.

The uniqueness of u follows from (4).

Theorem 2 of Imdad and Ahmad [4] is a special case of Theorem 2, since weakly commuting implies compatibility. We have improved Theorem 5 of Jeong and Rhoades [6] by removing the conditions of continuity and the completeness of X.

Theorem 3. *Let A, B, S and T be selfmaps of a metric space (X, d) satisfying $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and, for each $x, y \in X$, either*

$$d(Ax, By) \leq \frac{ad(Sx, Ax)d(Ty, By) + bd(Sx, By)d(Ty, Ax)}{d(Sx, Ax) + d(By, Ty)} + cd(Sx, Ty) \quad (7)$$

if $d(Sx, Ax) + d(By, Ty) \neq 0$, $a \geq 0$, $0 \leq c < 1$, $a + 2c < 2$, or

$$d(Ax,\ By) = 0 \ \ if \ \ d(Sx,\ Ax) + d(By,\ Ty) = 0.$$

If one of $A(X)$, $B(X)$, $S(X)$ *or* $T(X)$ *is complete subspace of* X, *and* $\{A,\ S\}$, $\{B,\ T\}$ *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

Proof. If $y_n = y_{n+1}$ for some n, then, using the same argument as in Theorem 6 of [6], A, B, S and T have a common fixed point.

From (7) we get, for all n,

$$d(y_{n+1},\ y_{n+2}) \le \frac{ad(y_n,\ y_{n+1})d(y_{n+1},\ y_{n+2})}{d(y_n,\ y_{n+1}) + d(y_{n+1},\ y_{n+2})} + cd(y_n,\ y_{n+1}). \tag{8}$$

Suppose that $y_n \neq y_{n+1}$ for each n. Then with $d_n := d(y_n,\ y_{n+1})$, it follows from (8) that

$$d_{n+1}^2 + (1 - a - c)d_n d_{n+1} - cd_n^2 \le 0.$$

The corresponding quadratic equation has one positive solution

$$k := \frac{-(1 - a - c) + \sqrt{(1 - a - c)^2 + 4c}}{2} \text{ with } 0 < k < 1.$$

Thus the above inequality implies that $d_{n+1} \le kd_n$. Therefore $\{y_n\}$ is Cauchy.

The remainder of the proof is similar to that of Theorem 1 and Theorem 2, and will therefore be omitted.

Theorem 1 of Ahmad et al. [2] is a special case of Theorem 3, and Theorem 3 is an improvement of Theorem 6 of Jeong and Rhoades [6].

There are two contractive forms for three maps. One is obtained by setting $T = S$ and the other is obtained by setting $B = A$. Also for three maps we can prove more general results.

For the situation in which $T = S$, set $x_0 \in X$ and define $\{x_n\}$ by

$$Ax_{2n} = Sx_{2n+1}, \quad Bx_{2n+1} = Sx_{2n+2},$$

$$y_{2n} := Sx_{2n}, \quad y_{2n+1} := Sx_{2n+1}.$$

Theorem 4. *Let A, B and S be three selfmaps of a metric space (X, d) such that, for each $x, y \in X$, either*

$$d(Ax, By) \leq \frac{ad(Sx, Ax)d(Sy, By) + bd(Sx, By)d(Sy, Ax)}{d(Sx, Ax) + d(Sy, By)} + c\left\{\frac{d(Sx, Ax)d(Sy, Ax) + d(Sy, By)d(Sx, By)}{d(Sx, By) + d(Sy, Ax)}\right\} \tag{9}$$

if $d(Sx, Ax) + d(Sy, By) \neq 0$ and $d(Sx, By) + d(Sy, Ax) \neq 0$, where a, b, c ≥ 0 with $a + 2c < 2$, or

$$d(Ax, By) = 0 \text{ if } d(Sx, Ax) + d(Sy, By) = 0 \text{ or}$$

$$d(Sx, By) + d(Sy, Ax) = 0. \tag{10}$$

If $A(X) \cup B(X) \subseteq S(X)$ and one of $A(X)$, $B(X)$ or $S(X)$ is complete subspace of X, and if $\{A, S\}$, $\{B, S\}$ are weakly compatible, then A, B and S have a unique common fixed point.

Proof. If $y_n = y_{n+1}$ for some n, then, as in the proof of Theorem 7 of [6], A, B and S have a common fixed point.

Suppose that $y_n \neq y_{n+1}$ for each n. From (9) we get, for all n,

$$d(y_n, y_{n+1}) \leq \frac{ad(y_{n-1}, y_n)d(y_n, y_{n+1})}{d(y_{n-1}, y_n) + d(y_n, y_{n+1})} + cd(y_n, y_{n+1}). \tag{11}$$

With $d_n := d(y_n, y_{n+1})$, it follows from (11) that

$$(1 - c)d_n^2 + (1 - a - c)d_{n-1}d_n \leq 0$$

or

$$d_n((1 - c)d_n + (1 - a - c)d_{n-1}) \leq 0.$$

Thus the above inequality implies that $d_n \leq kd_{n-1}$, where $k = \frac{a + c - 1}{1 - c}$ and $0 < k < 1$. Therefore $\{y_n\}$ is Cauchy.

Now suppose that $S(X)$ is complete. Then $\{y_n\}$ has a limit in $S(X)$. And $\{y_{2n}\}$, $\{y_{2n+1}\}$ have the same limit in $S(X)$. Let $u = \lim_{n\to\infty}\{y_n\} =$

$\lim_{n\to\infty}\{y_{2n}\} = \lim_{n\to\infty}\{y_{2n+1}\}$. Then there exists a $v \in X$ such that $Sv = u$ since $u \in S(X)$.

To prove that $Av = u$, let $r_1 = d(Av, u)$, and assume that $r_1 > 0$. Setting $x = v$, $y = x_{2n-1}$ in (9) gives

$$\begin{aligned}
&d(Av, y_{2n})\\
&= d(Av, Bx_{2n-1})\\
&\le \frac{ad(Sv, Av)d(Sx_{2n-1}, Bx_{2n-1}) + bd(Sv, Bx_{2n-1})d(Sx_{2n-1}, Av)}{d(Sv, Av) + d(Sx_{2n-1}, Bx_{2n-1})}\\
&\quad + c\left\{\frac{d(Sv, Av)d(Sx_{2n-1}, Av) + d(Sx_{2n-1}, Bx_{2n-1})d(Sv, Bx_{2n-1})}{d(Sv, Bx_{2n-1}) + d(Sx_{2n-1}, Av)}\right\}\\
&= \frac{ad(u, Av)d(y_{2n-1}, y_{2n}) + bd(u, y_{2n})d(y_{2n-1}, Av)}{d(u, Av) + d(y_{2n-1}, y_{2n})}\\
&\quad + c\left\{\frac{d(u, Av)d(y_{2n-1}, Av) + d(y_{2n-1}, y_{2n})d(u, y_{2n})}{d(u, y_{2n}) + d(y_{2n-1}, Av)}\right\}.
\end{aligned}$$

Taking the limit as $n \to \infty$ gives, $d(Av, u) \le cd(u, Av)$, a contradiction. Hence $Av = u$, and A and S have a coincidence point v.

To prove that $Bv = u$, let $r_2 = d(Bv, u)$, and assume that $r_2 > 0$. Setting $x = x_{2n}$, $y = v$ in (9) gives

$$\begin{aligned}
d(y_{2n+1}, Bv) &= d(Ax_{2n}, Bv)\\
&\le \frac{ad(Sx_{2n}, Ax_{2n})d(Sv, Bv) + bd(Sx_{2n}, Bv)d(Sv, Ax_{2n})}{d(Sx_{2n}, Ax_{2n}) + d(Sv, Bv)}\\
&\quad + c\left\{\frac{d(Sx_{2n}, Ax_{2n})d(Sv, Ax_{2n}) + d(Sv, Bv)d(Sx_{2n}, Bv)}{d(Sx_{2n}, Bv) + d(Sv, Ax_{2n})}\right\}\\
&= \frac{ad(y_{2n}, y_{2n+1})d(u, Bv) + bd(y_{2n}, Bv)d(u, y_{2n+1})}{d(y_{2n}, y_{2n+1}) + d(u, Bv)}\\
&\quad + c\left\{\frac{d(y_{2n}, y_{2n+1})d(u, y_{2n+1}) + d(u, Bv)d(y_{2n}, Bv)}{d(y_{2n}, Bv) + d(u, y_{2n+1})}\right\}.
\end{aligned}$$

Taking the limit as $n \to \infty$ yields, $d(u, Bv) \leq cd(u, Bv)$, a contradiction. Hence $Bv = u$. Thus B and S have a coincidence point v. Therefore $u = Sv = Av = Bv$.

If $A(X)$ is complete, then $u \in A(X) \subseteq S(X)$. Similarly if $B(X)$ is complete, then $u \in B(X) \subseteq S(X)$. Thus, by the previous cases, A and S and B have a coincidence point; i.e., $u = Sv = Av = Bv$.

Since A and S are weakly compatible, $Au = ASv = SAv = Su$. Since B and S are weakly compatible, $Bu = BSv = SBv = Su$. Thus $Au = Bu = Su$.

Since $d(Su, Au) + d(Sv, Bv) = 0$, it follows from (10) that $d(Au, Bv) = 0$; i.e., $Au = Bv$. But $Bv = u$. Thus $Au = u$. Therefore u is a common fixed point of A, B and S.

The uniqueness of u follows from (10).

Theorem 1 of Divicarro et al. [3] is a special case of Theorem 4. And Theorem 4 is an improvement of Theorem 7 of Jeong and Rhoades [6].

Theorem 5. *Let A, B and S be three selfmaps of a metric space (X, d) such that, for each $x, y \in X$, either*

$$d(Ax, By) \leq \alpha_1 \left\{ \frac{d(Sx, By) d(Sx, Sy)}{d(Sx, Sy) + d(Sy, By)} \right\} + \alpha_2 (d(Sx, Ax) + d(Sy, By))$$

$$+ \alpha_3 (d(Sx, By) + d(Sy, Ax)) + \alpha_4 d(Sx, Sy) \tag{12}$$

if $d(Sx, Sy) + d(Sy, By) \neq 0$, where $\alpha_i \geq 0$ with $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 < 1$, or

$$d(Ax, By) = 0 \text{ if } d(Sx, Sy) + d(Sy, By) = 0. \tag{13}$$

If $A(X) \cup B(X) \subseteq S(X)$ and one of $A(X)$, $B(X)$ or $S(X)$ is complete subspace of X, and if $\{A, S\}$, $\{B, S\}$ are weakly compatible, then A, B and S have a unique common fixed point.

Proof. If $y_n = y_{n+1}$ for some n, then, as in the proof of Theorem 9 of Jeong and Rhoades [6], it follows that A, B and S have a common fixed point.

Now we assume that $y_n \neq y_{n+1}$ for each n. From (12) we obtain, for all n,

$$d(y_n, y_{n+1}) \leq \alpha_1 \left\{ \frac{d(y_{n-1}, y_{n+1}) d(y_{n-1}, y_n)}{d(y_{n-1}, y_n) + d(y_n, y_{n+1})} \right\}$$

$$+ \alpha_2 (d(y_{n-1}, y_n) + d(y_n, y_{n+1}))$$

$$+ \alpha_3 d(y_{n-1}, y_{n+1}) + \alpha_4 d(y_{n-1}, y_n)$$

$$\leq \alpha_1 d(y_{n-1}, y_n) + (\alpha_2 + \alpha_3)(d(y_{n-1}, y_n) + d(y_n, y_{n+1}))$$

$$+ \alpha_4 d(y_{n-1}, y_n). \tag{14}$$

With $d_n := d(y_n, y_{n+1})$, it follows from (14) that $d_n \leq k d_{n-1}$, where $k = \dfrac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{1 - \alpha_2 - \alpha_3}$, and $0 < k < 1$. Therefore $\{y_n\}$ is Cauchy.

Suppose that $S(X)$ is complete. Then $\{y_n\}$ has a limit in $S(X)$. And $\{y_{2n}\}$, $\{y_{2n+1}\}$ have the same limit in $S(X)$. Let $u = \lim_{n \to \infty} \{y_n\} = \lim_{n \to \infty} \{y_{2n}\} = \lim_{n \to \infty} \{y_{2n+1}\}$. Then there exists a $v \in X$ such that $Sv = u$ since $u \in S(X)$.

To prove that $Bv = u$, let $r = d(Bv, u)$, and assume that $r > 0$. Setting $x = x_{2n}$, $y = v$ in (12) gives

$$d(y_{2n+1}, Bv) = d(Ax_{2n}, Bv)$$

$$\leq \alpha_1 \left\{ \frac{d(Sx_{2n}, Bv) d(Sx_{2n}, Sv)}{d(Sx_{2n}, Sv) + d(Sv, Bv)} \right\}$$

$$+ \alpha_2 (d(Sx_{2n}, Ax_{2n}) + d(Sv, Bv))$$

$$+ \alpha_3 (d(Sx_{2n}, Bv) + d(Sv, Ax_{2n})) + \alpha_4 d(Sx_{2n}, Sv)$$

$$= \alpha_1 \left\{ \frac{d(y_{2n}, Bv) d(y_{2n}, u)}{d(y_{2n}, u) + d(u, Bv)} \right\} + \alpha_2 (d(y_{2n}, y_{2n+1}) + d(u, Bv))$$

$$+ \alpha_3 (d(y_{2n}, Bv) + d(u, y_{2n+1})) + \alpha_4 d(y_{2n}, u).$$

Taking the limit as $n \to \infty$ gives $r \leq (\alpha_2 + \alpha_3) r$, a contradiction. Hence

$Bv = u$. Thus B and S have a coincidence point v. Since $d(Sv, Sv) + d(Sv, Bv) = 0$, by (13), $d(Av, Bv) = 0$. Hence $Av = Bv$. Thus $Av = Bv = Sv = u$. So A and S also have a coincidence point v. If $A(X)$ is complete, then $u \in A(X) \subseteq S(X)$. Similarly if $B(X)$ is complete, then $u \in B(X) \subseteq S(X)$. Thus, by the previous cases, A and S and B have a coincidence point v; i.e., $Av = Bv = Sv = u$. Since A and S are weakly compatible, $Au = ASv = SAv = Su$. Since B and S are weakly compatible, $Bu = BSv = SBv = Su$. Thus $Au = Bu = Su$.

Suppose that $Su \neq u$. Then, by (12),

$$\begin{aligned} d(Su, u) &= d(Au, Bv) \\ &\leq \alpha_1 \left\{ \frac{d(Su, Bv) d(Su, Sv)}{d(Su, Sv) + d(Sv, Bv)} \right\} + \alpha_2 (d(Su, Au) + d(Sv, Bv)) \\ &\quad + \alpha_3 (d(Su, Bv) + d(Sv, Au)) + \alpha_4 d(Su, Sv) \\ &= (\alpha_1 + 2\alpha_3 + \alpha_4) d(Su, u), \end{aligned}$$

a contradiction. Hence $Su = u$.

Therefore u is a common fixed point of A, B and S.

The uniqueness of u follows from (13).

The Theorem of Pande and Dubey [7] is a special case of Theorem 5. And Theorem 5 is an improvement of Theorem 9 of Jeong and Rhoades [6].

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