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ON THE NEGATIVE SOLUTION OF A CLASS OF *p*-LAPLACIAN BVP WITH NEUMANN-ROBIN CONDITIONS

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Abstract

In this paper, we consider the following Neumann-Robin boundary value problem

$$\begin{cases} -(\varphi_p(u'(x)))' = |u(x)|^p - \lambda, & x \in (0, 1), \\ u'(0) = 0, \\ u'(1) + \alpha u(1) = 0, \end{cases}$$

where $p > 1$, $\lambda > 0$ and $\alpha > 0$ are parameters. We study the negative solution of this problem with respect to a parameter ρ (i.e., $u(0) = \rho$). By using a quadrature method, the results are obtained.

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1. Introduction

In this paper, we consider the nonlinear two-point boundary value problem

$$-(\varphi_p(u'(x)))' = |u(x)|^p - \lambda, \quad x \in (0, 1), \tag{1}$$

$$u'(0) = 0, \tag{2}$$

$$u'(1) + \alpha u(1) = 0, \tag{3}$$

where $p > 1$, $\lambda > 0$ and $\alpha > 0$ are parameters, $p' = \frac{p}{p-1}$ is the conjugate exponent of p and $\varphi_p(s) := |s|^{p-2}s$ for all $s \neq 0$ and $\varphi_p(0) = 0$. Here $(\varphi_p(u'))'$ is the one dimensional p-Laplacian operator with $p > 1$ that was considered in several recent papers. We investigate the existence and nonexistence of negative solution of this problem with respect to a parameter ρ (that is, the value of the solution at zero, i.e., $u(0) = \rho$). Our approach is based on the quadrature method. In [3] problem (1) with Dirichlet boundary value conditions has been studied by Ammar-Khodja for the case Laplacian and in [1] the same problem with the same boundary value conditions has been extended by Addou to the general quasilinear case p-Laplacian with $p > 1$. In [4] Anuradha et al. considered a problem involving the one-dimensional Laplacian with Neumann-Robin boundary conditions by using a quadrature method. In [6] for semipositone problems, existence and multiplicity results have been established for the case $p = 2$ with Neumann boundary value conditions. In [2, 5] the existence of solutions has been studied with the p-Laplacian operator together with Robin condition.

The plan of the paper is as follows. In Section 2, we state a definition and our main result and finally in Section 3, we provide the proof of our main result that relies on quadrature method.

2. Notation and Main Result

We first define that u is a solution of problem (1)-(3) if

(i) $|u'|^{p-2}u'$ is absolutely continuous,

(ii) $-(\varphi_p(u'(x)))' = |u(x)|^p - \lambda$, a.e. on $(0, 1)$ and

$u'(0) = 0 = u'(1) + \alpha u(1)$.

Throughout this paper, we denote by ρ, the value of the solution at zero (i.e., $u(0) = \rho$). Now, we state the existence and nonexistence of negative solution to the problem (1)-(3) as described below:

Theorem 2.1. *Let* $\alpha > 0$, $p > 1$ *and* $\rho < 0$. *Then there exists a real number* $\rho^* < 0$ *such that*:

(a) *If* $1 < p \leq 2$, *then for any* $\rho \in (-\infty, \rho^*)$ *there exists a real number* $\lambda_\rho \in (|\rho|^p, \infty)$ *for which the problem* (1)-(3) *has a negative solution* u *at* $\lambda = \lambda_\rho$.

(b) *The problem* (1)-(3) *has no negative solution* u *with* $u(0) \in (\rho^*, 0)$ *at any* $\lambda \in (|\rho|^p, \infty)$.

3. Proof

Let u be a negative solution of problem (1)-(3) at λ with $u(0) = \rho$. Now multiplying (1) throughout by u' and integrating over $(0, x)$, we obtain $|u'(x)|^p = p'\left\{-\frac{u|u|^p}{p+1} + \lambda u + C\right\}$, where C is a constant. Applying conditions $u(0) = \rho$ and $u'(0) = 0$, we have $|u'(x)|^p = p'\{M(p, \rho, \lambda, u(s))\}$, where $M(p, \rho, \lambda, s) := \frac{\rho|\rho|^p}{p+1} - \frac{s|s|^p}{p+1} + \lambda(s - \rho)$. Since u has no interior critical point and $u' > 0$, hence

$$\{u'\}^p = p'\{M(p, \rho, \lambda, u)\}, \quad x \in (0, 1). \tag{4}$$

Now by integrating (4) on $(0, x)$, where $x \in [0, 1]$, we obtain

$$\int_\rho^{u(x)} \{M(p, \rho, \lambda, s)\}^{-1/p} ds = \{p'\}^{1/p} x, \quad x \in [0, 1]. \tag{5}$$

Remark 3.1. Let $p > 1$ and $\rho < 0$. Then:

(a) $M(p, \rho, \lambda, \rho) = 0$.

(b) If $\lambda > |\rho|^p$, then $M(p, \rho, \lambda, s) > 0$ for $s \to \rho^+$ and if $\lambda < |\rho|^p$, then $M(p, \rho, \lambda, s) < 0$ for $s \to \rho^+$.

Thus by Remark 3.1, for the existence of a negative solution u to problem (1)-(3) with $u(0) = \rho$ at λ, we must have $\lambda > |\rho|^p$.

Now, we provide a necessary condition for the existence of negative solution to problem (1)-(3) in the following Lemma 3.2:

Lemma 3.2. *The necessary condition for the existence of negative solution u to problem* (1)-(3) *at λ with $u(0) = \rho$ is the existence $m \in \Omega = (0, -\alpha\rho)$ such that satisfies the equations of the system*

$$G(m) = \{p'\}^{1/p} \text{ and } H(m) = \{p'\}^{1/p}, \tag{6}$$

where

$$G(m) = \int_{\rho}^{-\frac{m}{\alpha}} \{M(p, \rho, \lambda, s)\}^{-1/p} ds \text{ and } H(m) = m\left\{M\left(p, \rho, \lambda, -\frac{m}{\alpha}\right)\right\}^{-1/p}. \tag{7}$$

Proof of Lemma 3.2. By substituting $x = 1$ in (4) and (5), we have

$$u'(1)\{M(p, \rho, \lambda, u(1))\}^{-1/p} = \{p'\}^{1/p} = \int_{\rho}^{u(1)} \{M(p, \rho, \lambda, s)\}^{-1/p} ds.$$

By setting $u'(1) = m$, where $m > 0$, from (3), we have $u(1) = -\frac{m}{\alpha} \in (\rho, 0)$. Then

$$m\left\{M\left(p, \rho, \lambda, -\frac{m}{\alpha}\right)\right\}^{-1/p} = \{p'\}^{1/p} = \int_{\rho}^{-\frac{m}{\alpha}} \{M(p, \rho, \lambda, s)\}^{-1/p} ds.$$

Thus for such a solution to exist, there must exist an m such that $G(m) = \{p'\}^{1/p} = H(m)$. □

Now, we investigate whether such an m exists or not. For this mean, we first study the variations of functions $G(m)$ and $H(m)$.

Claim 3.3. The function $G(m)$ in (7) on $\Omega = (0, -\alpha\rho)$ is decreasing and $G(-\alpha\rho) = 0$ and $G(0) < \infty$ if and only if $p > 1$. Also there exists a real number $\rho^* \in (-\infty, 0)$ such that $G(m) < \{p'\}^{1/p}$ on Ω for any $\rho \in (\rho^*, 0)$ and $\lambda \in (|\rho|^p, \infty)$.

Proof of Claim 3.3. It is clear that $G'(m) = -\frac{1}{\alpha}\left\{M\left(p, \rho, \lambda, -\frac{m}{\alpha}\right)\right\}^{-1/p}$ < 0 on $m \in (0, -\alpha\rho)$ and $G(-\alpha\rho) = 0$. For showing, $G(0) < \infty$ if and only if $p > 1$, since $\{M(p, \rho, \lambda, s)\}^{-1/p} \approx \{\lambda - |\rho|^p\}^{-1/p}(s - \rho)^{-1/p}$ near ρ^+ and $\int_\rho^0 (s - \rho)^{-1/p} ds < \infty$ if and only if $p > 1$. Thus one can conclude that $G(0) < \infty$ if and only if $p > 1$. Easy computations show that

$$\{M(p, \rho, \lambda, s)\}^{-1/p} \le \{f(s)\}^{-1/p}, \text{ on } (\rho, 0), \tag{8}$$

where $f(s) = (\lambda - |\rho|^p)(s - \rho)$. Now, by integrating (8) on $(\rho, 0)$, one can conclude that

$$\begin{aligned} 0 < G(0) &= \int_\rho^0 \{M(p, \rho, \lambda, s)\}^{-1/p} ds \\ &\le \frac{1}{\{\lambda - |\rho|^p\}^{1/p}} \int_\rho^0 (s - \rho)^{-1/p} ds \\ &= \frac{p' |\rho|^{1/p'}}{\{\lambda - |\rho|^p\}^{1/p}} \to 0 \text{ as } \rho \to 0^-. \end{aligned} \tag{9}$$

Thus for any fixed $p > 1$, there exists a real number $\rho^* \in (-\infty, 0)$ such that $G(m) < G(0) \le \{p'\}^{1/p}$ on Ω for any $\rho \in (\rho^*, 0)$ and $\lambda \in (|\rho|^p, \infty)$. Hence the proof of Claim 3.3 is complete. □

Now, we investigate the variations of $H(m)$.

Claim 3.4 H is strictly increasing on $(0, -\alpha\rho)$ and $H(0) = 0$ and $H(-\alpha\rho) = \infty$.

Proof of Claim 3.4. Compute $H'(m) = h(m)\left\{M\left(p, \rho, \lambda, -\frac{m}{\alpha}\right)\right\}^{-\frac{p+1}{p}}$,

where $h(m) := \frac{\rho|\rho|^p}{p+1} + \frac{1}{p(p+1)}\left(-\frac{m}{\alpha}\right)\left|-\frac{m}{\alpha}\right|^p - \frac{\lambda m}{\alpha p'} - \lambda\rho$, on $(0, -\alpha\rho)$. Also, $h'(m) = -\frac{1}{\alpha p}\left(\left|-\frac{m}{\alpha}\right|^p + \lambda(p-1)\right) < 0$ on $(0, -\alpha\rho)$. Then $\min h(m) = h(-\alpha\rho) = \frac{\rho}{p}(|\rho|^p - \lambda) > 0$, and then $H'(m) > 0$ on $(0, -\alpha\rho)$. □

By Claims 3.3 and 3.4, it follows that there exists a unique $m^* = m^*(p, \alpha, \rho, \lambda) \in (0, -\alpha\rho)$ such that satisfies $H(m^*) = G(m^*)$. Now, it remains to find the values $\rho < 0$ and the values of $\lambda \in (|\rho|^p, \infty)$ such that one can obtain an $m^* = m^*(p, \alpha, \rho, \lambda)$ that satisfies $H(m^*) = \{p'\}^{1/p} = G(m^*)$. At first we consider the equation $H(m^*) = \{p'\}^{1/p}$ in the following Lemma 3.5:

Lemma 3.5. *Consider the equation in* $m \in \Omega = (0, -\alpha\rho)$

$$H(m) = \{p'\}^{1/p}, \tag{10}$$

where $\alpha > 0, \rho < 0, p > 1$ *and* $\lambda \in (|\rho|^p, \infty)$ *are real parameters. Then*:

(a) *For any fixed* $\alpha > 0, \rho < 0, p > 1$ *and* $\lambda \in (|\rho|^p, \infty)$, *the equation* (10) *admits a unique positive zero* $m^* = m^*(p, \alpha, \rho, \lambda)$.

(b) *The function* $\lambda \mapsto m^*(p, \alpha, \rho, \lambda)$ *is* C^1 *on* $(|\rho|^p, \infty)$ *and*

$$\frac{\partial m^*}{\partial \lambda} = \frac{-m^*\left\{\frac{m^*}{\alpha} + \rho\right\}}{p\{h(m)\}} > 0,$$

for all $\alpha > 0, \rho < 0, p > 1$ *and* $\lambda \in (|\rho|^p, \infty)$.

Proof of Lemma 3.5. (a) By similar argument in [1, Lemma 6], for any fixed $\alpha > 0, \rho < 0, p > 1$ and $\lambda \in (|\rho|^p, \infty)$, consider the function

$$m \mapsto F(p, \alpha, \rho, \lambda, m) := H(m) - \{p'\}^{1/p},$$

defined in Ω. By Claims 3.3 and 3.4 and the intermediate value theorem, one can conclude that the function (10) admits a unique positive zero $m^* = m^*(p, \alpha, \rho, \lambda) \in \Omega$.

(b) For any $\alpha > 0$, $p > 1$ and $\rho < 0$ given, consider the real-valued function

$$(\lambda, m) \mapsto F_+(\lambda, m) := H(m) - \{p'\}^{1/p},$$

defined on $\Omega_+ = (|\rho|^p, \infty) \times \Omega$. It is clear that $F_+ \in C^1(\Omega_+)$ and

$$\frac{\partial F_+}{\partial m} = h(m)\left\{M\left(p, \rho, \lambda, -\frac{m}{\alpha}\right)\right\}^{-\frac{p+1}{p}}, \text{ on } \Omega_+,$$

$$\frac{\partial F_+}{\partial \lambda} = \frac{m}{p}\left(\rho + \frac{m}{\alpha}\right)\left\{M\left(p, \rho, \lambda, -\frac{m}{\alpha}\right)\right\}^{-\frac{p+1}{p}} > 0, \text{ on } \Omega_+.$$

Hence by Claim 3.4, $\frac{\partial F_+}{\partial m}(\lambda, m) > 0$ on Ω_+, also $m^*(p, \alpha, \rho, \lambda) \in \Omega$ satisfies from its definition,

$$F_+(\lambda, m^*(p, \alpha, \rho, \lambda)) = 0. \tag{11}$$

So, one can make use of the implicit function theorem to show that the function $\lambda \mapsto m^*(p, \alpha, \rho, \lambda)$ is $C^1((|\rho|^p, \infty), \mathbf{R})$ and to obtain the expression for $\frac{\partial m^*}{\partial \lambda} = -\left(\frac{\partial F_+}{\partial \lambda}\right) \Big/ \left(\frac{\partial F_+}{\partial m^*}\right)$ given by (b). □

Now, for finding a solution to system (6), we solve the equation $G(m^*(p, \alpha, \rho, \lambda)) = \{p'\}^{1/p}$ in $\rho \in (-\infty, 0)$ and $\lambda \in (|\rho|^p, \infty)$ in the following Lemma 3.6.

Lemma 3.6. *Consider the equation* $G(m^*(p, \alpha, \rho, \lambda)) = \{p'\}^{1/p}$, *where* $\rho \in (-\infty, 0)$ *and* $\lambda \in (|\rho|^p, \infty)$, *then there exists a real number* $\rho^* < 0$ *such that*:

(a) *For any* $\rho \in (\rho^*, 0)$ *there exists no real number* $\lambda_\rho \in (|\rho|^p, \infty)$ *for which* $G(m^*(p, \alpha, \rho, \lambda_\rho)) = \{p'\}^{1/p}$.

(b) *If* $1 < p \leq 2$, *then there exists a unique real number* $\lambda_\rho \in (|\rho|^p, \infty)$ *for which for any* $\rho \in (-\infty, \rho^*)$ *there exists a unique real number* $\lambda_\rho \in (|\rho|^p, \infty)$ *for which* $G(m^*(p, \alpha, \rho, \lambda_\rho)) = \{p'\}^{1/p}$.

Proof of Lemma 3.6. (a) It is clearly follows from Claim 3.3.

(b) We first prove the following Claim 3.7:

Claim 3.7. 1. $\lim_{\lambda \to \infty} G(m^*(p, \alpha, \rho, \lambda)) = 0$.

2. $\dfrac{\partial G(m^*(p, \alpha, \rho, \lambda))}{\partial \lambda} < 0$.

3. $\lim_{\lambda \to (|\rho|^p)^+} G(m^*(p, \alpha, \rho, \lambda)) = \infty$ *if* $1 < p \leq 2$.

Proof of Claim 3.7. By (9), it is clear that $G(m^*(p, \alpha, \rho, \lambda)) \to 0$ as $\lambda \to \infty$. Also easy computations show that for any $\rho < 0$ and $\lambda \in (|\rho|^p, \infty)$,

$$\frac{\partial G(m^*)}{\partial \lambda} = \int_\rho^{-\frac{m^*}{\alpha}} \frac{(\rho - s)ds}{p\{M(p, \rho, \lambda, s)\}^{\frac{p+1}{p}}} - \frac{\frac{\partial m^*}{\partial \lambda}}{\alpha\left\{M\left(p, \rho, \lambda, -\frac{m^*}{\alpha}\right)\right\}^{\frac{1}{p}}}. \tag{12}$$

Hence, by Lemma 3.5, one can conclude that $\dfrac{\partial G(m^*)}{\partial \lambda} < 0$. By the monotone convergence theorem and the definition of $G(m^*(p, \alpha, \rho, \lambda))$ on $\Omega = (0, -\alpha\rho)$, one can conclude that

$$\lim_{\lambda \to (|\rho|^p)^+} G(m^*(p, \alpha, \rho, \lambda)) = \int_\rho^{-\frac{m^*}{\alpha}} \{M(p, \rho, |\rho|^p, s)\}^{-1/p} ds, \tag{13}$$

since $\{M(p, \rho, |\rho|^p, s)\}^{-1/p} \approx \left\{\frac{2}{p|\rho|^{p-1}}\right\}^{1/p}(s-\rho)^{-2/p}$ near ρ^+ and $\int_\rho^0 (s-\rho)^{-2/p} ds = \infty$ if $1 < p \leq 2$. Thus $\lim_{\lambda \to (|\rho|^p)^+} G(m^*(p, \alpha, \rho, \lambda)) = \infty$ if $1 < p \leq 2$. □

Now from the Claim 3.7 and the continuity of $G(m^*(p, \alpha, \rho, \lambda))$, one can conclude that for the case $1 < p \leq 2$, there exists a unique real number $\lambda_\rho \in (|\rho|^p, \infty)$ for which $G(m^*(p, \alpha, \rho, \lambda_\rho)) = \{p'\}^{1/p}$. Here the proof of Lemma 3.6 is complete. □

On the other hand by Lemma 3.5, $H(m^*(p, \alpha, \rho, \lambda_\rho)) = \{p'\}^{1/p}$, therefore in the case $1 < p \leq 2$ and $\rho < \rho^*$, by Lemma 3.6(b), we can conclude that the equations of the system (6) are simultaneously solvable in $m^* = m^*(p, \alpha, \rho, \lambda_\rho)$. Also by Lemmas 3.5 and 3.6(a), one can conclude that for any $\rho^* \in (\rho^*, 0)$ the equations of the system (6) are not simultaneously solvable in $m^* = m^*(p, \alpha, \rho, \lambda_\rho)$.

Hence from Lemma 3.2, one can prove Theorem 2.1.

Open problem. The study of the existence of negative solution to the problem (1)-(3) in the case $p > 2$ and $\rho < \rho^*$, is an open problem. Because one must compare the value of $G(m^*(p, \alpha, \rho, |\rho|^p))$ to the value of $\{p'\}^{1/p}$.

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