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# INFINITELY DIVISIBLE DISTRIBUTIONS OVER LOCALLY COMPACT NON-ARCHIMEDEAN FIELDS

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**Abstract**

The article is devoted to stochastic processes with values in finite-dimensional vector spaces over infinite locally compact fields of zero and positive characteristics with non-trivial non-Archimedean norms. Infinitely divisible distributions are investigated. Theorems about their characteristic functionals are proved. Particular cases are demonstrated as applications to non-Archimedean analogs of Gaussian and Poisson processes and their generalizations.

## 1. Introduction

It is well-known that infinitely divisible distributions play very important role in the theory of stochastic processes over fields of real and complex numbers [4, 9, 10, 11, 24, 28]. The main advantage of the results of Khinchin and Levy and their followers in this area was that they have taken into account the linear and bilinear functionals on linear spaces, that is, terms of the first and the second order, and have obtained characteristic functionals of infinitely divisible distributions and

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homogeneous stochastic processes with independent increments. That had permitted them to get interpretations of obtained results in particular cases of Gaussian and Poisson processes and their generalizations.

On the other hand, generalizations on locally compact Abelian groups were also considered [11, 22, 23]. Though in latter cases results were too general in comparison with those on linear spaces. For example, classes of measures corresponding to Gaussian processes on groups are much wider than those on linear spaces, that is, they do not take into account the field structure, because they operate with the additive group structure only (see [22] and Definition 6.1 and Theorem 6.1 [23]). Moreover, their approach on totally disconnected groups takes into account terms of the first order only and their terms of the second order copied from the real case vanish (see [22] and Section 3 particularly Example 3.4 [23]). But the terms of the second order are crucial for the Gaussian and Poisson processes.

Recently non-Archimedean analysis is being fast developed [26, 27, 30]. Below in this article to overcome difficulties met in previous works of other authors later results of non-Archimedean analysis were used.

Limit distributions on non-Archimedean local fields (of zero characteristic certainly) were studied in [13, 32] and in these articles results about representations of functionals of infinitely divisible distributions on locally compact Abelian groups from [22, 23] were used.

Nevertheless, infinitely divisible distributions over infinite fields with non-Archimedean non-trivial norms were not yet studied especially for fields of positive characteristics. This article is devoted to infinitely divisible distributions of stochastic processes in vector spaces over locally compact fields **K**. In this paper the new approach taking into account the field structure and terms of the first and the second order is developed (see Theorems 5, 7, 8, 10, 12 and 15). This permits to get non-Archimedean analogs of the Gaussian and Poisson processes, that is done below (see, for example, 16 and 17 in Section 2).

The locally compact non-Archimedean fields have non-Archimedean norms and their characteristics may be either zero such as for $\mathbf{Q_p}$ or for its finite algebraic extension, or positive characteristics $\mathrm{char}(\mathbf{K}) = p > 0$ such as $\mathbf{F_p}(\theta)$ of Laurent series over a finite field $\mathbf{F_p}$ with $p$ elements and an indeterminate $\theta$, where $p > 1$ is a prime number [31]. Multiplicative norms in such fields $\mathbf{K}$ satisfy stronger inequality, than the triangle inequality, $|x + y| \leq \max(|x|, |y|)$ for each $x, y \in \mathbf{K}$. Non-Archimedean fields are totally disconnected and balls in them are either non-intersecting or one of them is contained in another.

In works [2, 5]-[8, 12, 14] stochastic processes on spaces of functions with domains of definition in a non-Archimedean linear space and with ranges in the field of real $\mathbf{R}$ or complex numbers $\mathbf{C}$ were considered. Different variants of non-Archimedean stochastic processes are possible depending on a domain of definition, a range of values of functions, values of measures in either the real field or a non-Archimedean field [19, 21], a time parameter may be real or non-Archimedean and so on. That is, depending on considered problems different non-Archimedean variants arise.

Stochastic processes with values in non-Archimedean spaces appear while their studies for non-Archimedean Banach spaces, totally disconnected topological groups and manifolds [15-18, 20]. Also branching processes in graphs have very great importance [1, 10, 11]. For finite or infinite graphs with finite degrees of vertices it is possible to consider their embeddings into $p$-adic graphs, which can be embedded into locally compact fields. That is, a consideration of such processes reduces to processes with values in either the field $\mathbf{Q_p}$ of $p$-adic numbers or $\mathbf{F_p}(\theta)$.

In this article theorems about representations of characteristic functionals of infinitely divisible distributions with values in vector spaces over locally compact infinite fields with zero and positive characteristics with non-trivial non-Archimedean norms are formulated and proved. In these theorems characteristic functionals are obtained in the new form that was not got earlier. For this specific non-Archimedean classes of mappings are introduced. They are not linear or bilinear, but of the specific non-Archimedean form, because there is not any non-constant linear mapping from the field of real numbers into the field $\mathbf{Q_p}$ or $\mathbf{F_p}(\theta)$ or vice versa. They have permitted to overcome difficulties which were met earlier by another authors. For example, the cases of fields with positive characteristics are considered; not only the terms of the first, but also terms of the second order are taken into account below.

Special features of the non-Archimedean case are elucidated. Therefore, a part of definitions, formulations of theorems and their proofs are changed in comparison with the classical case. Some necessary facts from probability theory or non-Archimedean analysis are recalled (see, for example, 1-3 in Section 2), that to make reading easier. The results of this paper are complementary to those of preceding papers and develop them further (see also above). The main results of this paper, for example, Theorems 5, 7, 8 and 10, are obtained for the first time.

There is also an interesting interpretation of stochastic processes with values in $\mathbf{Q_p^n}$, for which a time parameter may be either real or $p$-adic. A random trajectory in $\mathbf{Q_p^n}$ may be continuous relative to the non-Archimedean norm in $\mathbf{Q_p}$, but its trajectory in $\mathbf{Q^n}$ relative to the usual metric induced by the real metric may be discontinuous. This gives new approach to spasmodic or jump or discontinuous stochastic processes with values in $\mathbf{Q^n}$, when the latter is considered as embedded into $\mathbf{R^n}$. On the other hand, stochastic processes with values in $\mathbf{F_p}(\theta)^n$ can naturally take into account cyclic stochastic processes in definite problems.

## 2. Infinitely Divisible Distributions

To avoid misunderstandings we first present our notations and definitions and recall the basic facts.

**1. Notations and definitions.** Let $(\Omega, \mathcal{A}, P)$-be a probability space, where $\Omega$ is a space of elementary events, $\mathcal{A}$ is a $\sigma$-algebra of events in $\Omega$, $P : \mathcal{A} \to [0, 1]$ is a probability.

Denote by $\xi$ a random vector (a random variable for $n = 1$) with values in $\mathbf{K^n}$ such that it has the probability distribution $P_\xi(A) = P(\{\omega \in \Omega : \xi(\omega) \in A\})$ for each $A \in \mathcal{B}(\mathbf{K^n})$, where $\xi : \Omega \to \mathbf{K^n}$, $\xi$ is $(\mathcal{A}, \mathcal{B}(\mathbf{K^n}))$-measurable. That is, $\xi^{-1}(\mathcal{B}(\mathbf{K^n})) \subset \mathcal{A}$, where $\mathbf{K}$ is a locally compact infinite field with a non-trivial non-Archimedean norm, $n \in \mathbf{N}$, $\mathbf{Q_p}$ is the field of $p$-adic numbers, $1 < p$ is a prime number.

Here $\mathbf{K}$ is either a finite algebraic extension of the field $\mathbf{Q_p}$ or the field $\mathbf{Q_p}$ itself for $\mathrm{char}(\mathbf{K}) = 0$, or $\mathbf{K} = \mathbf{F_p}(\theta)$ for $\mathrm{char}(\mathbf{K}) = p > 1$, $\mathcal{B}(\mathbf{K^n})$ is the $\sigma$-algebra of all Borel subsets in $\mathbf{K^n}$. Random vectors $\xi$ and $\eta$ with values in $\mathbf{K^n}$ are called *independent*, if $P(\{\xi \in A, \eta \in B\}) = P(\{\xi \in A\})P(\{\eta \in B\})$ for each $A, B \in \mathcal{B}(\mathbf{K^n})$.

A random vector (a random variable) $\xi$ is called *infinitely divisible*, if

(1) for each $m \in \mathbf{N}$ there exist random vectors (random variables) $\xi_1, \ldots, \xi_m$ such that $\xi = \xi_1 + \cdots + \xi_m$ and the probability distributions of $\xi_1, \ldots, \xi_m$ are the same.

If $\xi = \xi(t) = \xi(t, \omega)$ is a stochastic process with the real time, $t \in T$, $T \subset \mathbf{R}$, then it is called *infinitely divisible*, if Condition (1) is satisfied for each $t \in T$.

Introduce the notation $B(X, x, R) := \{y \in X : \rho(x, y) \le R\}$ for the ball in a metric space $(X, \rho)$ with a metric $\rho$, $0 < R < \infty$, $\xi_j(t)$ are stochastic processes, $j = 1, \ldots, m$.

**2. Lemma.** *If $\xi$ and $\eta$ are two independent random vectors with values in $\mathbf{K^n}$ with probability distributions $P_\xi$ and $P_\eta$, then $\xi + \eta$ has the probability distribution $P_{\xi+\eta}(A) = \int_{\mathbf{K^n}} P_\xi(A - dy)P_\eta(dy)$ for each $A \in \mathcal{B}(\mathbf{K^n})$.*

**Proof.** Since $\xi$ and $\eta$ are independent, $P(\{\omega \in \Omega : \xi(\omega) \in C,\ \eta(\omega) \in B\}) = P(\{\omega \in \Omega : \xi(\omega) \in C\})P(\{\omega \in \Omega : \eta(\omega) \in B\})$ for each $C,\ B \in \mathcal{B}(\mathbf{K^n})$.

Therefore, $P(\{\xi + \eta \in A\}) = P(\{\xi \in A - y,\ \eta = y,\ y \in \mathbf{K^n}\})$ for each $A \in \mathcal{B}(\mathbf{K^n})$, consequently, $P_{\xi+\eta}(A) = \int_{\mathbf{K^n}} P_\xi(A - dy)P_\eta(dy)$.

This means that $P_{\xi+\eta} = P_\xi * P_\eta$ is the convolution of measures $P_\xi$ and $P_\eta$.

**3. Corollary.** *If $\xi$ is an infinitely divisible random vector, then $P_\xi = P_{\xi_1}^{*m}$ for each $m \in \mathbf{N}$, where $P_\eta^{*m}$ denotes the m-fold convolution $P_\eta$ with itself.*

**Proof.** In view of Lemma 2 and Definition 1 $P_\xi = P_{\xi_1} * P_{\xi_2+\dots+\xi_m} = \dots = P_{\xi_1} * P_{\xi_2} * \dots * P_{\xi_m}$.

On the other hand, $\xi_1, \dots, \xi_m$ have the same probability distributions, hence $P_{\xi_1} * P_{\xi_2} * \dots * P_{\xi_m} = P_{\xi_1}^{*m}$.

**4. Notes and definitions.** Corollary 3 means that the equality $P_\xi = P_{\xi_1}^{*m}$ implies the relation:

$$P_\xi(A) = \int_{\mathbf{K^n}} \cdots \int_{\mathbf{K^n}} P_{\xi_1}(A - dy_2)P_{\xi_2}(dy_2 - dy_3)\cdots P_{\xi_{m-1}}(dy_{m-1} - dy_m)P_{\xi_m}(dy_m),$$

where $A \in \mathcal{B}(\mathbf{K^n})$. In the case of $\mathrm{char}(\mathbf{K}) = p > 1$ Corollary 3 means, that for $m = kp$, where $k \in \mathbf{N}$, if $P_\xi(\{0\}) = 0$, then $P_{\xi_1}(\{y\}) = 0$ for each singleton $y \in \mathbf{K^n}$, since $P(\xi = 0) \ge P(\xi_1 = \xi_2 = \dots = \xi_m) \ge P_{\xi_1}(\{y\})^m$. It is the restriction on the atomic property of $P_\xi$ and $P_{\xi_1}$.

For $p$-adic numbers $x = \sum_{k=N}^{\infty} x_k p^k$, where $x_k \in \{0, 1, ..., p-1\}$, $N \in \mathbf{Z}$, $N = N(x)$, $x_N \neq 0$, $x_j = 0$ for each $j < N$, put as usually $\mathrm{ord}_{\mathbf{Q_p}}(x) = N$ for the order of $x$, thus its norm is $|x|_{\mathbf{Q_p}} = p^{-N}$. Define the function $[x]_{\mathbf{Q_p}} := \sum_{k=N}^{-1} x_k p^k$ for $N < 0$, $[x]_{\mathbf{Q_p}} = 0$ for $N \geq 0$ on $\mathbf{Q_p}$. Therefore, the function $[x]_{\mathbf{Q_p}}$ on $\mathbf{Q_p}$ is considered with values in the segment $[0, 1] \subset \mathbf{R}$.

For the field $\mathbf{F_p}(\theta)$ put $|x|_{\mathbf{F_p}(\theta)} = p^{-N}$, where $N = \mathrm{ord}_{\mathbf{F_p}(\theta)}(x) \in \mathbf{Z}$, $x = \sum_{k=N}^{\infty} x_j \theta^j$, $x_j \in \mathbf{F_p}$ for each $j$, $x_N \neq 0$, $x_j = 0$ for each $j < N$. Then we define the mapping $[x]_{\mathbf{F_p}(\theta)} = x_{-1}/p$, where we consider elements of $\mathbf{F_p} = \{0, 1, ..., p-1\}$ embedded into $\mathbf{R}$, hence $[x]_{\mathbf{F_p}(\theta)}$ takes values in $\mathbf{R}$, where $1/p \in \mathbf{R}$, $x_{-1} = 0$ when $N = N(x) \geq 0$.

Consider a local field $\mathbf{K}$ as the vector space over the field $\mathbf{Q_p}$, then it is isomorphic with $\mathbf{Q_p^b}$ for some $b \in \mathbf{N}$, since $\mathbf{K}$ is a finite algebraic extension of the field $\mathbf{Q_p}$. In the case of $\mathbf{K} = \mathbf{F_p}(\theta)$ we take $b = 1$. Put

(i) $\mathbf{F} := \mathbf{Q_p}$ for $\mathrm{char}(\mathbf{K}) = 0$ with $\mathbf{K} \supset \mathbf{Q_p}$, while

(ii) $\mathbf{F} := \mathbf{F_p}(\theta)$ for $\mathrm{char}(\mathbf{K}) = p > 1$ with $\mathbf{K} = \mathbf{F_p}(\theta)$.

Let $(x, y) := (x, y)_{\mathbf{F}} := \sum_{j=1}^{b} x_j y_j$ for $x, y \in \mathbf{F^b}$, $x = (x_1, ..., x_b)$, $x_j \in \mathbf{F}$; $(x, y)_{\mathbf{K}} := \sum_{j=1}^{n} x_j y_j$ for $x, y \in \mathbf{K^n}$, $x = (x_1, ..., x_n)$, $x_j \in \mathbf{K}$.

Define the mapping $\langle q \rangle_{\mathbf{F}} := 2\pi[(e, q)]_{\mathbf{F}}$ for each $q \in \mathbf{K}$, which is considered in $(e, q)$ as the element from $\mathbf{F^b}$, $\langle q \rangle_{\mathbf{F}} : \mathbf{K} \to \mathbf{R}$, where $e := (1, ..., 1) \in \mathbf{F^b}$, particularly $e = 1$ for $b = 1$, that is, either in (i) $\mathbf{K} = \mathbf{Q_p}$ or in the case (ii) for $\mathbf{K} = \mathbf{F_p}(\theta)$. For the additive group $\mathbf{K^n}$ then there exists the character $\chi_s(z) := \exp(i\langle (s, z)_{\mathbf{K}} \rangle_{\mathbf{F}})$ with values in the field of complex numbers $\mathbf{C}$ for each value of the parameter $s \in \mathbf{K^n}$, since $s_j(z_j + v_j) = s_j z_j + s_j v_j$ for each $s_j, z_j, v_j \in \mathbf{K}$ and $(s, z + v)_{\mathbf{K}} = (s, z)_{\mathbf{K}} + (s, v)_{\mathbf{K}}$, $[x + y]_{\mathbf{F}} - [x]_{\mathbf{F}} - [y]_{\mathbf{F}} \in B(\mathbf{F}, 0, 1)$ for every $x, y \in \mathbf{F}$, while $[x]_{\mathbf{F}} = 0$ for each $x \in B(\mathbf{F}, 0, 1)$, where $i = (-1)^{1/2} \in \mathbf{C}$. In particular, $\chi_0(z) = 1$ for each $z \in \mathbf{K^n}$ for $s = 0$. The character is non-trivial for $s \neq 0$. At the same time $\chi_s(z) = \prod_{j=1}^n \chi_{s_j}(z_j)$, where $\chi_{s_j}(z_j)$ are characters of $\mathbf{K}$ as the additive group.

For a $\sigma$-additive measure $\mu : \mathcal{B}(\mathbf{K^n}) \to \mathbf{C}$ of a bounded variation the characteristic functional $\hat{\mu}$ is given by the formula: $\hat{\mu}(s) := \int_{\mathbf{K^n}} \chi_s(z)\mu(dz)$, where $s \in \mathbf{K^n}$ is the corresponding continuous $\mathbf{K}$-linear functional on $\mathbf{K^n}$ denoted by the same $s$.

In general the *characteristic functional* of the measure $\mu$ is defined in the space $C^0(\mathbf{K^n}, \mathbf{K})$ of continuous functions $f : \mathbf{K^n} \to \mathbf{K}$,

$$\hat{\mu}(f) := \int_{\mathbf{K^n}} \chi_1(f(z))\mu(dz), \quad \text{where } 1 \in \mathbf{K}.$$

Let $\mu$ be a $\sigma$-additive finite non-negative measure on $\mathcal{B}(\mathbf{K^n})$, $\mu(\mathbf{K^n}) < \infty$. Consider the class $\mathcal{C}_1 = \mathcal{C}_1(\mathbf{K})$ of continuous functions $A = A_\mu : \mathbf{K^n} \to \mathbf{R}$, satisfying Conditions (F1-F4).

(F1) $A(y+z) = A(y) + A(z) + 2\pi\int_{\mathbf{K^n}} f_1(y, z; x)\mu(dx)$ for each $y, z \in \mathbf{K^n}$,

(F2) $A(\beta y) = [\beta]_{\mathbf{F}} A(y) + 2\pi\int_{\mathbf{K^n}} f_2(\beta, (e, (y, x)_{\mathbf{K}})_{\mathbf{F}})\mu(dx)$ for each $y \in \mathbf{K^n}$, $\beta \in \mathbf{F}$, where either

(F3) if $\mathbf{F} = \mathbf{Q_p}$ for $\mathrm{char}(\mathbf{K}) = 0$, then $f_1 : (\mathbf{K^n})^3 \to \mathbf{Z}$ and $f_2 : \mathbf{Q_p^2} \to \mathbf{R}$

are locally constant continuous bounded functions, $f_1(y, z; x) \in \mathbf{Z}$ and $f_2(\alpha, \beta) p^{-N(\alpha,\beta)} \in \mathbf{Z}$ for $N(\alpha, \beta) < 0$ take only integer values, $N(\alpha, \beta) := \min(\mathrm{ord}_{\mathbf{Q_p}}(\alpha), \mathrm{ord}_{\mathbf{Q_p}}(\beta))$; or

(F4) if $\mathbf{F} = \mathbf{F_p}(\theta)$ for $\mathrm{char}(\mathbf{K}) = p > 0$, then $f_1 : (\mathbf{K^n})^3 \to \mathbf{R}$ and $f_2 : \mathbf{F}^2 \to \mathbf{R}$ are locally constant continuous bounded functions, $pf_1(y, z; x) \in \mathbf{Z}$ and $p^2 f_2(\alpha, \beta) \in \mathbf{Z}$ for $N(\alpha, \beta) < 0$ take only integer values, $N(\alpha, \beta) := \min(\mathrm{ord}_{\mathbf{F_p}(\theta)}(\alpha), \mathrm{ord}_{\mathbf{F_p}(\theta)}(\beta))$. While $f_1(y, z; x) = 0$ for $\max(|yx|_{\mathbf{K}}, |zx|_{\mathbf{K}}) \leq 1$, and $f_2(\alpha, \beta) = 0$ for $\max(|\alpha|_{\mathbf{F}}, |\beta|_{\mathbf{F}}) \leq 1$ in (F3, F4).

Denote by $\mathcal{C}_2 = \mathcal{C}_2(\mathbf{K})$ the class of continuous functions $B = B_\mu : (\mathbf{K^n})^2 \to \mathbf{R}$, satisfying Conditions (B1-B3):

(B1) $B(y, z) = B(z, y)$ for each $y, z \in \mathbf{K^n}$, where $B(y, y)$ is non-negative,

(B2) $B(q + y, z) = B(q, z) + B(y, z) + 2\pi \int_{\mathbf{K^n}} f_1(q, y; x) \langle (z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \mu(dx)$

for each $q, y, z \in \mathbf{K^n}$,

(B3) $B(\beta y, z) = [\beta]_{\mathbf{F}} B(y, z) + 2\pi \int_{\mathbf{K^n}} f_2(\beta, (e, (y, x)_{\mathbf{K}})_{\mathbf{F}}) \langle (z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \mu(dx)$

where $f_1$ and $f_2$ satisfy Condition either (F3) or (F4) depending on the characteristic $\mathrm{char}(\mathbf{K})$.

For $y = z$ we shall also write for short $B(y) := B(y, y)$.

**4.1. Lemma.** *If* $\chi_s(x) : \mathbf{F^n} \to \mathbf{C}$ *is a character of the additive group of* $\mathbf{F^n}$ *as in* 4 *of Section* 2, $\mu : \mathcal{B}(\mathbf{F^n}) \to [0, \infty]$ *is the Haar measure such that* $\mu(B(\mathbf{F^n}, 0, 1)) = 1$. *Then* $\int_{B(\mathbf{F^n}, 0, p^k)} \chi_s(x) \mu(dx) = J(s, k)$, *where* $J(s, k) = p^{kn}$ *for* $|s| \leq p^{-k}$, *while* $J(s, k) = 0$ *for* $|s| \geq p^{1-k}$.

**Proof.** The Haar measure $\mu$ on $\mathcal{B}(\mathbf{F^n})$ is the product of the Haar measures $\mu_1$ on $\mathcal{B}(\mathbf{F})$, $\mu(dx) = \otimes_{j=1}^{n}\mu_j(dx_j)$, $\mu_j = \mu_1$. Therefore, $\int_{B(\mathbf{F^n}, 0, p^k)} \chi_s(x)\mu(dx) = \prod_{j=1}^{n} \int_{B(\mathbf{F}, 0, p^k)} \chi_{s_j}(x_j)\mu_j(dx_j)$, where $\chi_j = \chi_1$, $\chi_{s_j}(x_j)$ is the character of $\mathbf{F}$.

Consider $n = 1$. Then

$$K := \int_{B(\mathbf{F^n}, 0, p^k)} \chi_s(x)\mu(dx) = \int_{B(\mathbf{F}, y, p^k)} \chi_s(x - y)\mu(dx)$$

for each $y \in B(\mathbf{F}, 0, p^k)$. Thus $K = \chi_s(-y)\int_{B(\mathbf{F}, 0, p^k)} \chi_s(x)\mu(dx)$, since $B(\mathbf{F}, 0, p^k) = B(\mathbf{F}, y, p^k)$ for each $y \in B(\mathbf{F}, 0, p^k)$, while $\mu(A - y) = \mu(A)$ for each $A \in \mathcal{B}(\mathbf{F})$. Take $|s|_{\mathbf{F}} \geq p^{-k+1}$ and $|y|_{\mathbf{F}} = p^k$ such that $[sy]_{\mathbf{F}} \neq 0$ is nonzero. Hence $K(1 - \chi_s(-y)) = 0$, but $\chi_s(-y) \neq 1$, consequently, $K = 0$.

On the other hand, if $|sx|_{\mathbf{F}} \leq 1$, then $\chi_s(x) = 1$ and inevitably $\int_{B(\mathbf{F}, 0, p^k)} \chi_s(x)\mu(dx) = p^k$, when $|s|_{\mathbf{F}} \leq p^k$ (see for comparison the case $\mathbf{F} = \mathbf{Q_p}$ in [30, Example 6, p. 62]).

**5. Theorem.** *Let* $\{\psi(v, y) : v \in V\}$ *be a family of characteristic functionals of* $\sigma$*-additive non-negative bounded measures on* $\mathcal{B}(\mathbf{K^n})$, *where* $V$ *is a monotonically decreasing sequence of positive numbers converging to zero. Suppose that there exists a limit* $g(y) = \lim_{v\downarrow 0}(\psi(v, y) - 1)/v$ *uniformly in each ball* $B(\mathbf{K^n}, 0, R)$ *for each given* $0 < R < \infty$. *Then in* $\{\mathbf{K^n}, \mathcal{B}(\mathbf{K^n})\}$ *there exists a* $\sigma$*-additive non-negative bounded measure* $\nu$, *functions* $A(y)$ *and* $B(y)$, *belonging to classes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$, *respectively, such that*

(i) $g(y) = iA(y) - B(y)/2 + \int_{\mathbf{K^n}} (\exp(i\langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}}) - 1 - i\langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}})(1 + |x|^2)^{-1} + \langle(y, x)_{\mathbf{K}}\rangle^2_{\mathbf{F}}((1 + |x|^2)^{-1}/2)[(1 + |x|^2)/|x|^2]\nu(dx)$, $\nu \geq 0$, $\nu(\{0\}) = 0$.

**Proof.** Let $\mu_v$ be a measure corresponding to the characteristic functional $\psi(v, y)$. Put $\lambda_v(A) := v^{-1}\int_A |z|^2/[1 + |z|^2]\mu_v(dz)$ for each $A \in \mathcal{B}(\mathbf{K^n})$, where $|z| := \max_{1 \leq j \leq n}|z_j|_{\mathbf{K}}$, $z = (z_1, ..., z_n) \in \mathbf{K^n}$, $z_j \in \mathbf{K}$ for every $j = 1, ..., n$. We prove a weak compactness of the family of measures $\{\lambda_v : v \in V\}$. That is, we need to prove that

(i) there exists $L = \text{const} > 0$ such that $\sup_{v \in V} \lambda_v(\mathbf{K^n}) \leq L$;

(ii) $\lim_{R \to \infty} \overline{\lim}_{v \downarrow 0} \lambda_v(\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, R)) = 0$.

The topologically dual space $\mathbf{K^n}'$ of all continuous $\mathbf{K}$-linear functionals on $\mathbf{K^n}$ is $\mathbf{K}$-linearly and topologically isomorphic with $\mathbf{K^n}$, since $n \in \mathbf{N}$. Since $\mathbf{K}$ is the locally compact field, it is spherically complete (see [26, Theorems 3.15, 5.36 and 5.39]). Since $\mathbf{K^n}$ as the linear space over $\mathbf{F}$ is isomorphic with $\mathbf{F^{bn}}$, it is sufficient to verify a weak compactness over the field $\mathbf{F}$, where either $\mathbf{F} = \mathbf{Q_p}$ for $\text{char}(\mathbf{K}) = 0$ with $\mathbf{K} \supset \mathbf{Q_p}$ and $b \in \mathbf{N}$, or $\mathbf{F} = \mathbf{F_p}(\theta)$ for $\text{char}(\mathbf{K}) = p > 0$ with $\mathbf{K} = \mathbf{F_p}(\theta)$ and $b = 1$. Indeed, apply the non-Archimedean variant of the Minlos-Sazonov theorem, due to which there exists the bijective correspondence between characteristic functionals and measures [20], where characteristic functionals are weakly continuous (see also Section IV.1.2 and Theorem IV.2.2 about the Minlos-Sazonov theorem on Hausdorff completely regular (Tychonoff) spaces [29]). They are positive definite on $(\mathbf{K^n})'$ or $C^0(\mathbf{K^n}, \mathbf{K})$, when $\mu$ is non-negative; $\hat{\mu}(0) = 1$ for $\mu(\mathbf{K^n}) = 1$. In the considered case $\mathbf{K^n}$ is a finite dimensional Banach space over $\mathbf{K}$. Since the multiplication in $\mathbf{K}$ is continuous, over $\mathbf{F}$ this gives the continuous mapping $f_0 : (\mathbf{F^b})^2 \to \mathbf{F^b}$. The composition of $f_0$ with all possible $\mathbf{K}$-linear continuous functionals $s : \mathbf{K^n} \to \mathbf{K}$ separates points in $\mathbf{K^n}$.

Let $|x| \le R_1$, where $0 < R_1 < \infty$ is an arbitrarily given number. Due to conditions of this theorem for each $\delta > 0$ there exists $v_0 = v_0(R_1, \delta) > 0$ such that for each $\varepsilon > 0$ there is satisfied the inequality:

(1) $-\mathrm{Reg}(y) + \delta \ge \int_{B(\mathbf{F^{bn}}, 0, \varepsilon)} [1 - \cos\langle (y, x)_{\mathbf{F}} \rangle_{\mathbf{F}}] |x|^2 \lambda_v(dx)$ for each $0 < v \le v_0$, since $e^{i\alpha} = \cos(\alpha) + i \sin(\alpha)$, $-\mathrm{Re}(e^{i\alpha} - 1) = 1 - \cos(\alpha)$ for each $\alpha \in \mathbf{R}$, while $1 + |x|^2 \ge 1$ and $[1 + |x|^2] |x|^{-2} \ge |x|^{-2}$.

If $\varepsilon > 1$ and $x \in \mathbf{F^{bn}} \setminus B(\mathbf{F^{bn}}, 0, \varepsilon)$, then from $|x|_{\mathbf{F}} > \varepsilon$ it follows $[1 + |x|^2] |x|^{-2} = 1 + |x|^{-2} \ge 1$ and then for each $\delta > 0$ there exists $v_0 > 0$ such that for each $\varepsilon > 1$ and each $0 < v \le v_0$ there is satisfied the inequality:

(2) $-\mathrm{Reg}(y) + \delta \ge \int_{\mathbf{F^{bn}} \setminus B(\mathbf{F^{bn}}, 0, \varepsilon)} (1 - \cos\langle (y, x) \rangle_{\mathbf{F}}) \lambda_v(dx)$.

Integrate these inequalities by $y \in B(\mathbf{F^{bn}}, 0, r)$ and divide on the volume (measure) $\mu(B(\mathbf{F^{bn}}, 0, r))$, where $\mu$ is the non-negative Haar measure on $\mathbf{F^{bn}}$ such that $\mu(B(\mathbf{F^{bn}}, 0, 1)) = 1$, $\mu(B(\mathbf{F^{bn}}, 0, r)) = r^{bn}$ for each $r = p^k$ with $k \in \mathbf{Z}$ [3, 31]. Then from (1) it follows:

(3) $-r^{-bn} \int_{B(\mathbf{F^{bn}}, 0, r)} \mathrm{Reg}(y) \mu(dy) + \delta \ge \int_{B(\mathbf{F^{bn}}, 0, r)} \Big( \int_{B(\mathbf{F^{bn}}, 0, \varepsilon)} |x|_{\mathbf{F}}^{-2} (1 - \cos\langle (y, x) \rangle_{\mathbf{F}}) \lambda_v(dx) \Big) \mu(dy) r^{-bn}$. From (2) we get:

(4) $-r^{-bn} \int_{B(\mathbf{F^{bn}}, 0, r)} \mathrm{Reg}(y) \mu(dy) + \delta \ge \int_{B(\mathbf{F^{bn}}, 0, r)} \Big( \int_{\mathbf{F^{bn}} \setminus B(\mathbf{F^{bn}}, 0, \varepsilon)} (1 -$

$\cos\langle (y,\, x)\rangle_{\mathbf{F}})\lambda_v(dx)\Big)\mu(dy)r^{-bn}$. On the other hand, $\cos(\langle (y,\, x)\rangle_{\mathbf{F}}) = \cos\Big(\sum_{j=1}^{bn} \langle x_j y_j\rangle_{\mathbf{F}}\Big)$, since $(y,\, x) = \sum_{j=1}^{bn} y_j x_j$, also $\langle a + b\rangle_{\mathbf{F}} = \langle a\rangle_{\mathbf{F}} + \langle b\rangle_{\mathbf{F}} + 2w\pi$ for each $a, b \in \mathbf{F}$, where $w$ is an integer number, $w = w(a, b) \in \mathbf{Z}$. For the characters integrals are known due to Lemma 4.1: $\int_{B(\mathbf{F^{bn}}, 0, p^k)} \chi_s(x)\mu(dx) = \prod_{j=1}^{bn} \int_{B(\mathbf{F}, 0, p^k)} \chi_{s_j}(x_j)\mu_j(dx_j) = J(s, k)$, where $J(s, k) = p^{kbn}$ for $|s|_{\mathbf{F}} \le p^{-k}$, $J(s, k) = 0$ for $|s|_{\mathbf{F}} \ge p^{-k+1}$. Since $(y, x) = (x, y)$ and $\cos(\alpha) = \mathrm{Re}(e^{i\alpha})$ for each $\alpha \in \mathbf{R}$, $\int_{B(\mathbf{F^{bn}}, 0, p^k)} \cos\langle (y,\, x)\rangle_{\mathbf{F}}\mu(dy) = J(x, k)$, since $J(x, k) \in \mathbf{R}$. Take in (3, 4) $r = p^k$, then

$$(5)\quad -p^{-kbn}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy) + \delta$$

$$\ge \Big(\int_{B(\mathbf{F^{bn}}, 0, \varepsilon)} |x|_{\mathbf{F}}^{-2}(1 - J(x,\, k)p^{-kbn})\lambda_v(dx)\Big),$$

$$(6)\quad -p^{-kbn}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy) + \delta$$

$$\ge \int_{\mathbf{F^{bn}}\setminus B(\mathbf{F^{bn}}, 0, \varepsilon)} (1 - J(x,\, k)p^{-kbn})\lambda_v(dx).$$

Since $J(x,\, k)p^{-kbn} = 1$ for $|x|_{\mathbf{F}} \le p^{-k}$, while $J(x,\, k)p^{-kbn} = 0$ for $|x|_{\mathbf{F}} \ge p^{-k+1}$, then for $\varepsilon > p^{-k+1}$ with $k \in \mathbf{Z}$, where $p \ge 2$, we get $(1 - J(x,\, k)p^{-kbn}) = 1$ for $p^{-k+1} \le |x|_{\mathbf{F}} \le \varepsilon$, then

$$-p^{-kbn}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy) + \delta$$

$$\ge \Big(\int_{B(\mathbf{F^{bn}}, 0, \varepsilon)\setminus B(\mathbf{F^{bn}}, 0, p^{-k})} |x|_{\mathbf{F}}^{-2}\lambda_v(dx)\Big)$$

$$\geq \varepsilon^{-2}[\lambda_v(B(\mathbf{F^{bn}}, 0, \varepsilon)) - \lambda_v(B(\mathbf{F^{bn}}, 0, p^{-k})],$$

hence

(7) $[\lambda_v(B(\mathbf{F^{bn}}, 0, \varepsilon)) - \lambda_v(B(\mathbf{F^{bn}}, 0, p^{-k})]$

$$\leq \varepsilon^2\left[\delta - p^{-kbn}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy)\right].$$

In particular, for $\varepsilon_k = p^{-k+2}$ with $\varepsilon_k \leq \varepsilon$ and $k \to \infty$, inequality (7) is satisfied. The summation of both parts of inequality (7) by such $k$ gives:

(8) $\lambda_v(B(\mathbf{F^{bn}}, 0, \varepsilon))$

$$\leq L_1\delta - \sum_{k=k_0}^{\infty} p^{-kbn-2k+4}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy),$$

where $L_1 = p^4\sum_{k=k_0}^{\infty} p^{-2k} = p^{4-2k_0}/(1-p^{-2})$, $k_0 \in \mathbf{Z}$ is fixed. At the same time from (6) it follows:

(9) $-p^{-kbn}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy) + \delta \geq \lambda_v(\mathbf{F^{bn}} \setminus B(\mathbf{F^{bn}}, 0, \varepsilon))$

for $\varepsilon > p^{-k+1}$. Therefore, due to inequalities (8) and (9) there exists $L =$ const $> 0$ such that $\lambda_v(\mathbf{F^{bn}}) = \lambda_v(B(\mathbf{F^{bn}}, 0, \varepsilon)) + \lambda_v(\mathbf{F^{bn}} \setminus B(\mathbf{F^{bn}}, 0, \varepsilon)) \leq L$, for each $v \in (0, v_0]$, where $L = \text{const} > 0$.

Due to conditions of this theorem the function $g(y)$ is continuous and $g(0) = 0$, consequently, for each $\delta > 0$ there exists sufficiently small $0 < R_1 = p^{k_1} < \infty$ such that $R_1^{-bn}\left|\int_{B(\mathbf{F^{bn}}, 0, R_1)} \mathrm{Reg}(y)\mu(dy)\right| < \delta$. In view of inequality (9) for each $\varepsilon > \max(p^{-k_1+1}, 1)$ there is satisfied the inequality $\lambda_v(\mathbf{F^{bn}} \setminus B(\mathbf{F^{bn}}, 0, \varepsilon)) < 2\delta$ for each $v \in (0, v_0]$, consequently, the family of measures $\{\lambda_v : v \in V\}$ is weakly compact.

Choose a sequence $h_n \downarrow 0$ such that $\lambda_{v_n}$ is weakly convergent to some measure $\nu$ on $\mathcal{B}(\mathbf{K^n})$. Due to conditions of this theorem and using the decomposition of exp into the series, we get:

$$(10)\ \ [\psi(v,\ y)-1]/v = \int_{\mathbf{K^n}} (\chi_y(x)-1)[1+|x|_{\mathbf{K}}^2]|x|_{\mathbf{K}}^{-2}\lambda_v(dx)$$

$$= iA_v(y) - B_v(y)/2 + \int_{\mathbf{K^n}} f(y,\ x)\lambda_v(dx),$$

where

$$A_v(y) = \int_{\mathbf{K^n}} \langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}} |x|_{\mathbf{K}}^{-2}\lambda_v(dx),$$

$$B_v(y) = \int_{\mathbf{K^n}} \langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}^{\mathbf{2}} |x|_{\mathbf{K}}^{-2}\lambda_v(dx).$$

$$(10')\ \ f(y,\ x) = (\exp(i\langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}) - 1 - i\langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}[1+|x|_{\mathbf{K}}^2]^{-1}$$

$$+ \langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}^2[1+|x|_{\mathbf{K}}^2]^{-1}/2)[1+|x|_{\mathbf{K}}^2]|x|_{\mathbf{K}}^{-2}.$$

The multiplier $[1+|x|_{\mathbf{K}}^2]|x|_{\mathbf{K}}^{-2}$ is continuous and bounded for $|x| \ge R$, where $0 < R < \infty$, $\langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}} = 0$ for $|y|_{\mathbf{K}}|x|_{\mathbf{K}} \le 1$, hence the function $f(y,\ x)$ is continuous, it is bounded, when $y$ varies in a bounded subset in $\mathbf{K^n}$, while $x \in \mathbf{K^n}$. Therefore, there exists

$$\lim_{k\to\infty} \int_{\mathbf{K^n}} f(y,\ x)\lambda_{v_k}(dx) = \int_{\mathbf{K^n}} f(y,\ x)\nu(dx).$$

The functions $\langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}|x|_{\mathbf{K}}^{-2}$ and $\langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}\langle (z,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}|x|_{\mathbf{K}}^{-2}$ are locally constant by $x$ for each given value of the parameters $y$ and $z$. These functions are zero, when $|y|_{\mathbf{K}}|x|_{\mathbf{K}} \le 1$, that is, they are defined in the continuous manner to be zero at the zero point $x = 0$. Since there exists the limit in the left hand side of (10), there exist

$$\lim_{k\to\infty} A_{v_k}(y) = A(y) = \int_{\mathbf{K^n}} \langle (y,\ x)_{\mathbf{K}}\rangle_{\mathbf{F}}|x|_{\mathbf{K}}^{-2}\nu(dx)$$

and

$$\lim_{k\to\infty} B_{\upsilon_k}(y) = B(y) = \int_{\mathbf{K^n}} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 | x |_{\mathbf{K}}^{-2} \nu(dx).$$

At the same time $B(y) \geq 0$ for each $y \in \mathbf{K^n}$.

Substitute the measure $\nu(U)$ on $\nu(U \setminus \{0\})$ and denote it by the same symbol, where $U \in \mathcal{B}(\mathbf{K^n})$. Due to the fact that $f(y, 0) = 0$, $\langle (y, 0)_{\mathbf{K}} \rangle_{\mathbf{F}} = 0$, then for such substitution of the measure the values of integrals

$$\int_{\mathbf{K^n}} f(y, x)\nu(dx), \qquad A(y) = \int_{\mathbf{K^n}} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} | x |_{\mathbf{K}}^{-2} \nu(dx)$$

and

$$B(y, z) = \int_{\mathbf{K^n}} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \langle (y, z)_{\mathbf{K}} \rangle_{\mathbf{F}} | x |_{\mathbf{K}}^{-2} \nu(dx)$$

do not change.

It is known that $[\alpha + \beta]_{\mathbf{F}} = [\alpha]_{\mathbf{F}} + [\beta]_{\mathbf{F}} + v(\alpha, \beta)$, where $v(\alpha, \beta) \in \mathbf{Z}$ for $\mathbf{F} = \mathbf{Q_p}$, $pv(\alpha, \beta) \in \mathbf{Z}$ for $\mathbf{F} = \mathbf{F_p}(\theta)$, $0 \leq [\alpha]_{\mathbf{F}} \leq 1$ for each $\alpha, \beta \in \mathbf{F}$. Also $[\alpha\beta]_{\mathbf{F}} = [\alpha]_{\mathbf{F}}[\beta]_{\mathbf{F}} + u(\alpha, \beta)$, where $p^{-N(\alpha, \beta)} u(\alpha, \beta) \in \mathbf{Z}$ for $\mathbf{F} = \mathbf{Q_p}$, $p^2 u(\alpha, \beta) \in \mathbf{Z}$ for $\mathbf{F} = \mathbf{F_p}(\theta)$, since

$$[\alpha]_{\mathbf{Q_p}}[\beta]_{\mathbf{Q_p}} = \sum_{k=N(\alpha)}^{-1} \sum_{l=N(\beta)}^{-1} \alpha_k \beta_l p^{k+l},$$

and

$$[\alpha\beta]_{\mathbf{Q_p}} = \sum_{N(\alpha) \leq k,\, N(\beta) \leq l,\, k+l \leq -1} \alpha_k \beta_l p^{k+l},$$

where $\alpha = \sum_{k=N(\alpha)}^{\infty} \alpha_k p^k \in \mathbf{Q_p}$, $\alpha_k \in \{0, 1, \dots, p-1\}$ for each $k \in \mathbf{Z}$, $\alpha_{N(\alpha)} \neq 0$, while

$$[\alpha]_{\mathbf{F_p}(\theta)}[\beta]_{\mathbf{F_p}(\theta)} = \alpha_{-1}\beta_{-1}p^{-2},$$

and

$$[\alpha\beta]_{\mathbf{F_p}(\theta)} = \sum\nolimits_{N(\alpha)\leq k,\, N(\beta)\leq l,\, k+l=-1} \alpha_k \beta_l p^{-1},$$

where

$$\alpha = \sum\nolimits_{k=N(\alpha)}^{\infty} \alpha_k \theta^k \in \mathbf{F_p}(\theta), \quad \alpha_k \in \mathbf{F_p}$$

for each $k \in \mathbf{Z}$, $\alpha_{N(\alpha)} \neq 0$ [30, 31]. At the same time $[\alpha]_{\mathbf{F}} = 0$, when $|\alpha|_{\mathbf{F}} \leq 1$, hence $v(\alpha, \beta) = 0$ and $u(\alpha, \beta) = 0$ for $\max(|\alpha|_{\mathbf{F}}, |\beta|_{\mathbf{F}}) \leq 1$. Then

(11) $\langle (y+z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} = \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + 2\pi f_1(y, z; x),$

where $f_1 \in \mathbf{Z}$ for $\mathbf{F} = \mathbf{Q_p}$, $pf_1 \in \mathbf{Z}$ for $\mathbf{F} = \mathbf{F_p}(\theta)$. Since $\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}$ is locally constant and $0 \leq [\alpha]_{\mathbf{F}} \leq 1$ for each $\alpha \in \mathbf{F}$, there is the inequality $-2 \leq f_1(y, z; x) \leq 1$ for each $x, y, z \in \mathbf{K^n}$ in (11). On the other hand,

(12) $\langle (\beta y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} = [\beta]_{\mathbf{F}} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + 2\pi f_2(\beta, (e, (y, x)_{\mathbf{K}})_{\mathbf{F}}),$

where $f_2(\alpha, \beta) = u(\alpha, \beta)$ for each $\alpha, \beta \in \mathbf{F}$, since $\mathbf{F}$ is naturally embedded into $\mathbf{K}$ and $\beta(e, (y, x)_{\mathbf{K}})_{\mathbf{F}} = (e, (\beta y, x)_{\mathbf{K}})_{\mathbf{F}}$. Since $[\alpha]_{\mathbf{F}} \in [0, 1]$ for each $\alpha \in \mathbf{F}$, $-1 \leq f_2(\alpha, \gamma) \leq 1$ for each $\alpha \in \mathbf{F}$ and $\gamma = (e, (y, x)_{\mathbf{K}})_{\mathbf{F}} \in \mathbf{F}$ in (12). In view of the continuity and the locally constant behavior of $\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}$ from this the continuity and local constantness of $f_1$ and $f_2$ follow. Thus, $f_1$ and $f_2$ satisfy Conditions (F3, F4) depending on $\mathrm{char}(\mathbf{K})$. Therefore, from (11) and (12) we get the properties:

(13) $A(y) = \int_{\mathbf{K^n}} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} |x|_{\mathbf{K}}^{-2} \nu(dx)$ and

(14) $B(y, z) = \int_{\mathbf{K^n}} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \langle (z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} |x|_{\mathbf{K}}^{-2} \nu(dx)$ with the measure $|x|_{\mathbf{K}}^{-2} \nu(dx)$ here instead of the measure $\mu$ in (F1-F4), (B1-B3). By the

construction given above the measures in the definitions of $A$ and $B$ are non-negative and the functions in integrals are non-negative, then $A(y)$ and $B(y, z)$ take non-negative values.

As the metric space $\mathbf{K^n}$ is complete separable and hence is the Radon space (see [4, Theorem 1.2]), that is, the class of compact subsets approximates from below each σ-additive non-negative finite measure on the Borel σ-algebra $\mathcal{B}(\mathbf{K^n})$. In view of the finiteness and the σ-additivity of the non-negative measure $|x|_{\mathbf{K}}^{-2}\nu(dx)$ on $\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, 1/|y|_{\mathbf{K}})$ for $|y|_{\mathbf{K}} > 0$, $\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} = 0$ for $|(x, y)_{\mathbf{K}}| \le 1$ and due to continuity and boundedness of the functions in integrals we have that the mappings $A(y)$ and $B(y, z)$ are continuous.

**6. Corollary.** *Let the conditions of Theorem* 5 *be satisfied and there exists* $J := \int_{\mathbf{K^n}} |x|_{\mathbf{K}}^{-2}\nu(dx) < \infty$. *Then*

$$A(y) = -i(\partial\phi(\beta, y)/\partial\beta)|_{\beta=0}$$

*and*

$$B(y) = -(\partial^2\phi(\beta, y)/\partial\beta^2)|_{\beta=0},$$

*where*

$$\phi(\beta, y) = \int_{\mathbf{K^n}} \exp(i\langle (y, x)\rangle_{\mathbf{F}}\beta)|x|_{\mathbf{K}}^{-2}\nu(dx), \quad -1 < \beta < 1.$$

**Proof.** In view of Theorem 5 there exist $A(y)$ and $B(y)$. At the same time the measure $\nu$ is non-negative as the weak limit of a weakly converging sequence of non-negative measures, consequently, the measure $\mu(dx) := |x|_{\mathbf{K}}^{-2}\nu(dx)$ is non-negative. In view of the supposition of this lemma $0 \le \mu(\mathbf{K^n}) = J < \infty$. If $J = 0$, then $A(y) = 0$, $B(y) = 0$ and $\phi(\beta, y) = 0$, then the statement of this lemma is evident. Therefore, there remains the case $J > 0$. Consider the random variable $\zeta := \langle (y, \eta)_{\mathbf{K}} \rangle_{\mathbf{F}}$ with values in $\mathbf{R}$, where $\eta$ is a random vector in $\mathbf{K^n}$ with the probability distribution $P(dx) := J^{-1} | x |_{\mathbf{K}}^{-2} \nu(dx)$, where $y \in \mathbf{K^n}$ is the given vector.

Then $\phi(\beta, y) = JM \exp(i\beta\zeta)$, where $MX$ denotes the mean value of the random variable $X$ with values in $\mathbf{C}$. That is,

$$M \exp(i\beta\zeta) = \int_{\mathbf{K^n}} \exp(i\beta\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) P(dx).$$

For $\zeta$ there exists the second moment, since there exists $B(y)$ for each $y \in \mathbf{K^n}$. In view of Theorem II.12.1 [28] about relations between moments of the random variable and values of derivatives of their characteristic functions at zero, we get the statement of this Corollary.

**7. Theorem.** *Let the conditions of Theorem* 5 *be satisfied and in addition measures* $\mu_v(dx)$ *posses finite moments of* $| x |_{\mathbf{K}}$ *of the second order*: $\int_{\mathbf{K^n}} | x |_{\mathbf{K}}^2 \mu_v(dx) < \infty \ \forall v \in V$. *Then for* $g(y)$ *there is the representation*:

(i) $$g(y) = i\widetilde{A}(y) - \widetilde{B}(y)/2 + \int_{B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}})$$

$$- 1 - i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 /2) \eta(dx)$$

$$+ \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1) \eta(dx),$$

*where* $\eta$ *is a non-negative* $\sigma$*-additive measure on* $\mathcal{B}(\mathbf{K^n})$, $\eta(\{0\}) = 0$, $\widetilde{A}(y) \in \mathcal{C}_1$, $\widetilde{B}(y, z) \in \mathcal{C}_2$.

**Proof.** Let

$$\eta_v(A) := v^{-1} \int_A | x |_{\mathbf{K}}^{-2} \mu_v(dx),$$

where $\{\mu_v : v\}$ is the family of measures corresponding to the characteristic functions $\psi(v, y)$. At first we prove the weak compactness of the family of measures $\{\Psi_B(x)\eta_v(dx) : v \in V\}$ for $B = B(\mathbf{K^n}, 0, R)$, $0 < R < \infty$, where $\Psi_B(x) = 1$ for $x \in B$, $\Psi_B(x) = 0$ for $x \notin B$, $\Psi_B(x)$ is the characteristic function of the set $B$. Using the non-Archimedean analog of the Minlos-Sazonov theorem as in 5 of Section 2 we reduce the proof to the case of measures on $\mathbf{F^{bn}}$. Take $0 < R_1 < \infty$. In view of the conditions of this theorem for each $\delta > 0$ there exists $v_0 = v_0(R_1, \delta) > 0$ such that for each $\varepsilon > 0$ and each $0 < v \le v_0$ there is accomplished the inequality

$$-\mathrm{Reg}(y) + \delta \ge \int_{\mathbf{F^{bn}}} [1 - \cos(\langle y, x\rangle_{\mathbf{F}})] |\, x\,|^{-2} \eta_v(dx)$$

due to the existence of $\lim_{v \downarrow 0} [\psi(v, y) - 1]/v = g(y)$ uniformly in the ball of the radius $0 < R_1 < \infty$, $\forall y \in \mathbf{F^{bn}} : |\, y\,| \le R_1$. Integrate this inequality by $y \in B(\mathbf{F^{bn}}, 0, r)$ and divide on the volume $\mu(B(\mathbf{F^{bn}}, 0, r)) = r^{bn}$ for $r \in \Gamma_{\mathbf{F}} := \{|\, x\,| : x \ne 0,\ x \in \mathbf{F}\} = \{p^k : k \in \mathbf{Z}\}$, where $\mu$ is the Haar non-negative non-trivial measure on $\mathbf{F^{bn}}$. Then

$$-r^{-bn}\int_{B(\mathbf{F^{bn}}, 0, r)} \mathrm{Reg}(y)\mu(dy) + \delta$$

$$\ge r^{-bn}\int_{B(\mathbf{F^{bn}}, 0, r)}\left(\int_{\mathbf{F^{bn}}} [1 - \cos\langle(y, x)\rangle_{\mathbf{F}}] |\, x\,|^{-2}\eta_v(dx)\right)\mu(dy)$$

$$\ge r^{-bn}\int_{B(\mathbf{F^{bn}}, 0, r)}\left(\int_{B(\mathbf{F^{bn}}, 0, \varepsilon)} [1 - \cos\langle(y, x)\rangle_{\mathbf{F}}] |\, x\,|^{-2}\eta_v(dx)\right)\mu(dy),$$

since $\eta_v \ge 0$ and $\mu \ge 0$ are non-negative measures. Since

$$\int_{B(\mathbf{F^{bn}}, 0, p^k)} \chi_s(x)\mu(dx) = J(s, k),$$

where $J(s, k) = p^{kbn}$ for $|s| \le p^{-k}$, $J(s, k) = 0$ for $|s| \ge p^{-k+1}$,

$$-p^{-bnk}\int_{B(\mathbf{F^{bn}},\, 0,\, p^k)} \mathrm{Reg}(y)\mu(dy) + \delta$$

$$\ge \int_{B(\mathbf{F^{bn}}, 0, \varepsilon)} [1 - p^{-bnk}J(x, k)]|x|^{-2}\eta_v(dx).$$

For $\varepsilon > p^{-k+1}$ we then get

$$[\eta_v(B(\mathbf{F^{bn}}, 0, \varepsilon) - \eta_v(B(\mathbf{F^{bn}}, 0, p^{-k}))]$$

$$\le \varepsilon^2\left[\delta - p^{-bnk}\int_{B(\mathbf{F^{bn}},\, 0,\, p^k)} \mathrm{Reg}(y)\mu(dy)\right].$$

Then for $\varepsilon = p^{-k_0+2}$ and $\varepsilon_k = p^{-k+2} \le \varepsilon$, $k \to \infty$ the summing of these inequalities leads to:

$$\eta_v(B(\mathbf{F^{bn}}, 0, \varepsilon) \le L_1\delta - \sum_{k=k_0}^{\infty} p^{-kbn-2k+4}\int_{B(\mathbf{F^{bn}}, 0, p^k)} \mathrm{Reg}(y)\mu(dy),$$

where $L_1 = p^{4-2k_0}/(1 - p^{-2})$, $k_0 \in \mathbf{Z}$ is fixed.

In view of the fact that the function $g(y)$ is continuous and $g(0) = 0$, then for each $\delta > 0$ there exists $0 < R_1 < \infty$ such that

$$R_1^{-bn}\left|\int_{B(\mathbf{F^{bn}}, 0, R_1)} \mathrm{Reg}(y)\mu(dy)\right| < \delta.$$

Then for $\varepsilon = p^{-k_0+2}$ there is accomplished the inequality: $\eta_v(B(\mathbf{F^{bn}}, 0, \varepsilon))$ $< 2L_1\delta$ for each $v \in (0, v_0]$. Since $\int_{\mathbf{K^n}\setminus B} \Psi_{\mathrm{B}}(x)\eta(dx) = 0$, the family of measures $\{\Psi_B\eta_v : v \in V\}$ is weakly compact for each given $0 < R < \infty$, $B = B(\mathbf{K^n}, 0, R)$.

Let $0 < \varepsilon < \infty$. Then

$$\left|\int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}}\rangle_{\mathbf{F}}\nu(dx)\right| < \infty.$$

and

$$\left| \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} | x |^{-2} \nu(dx) \right| < \infty.$$

Then

$$J_e := \int_{\mathbf{K^n}} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1 - i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} [1 + | x |^2]^{-1}$$

$$+ \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 [1 + | x |^2]^{-1} / 2) [1 + | x |^2] | x |^{-2} \nu(dx)$$

$$= \int_{\mathbf{K^n}} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1 - i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} [1 + | x |^2]^{-1}$$

$$+ \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 [1 + | x |^2]^{-1} / 2) \eta(dx),$$

where $\eta(A) := \int_A [1 + | x |^2] | x |^{-2} \nu(dx)$ for each $A \in \mathcal{B}(\mathbf{K^n})$. The measure $\eta \geq 0$ is non-negative, since $\nu \geq 0$ is non-negative. From $\nu(\{0\}) = 0$ it follows that $\eta(\{0\}) = 0$. The measure $\eta(A)$ is finite for each $A \in \mathcal{B}(\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon))$, when $0 < \varepsilon < \infty$, since $\nu(\mathbf{K^n}) < \infty$ and $| x | > \varepsilon$ for $x \in \mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)$. Therefore,

$$J_e := \left( \int_{B(\mathbf{K^n}, 0, \varepsilon)} + \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1 \right) \eta(dx)$$

$$+ \int_{\mathbf{K^n}} (-i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 / 2) | x |^{-2} \nu(dx).$$

At the same time

$$\int_{\mathbf{K^n}} (-i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 / 2) | x |^{-2} \nu(dx)$$

$$= \int_{B(\mathbf{K^n}, 0, \varepsilon)} (-i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 / 2) \eta(dx)$$

$$- \int_{B(\mathbf{K^n}, 0, \varepsilon)} (-i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 /2) [(1 + | x |^2) - 1] | x |^{-2} \nu(dx)$$

$$+ \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} (-i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 /2) | x |^{-2} \nu(dx),$$

hence

$$(1) \quad g(y) = i\widetilde{A}(y) - \widetilde{B}(y)/2 + \int_{B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1$$

$$- i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 /2) \eta(dx)$$

$$+ \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1) \eta(dx),$$

where

$$\widetilde{A}(y) = A(y) + \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \nu(dx)$$

$$- \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} | x |^{-2} \nu(dx),$$

$$\widetilde{B}(y) = B(y) + \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 \nu(dx)$$

$$- \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 | x |^{-2} \nu(dx).$$

Using the expressions for $A(y)$ and $B(y, z)$ from the proof of Theorem 5, we get

(2)

$$\widetilde{A}(y) = \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \nu(dx)$$

$$+ \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} | x |^{-2} \nu(dx),$$

(3)

$$\widetilde{B}(y, z) = \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \langle (z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \nu(dx)$$

$$+ \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} \langle (z, x)_{\mathbf{K}} \rangle_{\mathbf{F}} | x |^{-2} \nu(dx).$$

Due to identities 5(11, 12) with the measure $[1 + | x |^{-2}]\Psi_B \nu(dx)$ here as the measure $\mu$ in 4 of Section 2, with $B = B(\mathbf{K^n}, 0, \varepsilon)$, where $\Psi_B(x)$ is the characteristic function of the set $B$, $\Psi_B(x) = 1$ for $x \in B$, $\Psi_B(x) = 0$ for $x \in \mathbf{K^n} \setminus B$, we get, that $\widetilde{A}$ and $\widetilde{B}$ satisfy Conditions (F1-F4) and (B1-B3), respectively. Since the measures in the definition of $\widetilde{A}$ and $\widetilde{B}$ are non-negative and the functions in integrals are non-negative, $\widetilde{A}(y)$ and $\widetilde{B}(y, z)$ take non-negative values.

As the metric space $\mathbf{K^n}$ is complete and separable, hence it is the Radon space (see [4, Theorem 1.2]), that is the class of compact subsets approximates from below each $\sigma$-additive non-negative finite measure on the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{K^n})$. In view of the finiteness and $\sigma$-additivity of the non-negative measure $[1 + | x |^{-2}]\Psi_B \nu(dx)$ and the boundedness of the continuous functions in integrals the mappings $\widetilde{A}(y)$ and $\widetilde{B}(y, z)$ are continuous.

**8. Theorem.** *A characteristic function* $\psi(y)$ *of an infinitely divisible distribution in* $\mathbf{K^n}$ *has the form* $\psi(y) = \exp(g(y))$, *where* $g(y)$ *is given by Formula* 5(i). *If in addition distributions* $\mu_v(dx)$ *from Theorem* 5 *posses finite moments* $| x |_{\mathbf{K}}$ *of the second order*: $\int_{\mathbf{K^n}} | x |_{\mathbf{K}}^2 \mu_v(dx) < \infty$, *then* $g(y)$ *is given by Formula* 7(i).

**Proof.** Let $h_k = 1/k$, $k \in \mathbf{N}$, hence $g(y) = \lim_{k \to \infty} (\psi_k(y) - 1)/(1/k) = \lim_{k \to \infty} k(\psi_k(y) - 1) = \ln \psi(y)$, $\psi_k(y) = \psi(1/k, y)$, $\psi(y) = [\psi_k(y)]^k$. If fix

$\arg \psi(0) = 0$ and take such a continuous branch $\arg \psi(y)$, then $\psi(y) = \exp(g(y))$, where $g(y)$ is given by Theorem 5 or 7.

**9. Definitions.** Let there be a random function $\xi(t)$ with values in $\mathbf{K^n}$, $t \in T$, where $(T, \rho)$ is a metric space with a metric $\rho$. Then $\xi(t)$ is called *stochastically continuous* at a point $t_0$, if for each $\varepsilon > 0$ there exists $\lim_{\rho(t, t_0) \to 0} P(|\xi(t) - \xi(t_0)| > \varepsilon) = 0$. If $\xi(t)$ is stochastically continuous at each point of a subset $S$ in $T$, then it is called *stochastically continuous* on $S$.

If $\lim_{R \to \infty} \sup_{t \in S} P(|\xi(t)| > R) = 0$, then a random function $\xi(t)$ is called *stochastically bounded* on $S$.

Let $T = [0, a]$ or $T = [0, \infty)$, $a > 0$. A random process $\xi(t)$ with values in $\mathbf{K^n}$ is called a process with independent increments, if $\forall n$, $0 \le t_1 < \cdots < t_n$: random vectors $\xi(0), \xi(t_1) - \xi(0), ..., \xi(t_n) - \xi(t_{n-1})$ are mutually independent. At the same time the vector $\xi(0)$ is called the *initial state* (value), and its distribution $P(\xi(0) \in B)$, $B \in \mathcal{B}(\mathbf{K^n})$, is called the *initial distribution*. A process with independent increments is called *homogeneous*, if the distribution $P(t, s, B) := (\xi(t + s) - \xi(t) \in B)$, $B \in \mathcal{B}(\mathbf{K^n})$, of the vector $\xi(t + s) - \xi(t)$ is independent from $t$, that is, $P(t, s, B) = P(s, B)$ for each $t < t + s \in T$.

**10. Theorem.** *Let* $\psi(t, y)$ *be a characteristic function of the vector* $\xi(t + s) - \xi(s)$, $t > 0$, $s \ge 0$, *where* $\xi(t)$ *is the stochastically continuous random process with independent increments with values in* $\mathbf{K^n}$. *Then* $\psi(t, y) = \exp(tg(y))$, *where* $g(y)$ *is given by Formula* 5(i). *If in addition* $|\xi(t)|_{\mathbf{K}}$ *has the second order finite moments*, *then the function* $g(y)$ *is written by Formula* 7(i).

**Proof.** Let $\xi(t)$ be a homogeneous stochastically continuous process with independent increments with values in $\mathbf{K^n}$, where $t \in T \subset \mathbf{R}$. Let

$t > s$. Then

$$|\psi(t, y) - \psi(s, y)| = |M \exp(i\langle(y, \xi(t))_{\mathbf{K}}\rangle_{\mathbf{F}}) - M \exp(i\langle(y, \xi(s))_{\mathbf{K}}\rangle_{\mathbf{F}})|$$

$$= |M(\exp(i\langle(y, \xi(t) - \xi(s))_{\mathbf{K}}\rangle_{\mathbf{F}}) - 1)\exp(i\langle(y, \xi(s))_{\mathbf{K}}\rangle_{\mathbf{F}})|$$

$$\leq M|\exp(i\langle(y, \xi(t) - \xi(s))_{\mathbf{K}}\rangle_{\mathbf{F}}) - 1|.$$

Therefore, from the stochastic continuity of $\xi(t)$ it follows continuity of $\psi(t, y)$ by $t$. In view of being homogeneous and independency of increments the equalities are accomplished

$$\psi(t_1 + t_2, y) = M \exp(i\langle(y, \xi(t_1 + t_2) - \xi(t_1))\rangle_{\mathbf{F}} + i\langle(y, \xi(t_1) - \xi(0))\rangle_{\mathbf{F}})$$

$$= M \exp(i\langle(y, \xi(t_1) - \xi(0))\rangle_{\mathbf{F}})M \exp(i\langle(y, \xi(t_2) - \xi(0))\rangle_{\mathbf{F}})$$

$$= \psi(t_1, y)\psi(t_2, y),$$

for each $t_1, t_2 \in T$. On the other hand, a unique continuous solution of the equation $f(v + u) = f(v)f(u)$ for each $v, u \in \mathbf{R}$ has the form $f(v) = \exp(av)$, where $a \in \mathbf{R}$. Thus, $\psi(t, y) = \exp(tg(y))$, where $g(y) = \lim_{t\downarrow 0} (\psi(t, y) - 1)/t$. Applying Theorems 5 and 7, we get the statement of this theorem.

**11. Remark.** Consider auxiliary random process $\eta := [\xi]_p$ with values in $\mathbf{R}^{\mathbf{n}}$, where $[(q_1, ..., q_n)]_p := ([q_1]_p, ..., [q_n]_p)$ for $q = (q_1, .., q_n) \in \mathbf{K}^{\mathbf{n}}$. If $\xi(t)$ is a homogeneous process with independent increments, then such is also $\eta$. Let $a(t) := M\eta(t)$ is a mean value, while $R(t, s) := M[(\eta(t) - a(t))^*(\eta(s) - a(s))]$ is the correlation matrix, where $\eta = (\eta_1, ..., \eta_n)$ is the row-vector, $A^*$ denotes the transposed matrix $A$. For the process with independent increments and finite moments of the second order then $R(t, s) = B(\min(t, s))$, where the matrix $B(t)$ is symmetric and non-negative definite. If $\xi(t)$ is the homogeneous process with independent increments, $\eta$ has the finite second order moments, then as it is known $a(t) = at$, $R(t, s) = B\min(t, s)$, where $a$ is the vector, $B$ is the symmetric non-negative definite matrix [10].

**12. Theorem.** *Let $P$ and $Q$ be two non-negative finite $\sigma$-additive measures on the Borel $\sigma$-algebra $\mathcal{B}(\mathbf{K^n})$, where $\mathbf{K}$ is a locally compact infinite field with a non-trivial non-Archimedean norm, $n \in \mathbf{N}$. If their characteristic functions are equal $\hat{P}(y) = \hat{Q}(y)$ for each $y \in \mathbf{K^n}$, then $P(A) = Q(A)$ for each $A \in \mathcal{B}(\mathbf{K^n})$.*

**Proof.** The metric space $\mathbf{K^n}$ is complete and separable, consequently, it is the Radon space, then $P$ and $Q$ are Radon measures (see [4, Theorem 1.2]). Then for each $\delta > 0$ there exists the ball $B(\mathbf{K^n}, z, R)$, $0 < R < \infty$, $z \in \mathbf{K^n}$, such that $P(\mathbf{K^n} \setminus B(\mathbf{K^n}, z, R)) < \delta$ and $Q(\mathbf{K^n} \setminus B(\mathbf{K^n}, z, R)) < \delta$.

For each ball $B(\mathbf{K^n}, z, R_1)$, $z \in \mathbf{K^n}$, $0 < R_1 < \infty$, due to the Stone-Weierstrass theorem for each $\varepsilon > 0$ and each continuous bounded function $f : \mathbf{K^n} \to \mathbf{R}$ there exist $b_1, \ldots, b_k \in \mathbf{C}$ and $s_1, \ldots, s_k \in \mathbf{K^n}$ such that $\sup_{x \in B(\mathbf{K^n}, z, R_1)} |b_1\chi_{s_1}(x) + \cdots + b_k\chi_{s_k}(x) - f(x)| < \varepsilon$, where $\chi_s(x)$ is the character, $k \in \mathbf{N}$, since the family of all finite $\mathbf{C}$-linear combinations of characters forms the algebra which is the subalgebra of the algebra of all continuous functions on $B(\mathbf{K^n}, z, R_1)$, the complex conjugation preserves this subalgebra, this subalgebra contains all complex constants and separates points in $B(\mathbf{K^n}, z, R_1)$ (see [25, Theorem IV.10]).

The characteristic function $\Psi_{B(\mathbf{K^n}, z, R)}$ of the set $B(\mathbf{K^n}, z, R)$ is continuous on $\mathbf{K^n}$, since $\mathbf{K^n}$ is totally disconnected and the ball $B(\mathbf{K^n}, z, R)$ is clopen in $\mathbf{K^n}$ (simultaneously open and closed). Take $z \in \mathbf{K^n}$, $0 < \delta_k < 1/k$, $0 < \varepsilon_k < 1/k$, $R = R(\delta_k) \le R(\delta_{k+1})$ for each $k$. For an arbitrary vector $z_1 \in \mathbf{K^n}$ with $|z - z_1|_{\mathbf{K^n}} < R(\delta_1)$ take the function

$\Psi^{\varepsilon}(x) = b_1\chi_{s_1}(x) + \cdots + b_v\chi_{s_v}(x)$ such that

$$\sup_{x\in B(\mathbf{K^n},\, z_1,\, R_1)}|\,\Psi^{\varepsilon_k}(x) - \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)\,| < \varepsilon_k.$$

Then

$$\int_{\mathbf{K^n}} \Psi^{\varepsilon_k}(x)P(dx) = \int_{\mathbf{K^n}} \Psi^{\varepsilon_k}(x)Q(dx)$$

and

$$\int_{\mathbf{K^n}} \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)P(dx) = P(B(\mathbf{K^n},\, z_1,\, R_1)),$$

$$\int_{\mathbf{K^n}} \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)Q(dx) = Q(B(\mathbf{K^n},\, z_1,\, R_1)).$$

On the other hand,

$$\left|\int_{\mathbf{K^n}} \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)P(dx) - \int_{\mathbf{K^n}} \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)Q(dx)\right|$$

$$\le \left|\int_{\mathbf{K^n}} \Psi^{\varepsilon_k}(x)P(dx) - \int_{\mathbf{K^n}} \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)P(dx)\right|$$

$$+ \left|\int_{\mathbf{K^n}} \Psi^{\varepsilon_k}(x)Q(dx) - \int_{\mathbf{K^n}} \Psi_{B(\mathbf{K^n},\, z_1,\, R_1)}(x)Q(dx)\right|$$

$$+ \left|\int_{\mathbf{K^n}} \Psi^{\varepsilon_k}(x)P(dx) - \int_{\mathbf{K^n}} \Psi^{\varepsilon_k}(x)Q(dx)\right|$$

$$\le \varepsilon_k(P(\mathbf{K^n}) + Q(\mathbf{K^n})).$$

The right hand side of the latter inequality tends to zero while $k \to \infty$, consequently,

$$P(B(\mathbf{K^n},\, z_1,\, R_1)) = Q(B(\mathbf{K^n},\, z_1,\, R_1))$$

for each ball $B(\mathbf{K^n}, z_1, R_1)$ in $\mathbf{K^n}$, where $0 < R_1 < \infty$, $z_1 \in \mathbf{K^n}$, since $\lim_{k \to \infty} \delta_k = 0$ and

$$P(\mathbf{K^n} \setminus B(\mathbf{K^n}, z, R(\delta_k)) < \delta_k, \quad Q(\mathbf{K^n} \setminus B(\mathbf{K^n}, z, R(\delta_k)) < \delta_k.$$

Since balls form the base of the topology in $\mathbf{K^n}$, $P(A) = Q(A)$ for each $A \in \mathcal{B}(\mathbf{K^n})$.

**13. Theorem.** *Random vectors* $\eta_1, ..., \eta_k$ *in* $\mathbf{K^n}$ *are independent if and only if*

(1)

$$M \exp(i \langle (y_1, \eta_1)_{\mathbf{K}} + \cdots + (y_k, \eta_k)_{\mathbf{K}} \rangle_{\mathbf{F}})$$

$$= M \exp(i \langle (y_1, \eta_1)_{\mathbf{K}} \rangle_{\mathbf{F}}) \cdots M \exp(i \langle (y_k, \eta_k)_{\mathbf{K}} \rangle_{\mathbf{F}})$$

*for each* $y_1, ..., y_k \in \mathbf{K^n}$.

**Proof.** From the independence of $\eta_1, ..., \eta_k$ it follows the independence of $\langle (y_1, \eta_1)_{\mathbf{K}} \rangle_{\mathbf{F}}, ..., \langle (y_k, \eta_k)_{\mathbf{K}} \rangle_{\mathbf{F}}$, consequently, there is satisfied equality (1), since

$$\exp(i \langle (y_1, \eta_1)_{\mathbf{K}} + \cdots + (y_k, \eta_k)_{\mathbf{K}} \rangle_{\mathbf{F}})$$

$$= \exp(i \langle (y_1, \eta_1)_{\mathbf{K}} \rangle_{\mathbf{F}}) \cdots \exp(i \langle (y_k, \eta_k)_{\mathbf{K}} \rangle_{\mathbf{F}}).$$

Vice versa let (1) be satisfied. Denote by $P_{\eta_1, ..., \eta_k}$ the mutual probability distribution of random vectors $\eta_1, ..., \eta_k$, by $P_{\eta_j}$ denote the probability distribution of $\eta_j$. Then

$$\int_{\mathbf{K^n}} \exp(i \langle (y_1, x_1)_{\mathbf{K}} + \cdots + (y_k, x_k)_{\mathbf{K}} \rangle_{\mathbf{F}}) P_{\eta_1, ..., \eta_k}(dx)$$

$$= M \exp(i \langle (y_1, \eta_1)_{\mathbf{K}} + \cdots + (y_k, \eta_k)_{\mathbf{K}} \rangle_{\mathbf{F}})$$

$$= M\exp(i\langle(y_1,\ \eta_1)_{\mathbf{K}}\rangle_{\mathbf{F}})\cdots M\exp(i\langle(y_k,\ \eta_k)_{\mathbf{K}}\rangle_{\mathbf{F}})$$

$$= \prod_{j=1}^{k}\int_{\mathbf{K}^{\mathbf{n}}}\exp(i\langle(y_j,\ x_j)_{\mathbf{K}}\rangle_{\mathbf{F}})P_{\eta_j}(dx_j),$$

where $x = (x_1, \ldots, x_k)$, $y_1, \ldots, y_k$, $x_1, \ldots, x_k \in \mathbf{K}^{\mathbf{n}}$. Therefore, by Theorem 12 $P_{\eta_1,\ldots,\eta_k}(A_1 \times \cdots \times A_k) = P_{\eta_1}(A_1)\cdots P_{\eta_k}(A_k)$ for each $A_1, \ldots, A_k \in \mathcal{B}(\mathbf{K}^{\mathbf{n}})$, consequently, $\eta_1, \ldots, \eta_k$ are independent.

**14. Definitions.** A sequence of random vectors $\xi_m$ in $\mathbf{K}^{\mathbf{n}}$ is called *convergent* by the distribution to a random vector $\xi$, if for each continuous bounded function $f : \mathbf{K}^{\mathbf{n}} \to \mathbf{R}$ there exists $\lim_{m\to\infty} Mf(\xi_m) = Mf(\xi)$.

Let a metric space $(X, \rho)$ be given with a metric $\rho$ and a $\sigma$-algebra of Borel subsets $\mathcal{B}(X)$.

The family of probability measures $\mathcal{P} := \{P_\beta : \beta \in \Lambda\}$ on $(X, \mathcal{B}(X))$, where $\Lambda$ is a set, is called *relatively compact*, if an arbitrary sequence of measures from $\mathcal{P}$ contains a subsequence weakly converging to some probability measure.

A family of probability measures $\mathcal{P} := \{P_\beta : \beta \in \Lambda\}$ on $(X, \mathcal{B}(X))$ is called *dense*, if for each $\varepsilon > 0$ there exists a compact subset $C$ in $X$ such that $\sup_{\beta\in\Lambda} P_\beta(X/C) \le \varepsilon$.

A sequence $\{P_m : m \in \mathbf{N}\}$ of probability measures $P_m$ is called *weakly convergent* to a measure $P$ when $m \to \infty$, if for each continuous bounded function $f : X \to R$ there exists

$$\lim_{m\to\infty}\int_X f(x)P_m(dx) = \int_X f(x)P(dx).$$

**15. Theorem.** *A random vector* $\xi$ *in* $\mathbf{K}^{\mathbf{n}}$ *is a limit by a distribution of sums* $\tilde{\xi}_m := \sum_{k=1}^{m}\xi_{m,k}$ *of independent random vectors with the same* *probability distribution* $\xi_{m,k}$, $k = 1, \ldots, m$, *if and only if* $\xi$ *is infinitely divisible.*

**Proof.** If $\xi$ is infinitely divisible, then for each $m \geq 1$ there exists independent random vectors with the same distribution $\xi_{m,1}, \ldots, \xi_{m,k}$ such that the probability distributions of $\xi$ and of the sum $(\xi_{m,1} + \cdots + \xi_{m,k})$ are the same.

Let now $\widetilde{\xi}_m$ be a sequence of arbitrary vectors converging by the distribution to $\xi$ when $m \to \infty$. Take $k \geq 1$ and group the summands writing $\widetilde{\xi}_{mk}$ in the form:

$$\widetilde{\xi}_{mk} = \zeta_{m,1} + \cdots + \zeta_{m,k},$$

where

$$\zeta_{m,1} = \xi_{mk,1} + \cdots + \xi_{mk,m}, \ldots, \zeta_{mk,k} = \xi_{mk,m(k-1)+1} + \cdots + \xi_{mk,mk}.$$

Since the sequence $\widetilde{\xi}_{mk}$ converges by the distribution to $\xi$ while $m \to \infty$, the sequence of the probability distributions $P_{\widetilde{\xi}_{mk}}$ of random vectors $\widetilde{\xi}_{mk}$ is relatively compact, consequently, due to the Prohorov Theorem (see [11, Section VI.25] or [28, III. 2.1]) it is dense.

On the other hand, if $|\widetilde{\xi}_{mk}| > R$, then due to non-Archimedeanity of the norm in $\mathbf{K^n}$ there exists $j$ such that $|\zeta_{m,j}| > R$, consequently, $P(\zeta_{m,1} \in \mathbf{K^n} \setminus B(\mathbf{K^n}, 0, R)) \leq P(\widetilde{\xi}_{mk} \in \mathbf{K^n} \setminus B(\mathbf{K^n}, 0, R))$, since $\zeta_{m,j}$ are independent and have the same probability distribution. Therefore, $\{P_{\zeta_{m,1}} : m \in \mathbf{N}\}$ is the dense family of probability distributions. Then there exists the sequence $\{m_j : j \in \mathbf{N}\}$ and random vectors $\eta_1, \ldots, \eta_k$ such that $\zeta_{m_j,l}$ converges by the distribution to $\eta_l$ for each $l = 1, \ldots, k$ for $j \to \infty$. In view of the definition of convergence by the distribution this means in particular, that for each $b_1, \dots, b_k \in \mathbf{K}^\mathbf{n}$ there exists

$$\lim_{j\to\infty} M\exp(i\langle(b_1, \zeta_{m_j,1})_\mathbf{K} + \cdots + (b_k, \zeta_{m_j,k})_\mathbf{K}\rangle_\mathbf{F})$$

$$= M\exp(i\langle(b_1, \eta_1)_\mathbf{K} + \cdots + (b_k, \eta_k)_\mathbf{K}\rangle_\mathbf{F}).$$

In view of independency of random vectors $\zeta_{m_j,1}, \dots, \zeta_{m_j,k}$ there is satisfied the equality

$$M\exp(i\langle(b_1, \zeta_{m_j,1})_\mathbf{K} + \cdots + (b_k, \zeta_{m_j,k})_\mathbf{K}\rangle_\mathbf{F})$$

$$= M\exp(i\langle(b_1, \zeta_{m_j,1})_\mathbf{K}\rangle)\cdots M\exp(i\langle(b_k, \zeta_{m_j,k})_\mathbf{K}\rangle_\mathbf{F}),$$

since $\exp(i\langle y\rangle_\mathbf{F})$ is the character of the additive group of the field $\mathbf{K}$. Therefore,

$$\lim_{j\to\infty} M\exp(i\langle(b_1, \zeta_{m_j,1})_\mathbf{K} + \cdots + (b_k, \zeta_{m_j,k})_\mathbf{K}\rangle_\mathbf{F})$$

$$= M\exp(i\langle(b_1, \eta_1)_\mathbf{K}\rangle_\mathbf{F})\cdots M\exp(i\langle(b_k, \eta_k)_\mathbf{K}\rangle_\mathbf{F}),$$

thus

$$M\exp(i\langle(b_1, \eta_1)_\mathbf{K} + \cdots + (b_k, \eta_k)_\mathbf{K}\rangle_\mathbf{F})$$

$$= M\exp(i\langle(b_1, \eta_1)_\mathbf{K}\rangle_\mathbf{F})\cdots M\exp(i\langle(b_k, \eta_k)_\mathbf{K}\rangle_\mathbf{F})$$

for each $b_1, \dots, b_k \in \mathbf{K}^\mathbf{n}$. Then from Theorem 13 it follows, that the random vectors $\eta_1, \dots, \eta_k$ are independent.

Since $\tilde{\xi}_{m_jk} = \zeta_{m_j,1} + \cdots + \zeta_{m_j,k}$ converges by the distribution to $\eta_1 + \cdots + \eta_k$ and $\tilde{\xi}_{m_jk}$ converges by the distribution to $\xi$, $\xi$ is equal to $\eta_1 + \cdots + \eta_k$ by the distribution, since

$$Mf(\xi) = \lim_{j\to\infty} Mf(\tilde{\xi}_{m_jk})$$

$$= \lim_{j\to\infty} Mf(\zeta_{m_j,1} + \cdots + \zeta_{m_j,k})$$

$$= Mf(\eta_1 + \cdots + \eta_k)$$

for each continuous bounded function $f : \mathbf{K^n} \to \mathbf{R}$.

**16. Particular cases of Theorem 10.** 1. If $A(y) = q\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}}$, $B = 0$, $\nu = 0$, where $a \in \mathbf{K^n}$ is some vector, $q = \text{const} > 0$, then $\psi(t, y) = \exp(itq\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}})$. The random function $\eta(t) = \langle(\xi(t), y)_{\mathbf{K}}\rangle_{\mathbf{F}}$ has the form $\eta(t) = \eta(0) + tq$, where $\xi$ is the initial random vector with values in $\mathbf{K^n}$. That is, $\eta(t)$ corresponds to the uniform motion of the point in $\mathbf{R}$ with the velocity $q$.

In the case, when $A(y) = q(v, \langle y\rangle_{\mathbf{F}})$, $B = 0$, $\nu = 0$, where $v \in \mathbf{R^n}$ is a given vector, $0 \le v_j \le 1$ for each $j = 1, \dots, n$, $v = (v_1, \dots, v_n)$, $q = \text{const} > 0$, then $\psi(t, y) = \exp(itq(v, \langle y\rangle_{\mathbf{F}}))$. Therefore, the random variable $\eta(t) = (\langle\xi(t)\rangle_{\mathbf{F}}, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}}$ has the form $\eta(t) = \eta(0) + tq$.

2. It is possible to consider in formulas for $A(y)$ and $B(y, z)$ in 5 and 7 of Section 2 in particular atomic measures, denoting $\widetilde{A}$ by $A$ and $\widetilde{B}$ by $B$ here for the uniformity, then there are the expressions of the form $\sum_j q_j\langle(x_j, y)_{\mathbf{K}}\rangle_{\mathbf{F}}$ and $\sum_j q_j\langle(x_j, y)_{\mathbf{K}}\rangle_{\mathbf{F}}\langle(x_j, z)_{\mathbf{K}}\rangle_{\mathbf{F}}$, where $q_j = \nu(\{x_j\}) > 0$ or $q_j = \nu(\{x_j\})|x_j|^{-2} > 0$ depending on the considered case, $x_j \ne 0$. In particular, there may be $x_j = e_j = (0, \dots, 0, 1, 0, \dots) \in \mathbf{K^n}$ with the unity on the $j$-th place. These expressions may be transformed using Conditions (F1-F4) or (B1-B3) (see Formulas 5(i, 10, 13, 14) or 7(i, 1-3)). Then there are possible cases

$$A(y) = q\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}},$$

$$A(y) = (v, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}},$$

$$B(y, z) = \sum_{j=1}^{n} \langle s_j y_j z_j\rangle_{\mathbf{F}},$$

$$B(y, z) = \sum_{j=1}^{n} q_j \langle y_j \rangle_{\mathbf{F}} \langle z_j \rangle_{\mathbf{F}},$$

where

$$\langle y \rangle_{\mathbf{F}} = (\langle y_1 \rangle_{\mathbf{F}}, \ldots, \langle y_n \rangle_{\mathbf{F}}),$$

$$y = (y_1, \ldots, y_n) \in \mathbf{K^n},$$

$$y_k \in \mathbf{K}$$

for each $k, v \in \mathbf{R^n}$, $(*, *)_{\mathbf{R}}$ is the scalar product in $\mathbf{R^n}$, $s_j \in \mathbf{K}$, $a \in \mathbf{K^n}$. The consideration of the transition matrix $Y$ from one basis in $\mathbf{K^n}$ to another or the matrix $X$ of transition from one basis in $\mathbf{R^n}$ into another leads to the more general expressions for $B(y, z)$ such as $B(y, z) = (b\langle y \rangle_{\mathbf{F}}, \langle z \rangle_{\mathbf{F}})_{\mathbf{R}}$, $B(y, z) = \langle (hy, z)_{\mathbf{K}} \rangle_{\mathbf{F}}$, where $b$ is the symmetric non-negative definite $n \times n$ matrix with elements in the field of real numbers $\mathbf{R}$, $h$ is the symmetric $n \times n$ matrix with elements in the locally compact field $\mathbf{K}$.

3. If $A(y) = q\langle (a, y)_{\mathbf{K}} \rangle_{\mathbf{F}}$, $B(y, z) = \langle (hy, z)_{\mathbf{K}} \rangle_{\mathbf{F}}$, where $a \in \mathbf{K^n}$, $h$ is the symmetric $n \times n$ matrix with elements in the field $\mathbf{K}$, if the correlation term $\int_{\mathbf{K^n}} f(y, x)\nu(dx) = 0$ from 5 of Section 2 or

$$\int_{B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1$$

$$- i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}} + \langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}^2 / 2)\eta(dx)$$

$$+ \int_{\mathbf{K^n} \setminus B(\mathbf{K^n}, 0, \varepsilon)} (\exp(i\langle (y, x)_{\mathbf{K}} \rangle_{\mathbf{F}}) - 1)\eta(dx) = 0$$

from 7 of Section 2 is zero, then

$$\psi(t, y) = \exp(itq\langle (a, y)_{\mathbf{K}} \rangle_{\mathbf{F}} - t\langle (hy, y)_{\mathbf{K}} \rangle_p / 2).$$

Then $\xi(t)$ is one of the non-Archimedean variants of the Gaussian process.

4. In the case, when $A(y) = (v, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}}$, $B(y, z) = (b\langle y\rangle_{\mathbf{F}}, \langle z\rangle_{\mathbf{F}})_{\mathbf{R}}$ (see paragraph 2), while the correlation term is zero, then $\psi(t, y) = \exp(it(v, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}} - t(b\langle y\rangle_{\mathbf{F}}, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}}/2)$ and again $\xi(t)$ is one of the analogs of the Gaussian process. Though Gaussian processes in the non-Archimedean case do not exist. That is, we can satisfy a part of properties of the Gaussian type in the non-Archimedean case, but not all (see also [19]).

5. When $A = 0$, $B = 0$ (taking into account (F1-F4) and (B1-B3); see Formulas 5(i, 10, 13, 14) or 7(i, 1-3)) where $\nu$ is the purely atomic measure, concentrated at the point $z_0$, $\nu(\{z_0\}) = q > 0$, then $\psi(t, y) = \exp(qt(\exp(i\langle(y, z_0)_{\mathbf{K}}\rangle_{\mathbf{F}}) - 1)$. Therefore, $\xi(t)$ is the non-Archimedean analog of the Poisson process.

6. If

$$\widetilde{A}(y) = q\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}} + \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}}\, \eta(dx),$$

$$\widetilde{B}(y) = -\int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle(y, x)_{\mathbf{K}}\rangle^2_{\mathbf{F}}\, \eta(dx)/2, \ \eta(B(\mathbf{K^n}, 0, \varepsilon)) < \infty,$$

then

$$g(y) = i\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}} + w\int_{\mathbf{K^n}} (\exp(i\langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}}) - 1)\lambda(dx),$$

where $\lambda$ is the probability measure on $(\mathbf{K^n}, \mathcal{B}(\mathbf{K^n}))$, $0 < w = \eta(\mathbf{K^n}) < \infty$, $\eta(dx) = w\lambda(dx)$ (see Formulas 7(i, 1-3) and (F1-F4), (B1-B3)). Therefore,

$$\psi(t, y) = \exp(itq\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}})\sum_{k=0}^{\infty} \exp(-wt)\,((wt)^k/k!)$$

$$\left[\int_{\mathbf{K^n}} \exp(i\langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}})\lambda(dx)\right]^k.$$

This expression of the characteristic function of the random process $\xi(t) = \rho(t) + \xi_1 + \cdots + \xi_{\zeta(t)}$, where $\rho(t)$ is the random process in $\mathbf{K^n}$ with the characteristic function $\exp(itq\langle(a, y)_{\mathbf{K}}\rangle_{\mathbf{F}})$, where $\xi_1, \ldots, \xi_k, \ldots$ are independent random vectors in $\mathbf{K^n}$ with the same probability distribution $\lambda(dx)$, $\zeta(t)$ is the Poisson process with a parameter $w$ independent from $\rho, \xi_1, \ldots, \xi_k, \ldots$. Then there arises the non-Archimedean analog $\xi(t)$ of the generalized Poisson process.

If

$$\widetilde{A}(y) = (v, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}} + \int_{B(\mathbf{K^n}, 0, \varepsilon)} \langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}}\, \eta(dx),$$

where $\widetilde{B}(y)$ is the same as at the beginning of the given paragraph, then

$$\psi(t, y) = \exp(it(v, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}})\sum_{k=0}^{\infty} \exp(-wt)\,((wt)^k/k!)$$

$$\left[\int_{\mathbf{K^n}} \exp(i\langle(y, x)_{\mathbf{K}}\rangle_{\mathbf{F}})\lambda(dx)\right]^k,$$

where $\rho(t)$ has the characteristic function $\exp(it(v, \langle y\rangle_{\mathbf{F}})_{\mathbf{R}})$.

**17. Remark.** Let a branching random process be realized with values in the ring $\mathbf{Z_p}$ of integer $p$-adic numbers or in the ring $B(\mathbf{F_p}(\theta), 0, 1)$, denote it by $\mathbf{B}$. In the particular case of the uniform distribution $|x|^{-2}\nu(dx)$ in $\mathbf{B}$ the measure $\nu$ is proportional to the Haar measure $\mu$, $|x|^{-2}\nu(dx) = q\mu(dx)$, where $q > 0$, $\mu(\mathbf{B}) = 1$, $\nu(\mathbf{F}\setminus\mathbf{B}) = 0$, $\mathbf{K} = \mathbf{F} = \mathbf{Q_p}$ or $\mathbf{K} = \mathbf{F} = \mathbf{F_p}(\theta)$, respectively here, $n = 1$. Then it is possible to calculate $A(y)$ and $B(y)$. In view of 5 in Section 2 in this particular case $A(y) = q\int_{\mathbf{B}} \langle yx\rangle_{\mathbf{F}}\mu(dx)$ and $B(y) = q\int_{\mathbf{B}} \langle yx\rangle_{\mathbf{F}}^2\mu(dx)$. If $y = 0$, then $A(0) = 0$ and $B(0) = 0$, therefore, consider the case $y \neq 0$. The function $\langle yx\rangle_{\mathbf{F}}$ takes the zero value when $|yx|_{\mathbf{F}} \leq 1$ and is different from zero when $|x|_{\mathbf{F}} > 1/|y|_{\mathbf{F}}$.

In the considered case the support of the measure $\nu$ is contained in $\mathbf{B}$, then $A(y)$ and $B(y)$ are equal to zero when $|y|_{\mathbf{F}} \leq 1$. But the Haar measure is invariant relative to shifts $\mu(A+z) = \mu(A)$ for each Borel subset in $\mathbf{F}$ with the finite measure $\mu(A) < \infty$ and each $z \in \mathbf{F}$. Moreover, $\mu(zdx) = |z|_{\mathbf{F}}\mu(dx)$, where $|z|_{\mathbf{F}} = p^{-\mathrm{ord}_{\mathbf{F}}(z)}$ (see [31]). Then

$$A(y) = q\int_{z\in\mathbf{F}, |y|_{\mathbf{F}} \geq |z|_{\mathbf{F}} > 1} \langle z\rangle_{\mathbf{F}}\mu(dz)/|y|_{\mathbf{F}}$$

and

$$B(y) = q\int_{z\in\mathbf{F}, |y|_{\mathbf{F}} \geq |z|_{\mathbf{F}} > 1} \langle z\rangle_{\mathbf{F}}^2\mu(dz)/|y|_{\mathbf{F}},$$

where $|y|_{\mathbf{F}} > 1$. At the same time $z = \sum_{k=N(x)}^{\infty} z_k p^k$ for $\mathbf{F} = \mathbf{Q_p}$ or $z = \sum_{k=N(x)}^{\infty} z_k \theta^k$ for $\mathbf{F} = \mathbf{F_p}(\theta)$, where $N(z) = \mathrm{ord}_p(z)$, $z_k \in \{0, 1, \ldots, p-1\}$ or $z_k \in \mathbf{F_p}$. If $\nu(dx) = q\mu(dx)$, then

$$A(y) = q|y|_{\mathbf{F}}\int_{z\in\mathbf{F}, |y|_{\mathbf{F}} \geq |z|_{\mathbf{F}} > 1} \langle z\rangle_{\mathbf{F}}|z|_{\mathbf{F}}^{-2}\mu(dz)$$

and

$$B(y) = q|y|_{\mathbf{F}}\int_{z\in\mathbf{F}, |y|_{\mathbf{F}} \geq |z|_{\mathbf{F}} > 1} \langle z\rangle_{\mathbf{F}}^2|z|_{\mathbf{F}}^{-2}\mu(dz).$$

These integrals are expressible in the form of finite sums, since $\mu(B(\mathbf{F}, x, p^k)) = p^k$ for each $k \in \mathbf{Z}$ and $z \in \mathbf{F}$, where the functions in the integrals are locally constant.

The measure $\nu$ is Borelian, $\nu : \mathcal{B}(\mathbf{K^n}) \to [0, \infty)$, therefore each of its atom may be only a singleton. More generally (see Formulas 5(i, 13, 14)), if $\nu = \nu_1 + \nu_2$, where $\nu_2$ is the atomic measure, while $\nu_1(dx) = f(x)\mu(dx)$, where

$$f(x) = g(|x|_{\mathbf{F}}, \langle x \rangle_{\mathbf{F}}),\ g : \mathbf{R}^2 \to [0, \infty)$$

is a continuous function, then

$$A(y) = \sum_j \langle yx_j \rangle_{\mathbf{F}} |x_j|^{-2} \nu_2(\{x_j\}) + \int_{\mathbf{F}} \langle yx \rangle_{\mathbf{F}} f(x) |x|_{\mathbf{F}}^{-2} \mu(dx),$$

$$B(y) = \sum_j \langle yx_j \rangle_{\mathbf{F}}^2 |x_j|^{-2} \nu_2(\{x_j\}) + \int_{\mathbf{F}} \langle yx \rangle_{\mathbf{F}}^2 f(x) |x|_{\mathbf{F}}^{-2} \mu(dx),$$

where $\{x_j\}$ are atoms of the measure $\nu_2$, $\nu_2(\{x_j\}) > 0$, each $x_j \neq 0$ is nonzero. At the same time integrals by the Haar measure $\mu$ on $\mathbf{F}$ with functions $\langle yx \rangle_{\mathbf{F}} f(x) |x|_{\mathbf{F}}^{-2}$ and $\langle yx \rangle_{\mathbf{F}}^2 f(x) |x|_{\mathbf{F}}^{-2}$, where $f(x) = g(|x|_{\mathbf{F}}, \langle x \rangle_{\mathbf{F}})$, are expressible in the form of series, since $|x|_{\mathbf{F}}$ and $\langle x \rangle_{\mathbf{F}}$ are locally constant, hence $f$ is locally constant.

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