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ON SOME NUMERICAL RADIUS INEQUALITIES AND BOUNDS FOR THE ZEROS OF POLYNOMIALS

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Abstract

It is shown that, if $A \in M_n$, then

$$\frac{\|A^2\|^{\frac{1}{2}}}{\sqrt{2}} \leq w(A) \leq \sqrt{\frac{1}{2}(\|A^2\| + \|A\|^2)},$$

and also that, if A and B are in M_n, then

$$\begin{aligned} 2\|AB\| &\leq \|A^*A + BB^*\| \\ &\leq \frac{1}{2}[\|A\|^2 + \|B\|^2 + \sqrt{(\|A\|^2 - \|B\|^2)^2 + 4\|AB\|^2}], \end{aligned}$$

where $w(\cdot)$ and $\|\cdot\|$ are the numerical radius and the usual operator norm, respectively. We apply the numerical radius inequality to derive new bounds for the zeros of these polynomials.

1. Introduction

In this paper, we are concerned with the problem of locating the zeros of polynomials by employing the numerical radius inequality. Numerical radii estimate of companion matrices have been invoked by Linden [10]

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and Kittaneh [7, 9]. In addition, it has been shown by Kittaneh [9], that if $A \in M_n(\mathbb{C})$, then

$$\frac{1}{4}\|AA^* + A^*A\| \le w^2(A) \le \frac{1}{2}\|AA^* + A^*A\|.$$

In addition, we know that

$$\frac{\| A \|}{2} \le w(A) \le \| A \|.$$

In this paper, it is shown that, if $A \in M_n(\mathbb{C})$, then

$$\frac{\|A^2\|^{\frac{1}{2}}}{\sqrt{2}} \le w(A) \le \sqrt{\frac{1}{2}(\|A^2\| + \| A \|^2)},$$

and it shows that, if A and B are in $M_n(\mathbb{C})$, then

$$\begin{aligned} 2\| AB \| &\le \|A^*A + BB^*\| \\ &\le \frac{1}{2}[\| A \|^2 + \| B \|^2 + \sqrt{(\| A \|^2 - \| B \|^2)^2 + 4\| AB \|^2}], \end{aligned}$$

where $w(\cdot)$ and $\|\cdot\|$ are the numerical radius and the usual operator norm, respectively.

We apply the numerical radius inequality to the Frobenius companion matrices of monic polynomials to derive new bounds for the zeros of these polynomials.

In this work, let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices.

Definitions 1.1 [3]**.** If $A \in M_n(\mathbb{C})$, then:

(I) The *spectral norm* (or the *operator norm*) is defined by

$$\| A \| = \max\left\{\frac{\| Ax \|}{\| x \|} : \| x \| \ne 0\right\}, \quad \| A \| = \max\{\| Ax \| : \| x \| = 1\}. \tag{1}$$

(II) The *numerical radius* of A is defined by

$$w(A) = \max\{| \langle Ax, x\rangle | : x \in \mathbb{C}^n, \| x \| = 1\}. \tag{2}$$

Now, we list some known results as the pre-requisite.

Lemma 1.1 [13]. *If A and B are in* $M_n(\mathbb{C})$, *then*

$$2s_j(A^*B) \le s_j(AA^* + BB^*) \quad \textit{for } 1 \le j \le n, \tag{3}$$

which is, Zhan's inequality (see [13] *and* [1]).

Lemma 1.2 [3]. *If* $A \in M_n(\mathbb{C})$, *then*

$$\frac{\| A \|}{2} \le w(A) \le \| A \|. \tag{4}$$

Lemma 1.3 [7]. *If* $A \in M_n(\mathbb{C})$, *then*

$$w(A) \le \frac{1}{2}(\| A \| + \| A^2 \|^{\frac{1}{2}}). \tag{5}$$

Lemma 1.4 [9]. *If* $A \in B(H)$, *then*

$$\frac{1}{4}\| AA^* + A^*A \| \le w^2(A) \le \frac{1}{2}\| AA^* + A^*A \|. \tag{6}$$

Lemma 1.5 [6]. *If A and B are positive operators, then*

$$\| A + B \| \le \frac{1}{2}[\| A \| + \| B \| + \sqrt{(\| A \| - \| B \|)^2 + 4\| A^{\frac{1}{2}} B^{\frac{1}{2}} \|^2}]. \tag{7}$$

Lemma 1.6 [11]. *If* $X = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ *is positive, then*

$$2s_j(C) \le s_j(X) \quad \textit{for } 1 \le j \le n. \tag{8}$$

Lemma 1.7 [5]. *Let*

$$C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

be a companion matrix of

$$p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1 \tag{9}$$

with $a_i \in \mathbb{C}$ *for* $1 \le i \le n$ *and* $a_1 \ne 0$. *Then*

$$\| C(p) \| = \sqrt{\frac{(\alpha + 1) + \sqrt{(\alpha + 1)^2 - 4|a_1|^2}}{2}}, \tag{10}$$

where $\alpha = \sum_{j=1}^{n} |a_j|^2$.

Lemma 1.8 [8]. *If A and B are in* $M_n(\mathbb{C})$, *then*

$$\|A^* A + BB^*\| \le \max\{\| A \|^2, \| B \|^2\} + \| AB \|, \tag{11}$$

which is, *Kittaneh's inequality* (*see* [8]).

2. Main Results

Now we state Lemma 1 in another form as follows:

Lemma 2.1. *If A and B are in* $M_n(\mathbb{C})$, *then*

$$2s_j(AB) \le s_j(AA^* + BB^*) \quad \text{for } 1 \le j \le n. \tag{12}$$

Proof. Let $M = \begin{bmatrix} A^* & B \\ 0 & 0 \end{bmatrix}$. Then

$$X = MM^* = \begin{bmatrix} A^* A + BB^* & 0 \\ 0 & 0 \end{bmatrix}, Y = M^* M = \begin{bmatrix} AA^* & AB \\ (AB)^* & B^* B \end{bmatrix}.$$

From the inequality (8), we have

$$2s_j(AB) \le s_j(X) = s_j(Y) = s_j(AA^* + BB^*) \text{ for } 1 \le j \le n.$$

Lemma 2.2. *If A and B are in* $M_n(\mathbb{C})$, *then*

$$\|A^* A + BB^*\| \le \frac{1}{2} [\| A \|^2 + \| B \|^2 + \sqrt{(\| A \|^2 - \| B \|^2)^2 + 4\| AB \|^2}]. \tag{13}$$

Proof. The desired inequality (13) follows from the inequality (7), by substituting $A^* A$ instead of A and BB^* instead of B, respectively.

The inequality (13) is sharper than the inequality (11), because

$$\frac{1}{2}[\| A \|^2 + \| B \|^2 + \sqrt{(\| A \|^2 - \| B \|^2)^2 + 4\| AB \|^2}]$$

$$\leq \frac{1}{2}[\| A \|^2 + \| B \|^2 + |(\| A \|^2 - \| B \|^2)|] + \| AB \|$$

$$\leq \max\{\| A \|^2, \| B \|^2\} + \| AB \|.$$

Corollary 2.3. *Let* $A, B \in M_n(\mathbb{C})$ *be positive semidefinite. Then*

$$\| A^2 + B^2 \| \leq \frac{1}{2}[\| A \|^2 + \| B \|^2 + \sqrt{(\| A \|^2 - \| B \|^2)^2 + 4\| AB \| \| BA \|}]. \quad (14)$$

Corollary 2.4. *Letting* $A = B \in M_n(\mathbb{C})$ *in the inequality* (13), *we have*

$$\| A^*A + AA^* \| \leq \| A \|^2 + \| A^2 \|, \quad (15)$$

which is, Kittaneh's inequality see [8].

Theorem 2.5. *If* $A \in M_n(\mathbb{C})$, *then*

$$\frac{\| A^2 \|^{\frac{1}{2}}}{\sqrt{2}} \leq w(A) \leq \sqrt{\frac{1}{2}(\| A^2 \| + \| A \|^2)}. \quad (16)$$

Proof. The right hand side of the inequality (16) follows from the inequalities (6) and (15).

The left hand side of the inequality (16) follows from the inequality (12).

The inequality (16) is sharper than the inequality (4), because

$$\sqrt{\frac{1}{2}(\| A^2 \| + \| A \|^2)} \leq \sqrt{\frac{1}{2}(\| A \|^2 + \| A \|^2)} \leq \| A \|,$$

and $\| A^2 \| \leq \| A \|^2$.

Also

$$\frac{\| A^2 \|^{\frac{1}{2}}}{\sqrt{2}} \geq \frac{\| A \|}{2}, \text{ if } \| A^2 \| \geq \frac{\| A \|^2}{2}.$$

Now, we use previous theorem to present new bounds for the zeros of polynomials as follows:

Theorem 2.6. *If z is a zero of*

$$p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \cdots + a_2 z + a_1$$

with $a_i \in \mathbb{C}$ for $1 \leq i \leq n$ and $a_1 \neq 0$, then

$$|z| \leq \frac{1}{\sqrt{2}} \sqrt{\frac{(\alpha+1)+\sqrt{(\alpha+1)^2 - 4|a_1|^2}}{2} + \sqrt{\frac{(\delta+1)+\sqrt{(\delta-1)^2 + 4\delta'}}{2}}}, \quad (17)$$

where

$$\delta = \frac{1}{2}[(\alpha + \beta) + ((\alpha - \beta)^2 + 4|\zeta|^2)^{\frac{1}{2}}],$$

and

$$\alpha = \sum_{j=1}^{n} |a_j|^2, \quad \beta = \sum_{j=1}^{n} |L_j|^2, \quad \zeta = -\sum_{j=1}^{n} \overline{a_j} L_j, \quad L_j = a_n a_j - a_{j-1}$$

for $1 \leq j \leq n$, with $a_0 = 0$ and

$$\delta' = \frac{1}{2}[(\alpha' + \beta') + ((\alpha' - \beta')^2 + 4|\zeta'|^2)^{\frac{1}{2}}],$$

where

$$\alpha' = \sum_{j=3}^{n} |a_j|^2, \quad \beta' = \sum_{j=3}^{n} |L_j|^2, \quad \zeta = \zeta' = -\sum_{j=1}^{n} \overline{a_j} L_j.$$

Proof. The desired inequality (17) follows from the inequality (16)

$$w(C(p)) \leq \frac{1}{2}\sqrt{(\|(C(p))^2\| + \|C(p)\|^2)},$$

where $C(p)$ is the companion matrix of $p(z)$, such that

$$\|(C(p))^2\| \leq \sqrt{\frac{(\delta+1)+\sqrt{(\delta-1)^2 + 4\delta'}}{2}},$$

and

$$\| C(p) \| = \sqrt{\frac{(\alpha + 1) + \sqrt{(\alpha + 1)^2 - 4| a_1 |^2}}{2}},$$

where

$$\delta = \frac{1}{2}[(\alpha + \beta) + ((\alpha - \beta)^2 + 4| \zeta |^2)^{\frac{1}{2}}],$$

and

$$\alpha = \sum_{j=1}^{n} |a_j|^2, \quad \beta = \sum_{j=1}^{n} |L_j|^2, \quad \zeta = -\sum_{j=1}^{n} \overline{a_j} L_j, \quad L_j = a_n a_j - a_{j-1},$$

for $1 \leq j \leq n,$ with $a_0 = 0$ and

$$\delta' = \frac{1}{2}[(\alpha' + \beta') + ((\alpha' - \beta')^2 + 4| \zeta' |^2)^{\frac{1}{2}}],$$

where

$$\alpha' = \sum_{j=3}^{n} |a_j|^2, \quad \beta' = \sum_{j=3}^{n} |L_j|^2,$$

and

$$\zeta = \zeta' = -\sum_{j=1}^{n} \overline{a_j} L_j,$$

(see [5], and [7]).

3. Open Problems

The first open problem is possible to complement the upper bound (6) by giving an upper bound estimate for the zeros of p.

The second open problem is possible to complement the upper bound (6) or the bound (16) by giving a lower bound estimate for the zeros of p. To see this, observe that the zeros of the polynomial

$$q(z) = \frac{z^n}{a_1} p\left(\frac{1}{z}\right)$$

are the reciprocals of those of p. Thus, applying the upper bound (6) or the bound (16) to the zeros of q yields the desired lower bound estimate for the zeros of p. This enables us to present a new annulus containing the zeros of p.

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