Far East Journal of Mathematical Sciences (FJMS)
Volume 32, Issue 3, 2009, Pages 347-357
Published online: March 28, 2009
This paper is available online at http://www.pphmj.com

LOCAL COMMUTATIVE RESIDUATED LATTICES

MICHIRO KONDO

School of Information Environment
Tokyo Denki University
Inzai, 270-1382, Japan
e-mail: kondo@sie.dendai.ac.jp

Abstract

In this paper, we prove general properties of local commutative residuated lattices, which are extended results proved in [5]:

(1) For a commutative residuated lattice L, it is local if and only if $ord(x) < \infty$ or $ord(x^*) < \infty$ for all $x \in L$.

(2) If D is a perfect deductive system and $D \subseteq F$ for a deductive system F, then F is also a perfect deductive system.

(3) For a deductive system D, it is an ultra ds if and only if it is a maximal and perfect ds.

As a consequence, our results hold at least for the local BL-algebras, local ML algebras, local MTL algebras, local MV algebras and so on.

1. Introduction

BL-algebras were invented by Hájek [2] in order to prove the completeness theorem of basic fuzzy logic, BL-logic in short. Soon after Cignoli et al. [1] proved that Hájek's logic really is the logic of continuous t-norms as conjectured by Hájek. At the same time started a systematic study of BL-algebras, too. Indeed, Turunen [4] published where

2000 Mathematics Subject Classification: 03G25, 03B52, 06D35.

Keywords and phrases: local, commutative residuated lattices.

This work was supported by Tokyo Denki University Science Promotion Fund (Q08J-06).

Received December 7, 2008

BL-algebras were studied by deductive systems. Deductive systems correspond to subsets closed with respect to Modus Ponens and they are called *filters*, too. In [4], Boolean deductive systems and implicative deductive systems were introduced.

On the other hand, some fundamental results about local BL-algebras are obtained by Turunen and Seesa in [5]. They proved that, for example,

(1) Every linear BL-algebra is local and, since every BL-algebra is isomorphic to a subdirect product of linear (totally ordered) BL-algebras.

(2) Any BL-algebra is isomorphic to a subdirect product of local BL-algebras.

This means that for every equation $s = t$, it holds in any BL-algebra if and only if it does in any local BL-algebra. Thus it is very important to investigate properties of local BL-algebras to develop the theory of BL-algebras. In general, a class $\mathcal{V}$ of algebras is called *representable* if any member of $\mathcal{V}$ is isomorphic to a subdirect product of totally ordered members of $\mathcal{V}$. From this point of view, we can say that the class $\mathcal{BL}$ of all BL-algebras is representable. Thus it is worth thinking about properties of representable algebras. Considering deeply proofs of results in [5], we see that many results hold for more weaker algebras such as commutative residuated lattices, although the class $\mathcal{CRL}$ of all commutative residuated lattices is not representable. In this paper we prove general properties of local commutative residuated lattices.

(1) For a commutative residuated lattice L, it is local if and only if $ord(x) < \infty$ or $ord(x^*) < \infty$ for all $x \in L$.

(2) If D is a perfect deductive system (ds) and $D \subseteq F$ for a deductive system F, then F is also a perfect deductive system.

(3) For a ds D, it is an ultra ds if and only if it is a maximal and perfect ds.

As a consequence, our results hold at least for the local BL-algebras, local ML algebras, local MTL algebras, local MV algebras and so on, because such algebras are axiomatic extensions of commutative residuated lattices.

2. Preliminaries

At first we recall the definition of commutative residuated lattices. By a *commutative residuated lattice* (CRL), we mean an algebraic structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$, where

(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;

(2) $(L, \odot, 1)$ is a commutative monoid with a unit element 1;

(3) For all $x, y, z \in L$, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

Let L be a CRL. For any element $x \in L$, we define $x^* = x \rightarrow 0$ and x^n by $x^0 = 1$, $x^{n+1} = x^n \odot x$. If there is a non-negative integer n such that $x^n = 0$, then we denote the smallest such integer by $ord(x)$. If there is no such integer we define $ord(x) = \infty$. Let $D(L)$ be the set of all elements x such that $ord(x) = \infty$, that is,

$$D(L) = \{x \in L \,|\, ord(x) = \infty\} = \{x \in L \,|\, x^n > 0 \ (\forall n)\}.$$

Proposition 1. *Let L be a CRL. Then for all* $x, y, z \in L$, *we have*

(a) $x \rightarrow (y \rightarrow z) = x \odot y \rightarrow z$,

(b) $x \odot (x \rightarrow y) \leq y$,

(c) $x \leq y \Rightarrow y^* \leq x^*$,

(d) $x \leq x^{**}$, $x^* = x^{***}$.

A non-empty subset $D \subseteq L$ is called a *deductive system* (ds) if:

(1) If $x, y \in D$, then $x \odot y \in D$.

(2) If $x \in D$ and $x \leq y$, then $y \in D$.

It is easy to prove that for a non-empty subset D of a commutative residuated lattice L, it is a deductive system if and only it satisfies the conditions:

(ds1) $1 \in D$ and

(ds2) If $x, x \to y \in D$, then $y \in D$.

For any non-empty subset $S \subseteq L$, we define a subset

$$\langle S \rangle = \{x \in L \mid \exists s_i \in S;\ s_1 \odot \cdots \odot s_n \leq x\}.$$

Proposition 2. *For any non-empty subset $S \subseteq L$, $\langle S \rangle$ is the least deductive system including S.*

Proof. It is obvious that $S \subseteq \langle S \rangle$ and $\langle S \rangle$ is upwards closed, that is, $x \in \langle S \rangle$ and $x \leq y$ imply $y \in \langle S \rangle$.

For all $x, y \in \langle S \rangle$, there are $s_i, s'_j \in S$ such that $s_1 \odot \cdots \odot s_n \leq x$ and $s'_1 \odot \cdots \odot s'_m \leq y$. Since $s_1 \odot \cdots \odot s_n \odot s'_1 \odot \cdots \odot s'_m \leq x \odot y$, we have $x \odot y \in \langle S \rangle$.

Let D be a deductive system such that $S \subseteq D$. For any $x \in \langle S \rangle$, there exists $s_i \in S$ such that $s_1 \odot \cdots \odot s_n \leq x$. Since $s_i \in S \subseteq D$ and D is the deductive system, we have $s_1 \odot \cdots \odot s_n \in D$ and thus $x \in D$. This means that $\langle S \rangle \subseteq D$. Hence $\langle S \rangle$ is the least deductive system including S. □

Let L be a CRL. It is called *local* if there exists a unique maximal deductive system [5]. It is also called *locally finite* if every non-unit element has a finite order, that is, $ord(x) < \infty$ for all $x \neq 1$. Moreover it is called *perfect* if $ord(x) < \infty$ if and only if $ord(x^*) = \infty$ for all $x \in L$.

Lemma 1. *For every proper ds D of L, we have $D \subseteq D(L)$.*

Proof. For every $x \in D$, since D is a proper ds and $x^n \in D$ for every n, we have $x^n \neq 0$. This means that $x \in D(L)$ and hence $D \subseteq D(L)$. □

Lemma 2. *For every proper ds D, there is a maximal ds M including D.*

Proof. Let D be a proper ds. If we take $\Gamma = \{D_\lambda \mid D \subseteq D_\lambda, D_\lambda \text{ is a proper ds}\}$, since $D \in \Gamma$, then we have $\Gamma \neq \varnothing$. It is easy to prove that Γ

has a maximal element M by Zorn's lemma. That element M is a maximal ds containing D. □

Proposition 3. *Let L be a CRL. Then the following conditions are equivalent*:

(1) *L is local.*

(2) *Unique maximal ds is $D(L)$.*

(3) *$\langle D(L)\rangle$ is a proper ds.*

(4) *$D(L)$ is a ds.*

Proof. (1) $\Rightarrow$ (2) If L is local, then there is unique maximal ds M by definition. We take a subset $D_x = \langle x\rangle = \{u \in L \mid x^n \leq u \text{ for some } n\}$ for all $x \in D(L)$. Since $0 \notin D_x$ and D_x is a proper ds, there is a maximal ds containing D_x. It follows from supposition that it is exactly M. Thus $x \in D_x \subseteq M$ and $x \in M$ for all $x \in D(L)$. This means that $D(L) \subseteq M$. On the other hand, since M is the proper ds, it follows that $M \subseteq D(L)$. Thus we have $M = D(L)$.

(2) $\Rightarrow$ (3) Since $D(L)$ is proper, it follows that $0 \notin D(L) = \langle D(L)\rangle$ and hence that $\langle D(L)\rangle$ is the proper ds.

(3) $\Rightarrow$ (4) Suppose that $\langle D(L)\rangle$ is a proper ds. We have $\langle D(L)\rangle \subseteq D(L)$ by Lemma 1. On the other hand, it is obvious that $D(L) \subseteq \langle D(L)\rangle$. Thus we get $\langle D(L)\rangle = D(L)$ and $D(L)$ is the deductive system.

(4) $\Rightarrow$ (1) Let $D(L)$ be a ds. Since $D \subseteq D(L)$ for every proper ds D, this implies that $D(L)$ is the greatest proper ds. Hence L has a unique maximal ds. □

Theorem 1. *Let L be a CRL. Then L is local if and only if $ord(x) < \infty$ or $ord(x^*) < \infty$ for all $x \in L$.*

Proof. Suppose that L is a local commutative residuated lattice. If $ord(x) = ord(x^*) = \infty$ for some $x \in L$, then we have $x, x^* \in D(L) \subseteq$

$\langle D(L)\rangle$ and $0 = x \odot x^* \in \langle D(L)\rangle$. But, since L is local, this contradicts to the fact that $\langle D(L)\rangle$ is the proper ds. Thus we have $ord(x) < \infty$ or $ord(x^*) < \infty$ for all $x \in L$.

Conversely, we assume that $ord(x) < \infty$ or $ord(x^*) < \infty$ for all $x \in L$. It is enough to show that $\langle D(L)\rangle$ is the proper ds. If $0 \in \langle D(L)\rangle$, then there exists $x_i \in D(L)$, such that $x_1 \odot \cdots \odot x_n = 0$. Thus, $x_1 \odot \cdots \odot x_{n-1} \leq x_n^*$. Since $x_n \in D(L)$, it has an infinite order. It follows from assumption that $ord(x_n^*) = m_n < \infty$ for some m_n. This implies that $x_1^{m_n} \odot \cdots \odot x_{n-1}^{m_n} = (x_1 \odot \cdots \odot x_{n-1})^{m_n} \leq (x_n^*)^{m_n} = 0$ and hence that $x_1^{m_n} \odot \cdots \odot x_{n-1}^{m_n} = 0$. We also have $x_1^{m_n} \odot \cdots \odot x_{n-2}^{m_n} \leq (x_{n-1}^{m_n})^*$. Since $x_{n-1} \in D(L)$, we get $x_{n-1}^{m_n} \in D(L)$. This means that $x_{n-1}^{m_n}$ has an infinite order and thus $ord((x_{n-1}^{m_n})^*) = m_{n-1} < \infty$ for some m_{n-1}. We have $x_1^{m_n m_{n-1}} \odot \cdots \odot x_{n-2}^{m_n m_{n-1}} \leq [(x_{n-1}^{m_n})^*]^{m_{n-1}} = 0$. Iterating this process we have $x_1^{m_n m_{n-1} \cdots m_2} = 0$ for some $m_n m_{n-1}, \ldots, m_2$. This means that x_1 has the finite order. But this is a contradiction. Hence we have $0 \notin \langle D(L)\rangle$. □

For any ds D, we define a relation induced by D as follows [2, 5]:

$$x \sim_D y \text{ if and only if } (x \to y) \odot (y \to x) \in D.$$

It is easy to prove that the relation $\sim_D$ is a congruence on L and the quotient set $L/D = \{x/D \mid x \in L\}$ defined by $\sim_D$ is a commutative residuated lattice, where $x/D = \{y \in L \mid x \sim_D y\}$ and $(x/D) \circ (y/D) = (x \circ y)/D$ for $\circ \in \{\wedge, \vee, \to, \odot\}$. Moreover from any congruence θ on L we can construct a ds D_θ by

$$D_\theta = \{x \mid (x, 1) \in \theta\}.$$

Thus there is a lattice isomorphism between the class of all deductive systems and the class of all congruences on L.

For proper deductive systems D and F such that $D \subseteq F$, we can also define a relation $\sim_{F/D}$ on L/D as follows: For x/D, $y/D \in L/D$,

$$x/D \sim_{F/D} y/D \Leftrightarrow x \sim_F y.$$

It follows from the general theory of universal algebras that

Proposition 4. *For proper deductive systems D, F of L, we have*

(1) $\sim_{F/D}$ *is a congruence relation on* L/D;

(2) $(L/D)/(F/D) \cong L/F$.

Proposition 5. *Let L be a local commutative residuated lattice and F be a proper ds. We have*

$$D(L/F) = D(L)/F.$$

Proof. Suppose that there is $x/F \in L/F$ such that $x/F \in D(L/F)$ but $x/F \notin D(L)/F$. It follows from $x/F \in D(L/F)$ that $ord(x/F) = \infty$. Since $x/F \notin D(L)/F$, we also have $x \notin D(L)$ and hence $x^n = 0$ for some n. This means that $(x/F)^n = 0$ and $ord(x/F) < \infty$. But this is a contradiction. We conclude that $D(L/F) \subseteq D(L)/F$. The converse relation can be proved similarly. Thus we have $D(L)/F = D(L/F)$. □

In [5], a deductive system P is called *primary* when it satisfies the condition:

For all $x, y \in L$, if $(x \odot y)^* \in P$, then $(x^n)^* \in P$ or $(y^n)^* \in P$ for some n.

It is easy to prove the next result for commutative residuated lattices as in the case of BL-algebras in [5].

Proposition 6. *Let L be a CRL and P be a proper ds. Then we have P is primary if and only if L/P is local.*

Corollary 1. *For a proper ds P, we have P is primary if and only if for every x there exists a positive integer n such that $(x^n)^* \in P$ or $((x^*)^n)^* \in P$.*

Proof. Suppose that P is a primary ds. It follows from $(x \odot x^*)^* = 0^* = 1 \in P$ that there exists a positive integer n such that $(x^n)^* \in P$ or $((x^*)^n)^* \in P$.

Conversely, if $(x \odot y)^* \in P$, since $(x \odot y)^* = x \to y^* = y \to x^* \in P$, then we have $y/P \leq x^*/P$. If $(x^n)^* \notin P$ for all n, since $(x^n)^*/P \neq 1/P$, then $x^n/P \neq 0/P$ for all n. That is, $ord(x/P) = \infty$. Since L/P is local, we have $(x^*/P)^m = 0/P$ for some m. This implies that $(y/P)^m \leq (x^*/P)^m = 0/P$ and $y^m/P = 0/P$. It follows that $(y^m)^* \in P$ for some m. This shows that P is primary. □

Proposition 7. *Every CRL-chain (i.e., totally ordered commutative residuated lattice) is local.*

Proof. Let L be a CRL-chain and $x \in L$. Since L is the chain, we have $x \leq x^*$ or $x^* \leq x$. In the first case, it follows that $x^2 = x \odot x \leq x \odot x^* = 0$ and thus $x^2 = 0$. This means that $ord(x) < \infty$. For the case of $x^* \leq x$, since $(x^*)^2 = x^* \odot x^* \leq x \odot x^* = 0$ and thus $ord(x^*) \leq \infty$.

Thus every CRL-chain is local. □

A proper ds $D \subseteq L$ is called *perfect* if, for all $x \in L$, $(x^n)^* \in D$ for some n if and only if $[(x^m)^*]^* \notin D$ for all m. It is proved in [5] that, for a perfect ds D of a BL-algebra L, D is perfect if and only if L/D is a perfect BL-algebra. The results hold for the case of commutative residuated lattice.

Proposition 8. *Let L be a CRL and D be a ds. D is perfect if and only if L/D is a perfect commutative residuated lattice.*

Proof. The proof comes from the following sequence of statements:

L/D is perfect $\Leftrightarrow$ $ord(x/D) < \infty$ iff $ord(x^*/D) = \infty$

$\Leftrightarrow x^n/D = 0/D$ for some n iff $(x^*)^m/D \neq 0/D$ for all m

$\Leftrightarrow (x^n)^* \in D$ for some n iff $[(x^*)^m]^* \notin D$ for all m

$\Leftrightarrow D$ is perfect. □

Proposition 9. *If L is a perfect commutative residuated lattice, then any ds D of L is a perfect ds.*

Proof. For any $x/D \in L/D$, suppose $ord(x/D) < \infty$. There is an n such that $(x/D)^n = 0/D$ and that $(x^n)^* \in D$. Then we have $[(x^*)^m]^* \notin D$ for all m by assumption. This means that $((x^*)^m/D)^* \neq 1/D$ and that $(x^*/D)^m \neq 0/D$ for all m. This indicates that $ord(x^*/D) = \infty$. The converse can be proved similarly. We can conclude that L/D is perfect and hence that D is the perfect ds. □

Moreover we can show that

Theorem 2. *If D is a perfect ds and* $D \subseteq F$ *for a ds F, then F is a perfect ds.*

Proof. Suppose that D is a perfect ds and $D \subseteq F$ for a ds F. Since L/D is a perfect commutative residuated lattice, any ds of L/D is perfect. Moreover it is clear that any ds of L/D has a form S/D for some ds S such that $D \subseteq S$. This means that F/D is a ds of L/D and L/F is the perfect commutative residuated lattice by $(L/D)/(F/D) \cong L/F$. Thus F is the perfect ds. □

We define a new kind of deductive system, called *ultra ds* according to the theory of Boolean algebras. A deductive system D is called *ultra* if $x \in D$ or $x^* \in D$ for all $x \in L$. As to the ultra deductive systems we have the following result.

Proposition 10. *Let D be a proper ds. D is an ultra ds if and only if* $L/D \cong \{0, 1\}$.

Proof. For any $x/D \in L/D$, since $x \in L$, we have $x \in D$ or $x^* \in D$.

This implies $x/D = 1/D$ or $x^*/D = 1/D$. If $x^*/D = 1$, since $x/D \leq (x/D)^{**} = (1/D)^* = 0/D$, then we have $x/D = 0/D$. Thus we have $x/D = 1/D$ or $x/D = 0/D$ for all $x/D \in L/D$ and thus $L/D \cong \{0, 1\}$.

Conversely, suppose that $L/D \cong \{0, 1\}$. Since $x/D \in L/D \cong \{0, 1\}$, $x/D = 1/D$ or $x/D = 0/D$. Thus we get that $x \in D$ or $x^* \in D$ and hence that D is the ultra ds. □

We have a characterization of ultra deductive systems as follows.

Theorem 3. *Let D be a deductive system. Then D is an ultra ds if and only if it is a maximal perfect ds.*

Proof. Suppose that D is an ultra ds. If there is a ds F such that it contains D properly, then there is an element $a \in F$ such that $a \notin D$. Since D is the ultra ds, we have $a^* \in D \subset F$. It follows that $0 = a \odot a^* \in F$. This implies that D is the maximal ds. Moreover, since $L/D \cong \{0, 1\}$ and $\{0, 1\}$ is a perfect commutative residuated lattice, L/D is the perfect commutative residuated lattice, which means that D is perfect.

Conversely, suppose that D is a maximal and perfect ds. Since D is maximal, L/D is locally finite and thus $ord(x/D) < \infty$ for all $x/D \neq 1/D$. Since L/D is perfect, this means that $ord(x^*/D) = \infty$ and $x^*/D = 1/D$. We get $x^* \in D$ for all $x/D \neq 1/D$ and thus $x \in D$ or $x^* \in D$. This indicates that D is the ultra ds. □

Lemma 3. *Let L be a local commutative residuated lattice. Then $D(L)$ is an ultra ds if and only if $\{1\}$ is a perfect ds, that is, L is a perfect commutative residuated lattice.*

Proof. Suppose that $D(L)$ is an ultra ds. We have the following sequences of equivalent equations:

$$(x^n)^* = 1 \text{ for some } n \Leftrightarrow x^n = 0 \text{ for some } n$$
$$\Leftrightarrow x \notin D(L)$$

$$\Leftrightarrow x^* \in D(L)$$

$$\Leftrightarrow (x^*)^m \in D(L) \text{ for all } m$$

$$\Leftrightarrow [(x^*)^m]^* \notin D(L) \text{ for all } m.$$

This means that $\{1\}$ is the perfect ds.

Conversely, if $\{1\}$ is the perfect ds, since $\{1\} \subseteq D(L)$, then $D(L)$ is the perfect ds. It is obvious that $D(L)$ is maximal. Thus we have $D(L)$ is the ultra ds. □

As proved in the case of BL-algebras (Proposition 22 in [5]), we have a similar result in the case of commutative residuated lattice.

Theorem 4. *Let* L *be a CRL. Then the following conditions are equivalent*:

(a) L *is a perfect commutative residuated lattice*.

(b) $\{1\}$ *is a perfect ds*.

(c) $D(L)$ *is an ultra ds*.

(d) $L/D \cong \{0, 1\}$.

References

[1] R. Cignoli, F. Esteva, L. Godo and A. Torrens, Basic fuzzy logic is the logic of continuous t-norm and their residua, Soft Computing 4 (2000), 106-112.

[2] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, 1998.

[3] M. Kondo and W. A. Dudek, Filter theory of BL-algebras, Soft Computing 12 (2008), 419-423.

[4] E. Turunen, BL-algebras of basic fuzzy logic, Mathware and Soft Computing 6 (1999), 49-61.

[5] E. Turunen and S. Seesa, Local BL-algebras, Multiple Valued Logic 6 (2001), 229-249.