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GENERAL SUBMANIFOLDS OF A KAEHLER MANIFOLD-II

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Abstract

In this paper, we continue the study of most general class of submanifolds of a Kaehler manifold initiated in [4], which includes all existing classes of submanifolds (complex submanifolds, totally real submanifolds, CR-submanifolds, slant submanifolds). Such a submanifold M of a Kaehler manifold $\overline{M}$ has naturally defined operators ϕ, F, ψ and G. We study the geometry of a general submanifold with parallel F and obtain a condition under which a general submanifold is a complex submanifold.

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1. Introduction

The geometry of submanifolds of a Kaehler manifold is interesting because of the influence of the complex structure of the Kaehler manifold on the submanifold. Accordingly, there are various types of special submanifolds of a Kaehler manifold namely, complex submanifolds [5], totally real submanifolds, CR-submanifolds [1] (this class includes both complex submanifolds and totally real submanifolds), slant submanifolds [2]. We have initiated the study of most general submanifolds of a Kaehler manifold which includes all the existing types of submanifolds (cf. [4]). A general submanifold of a Kaehler manifold naturally carries four operators ϕ, F, ψ and G defined on this submanifold. In [4], it has been shown that a general submanifold of a Kaehler manifold with parallel ϕ is essentially a CR-submanifold and there are examples of general submanifold where ϕ is not parallel. There are lot many questions about a general submanifold of a Kaehler manifold to be answered, for instance the impact of conditions that one of the structure operators F, ψ and G is parallel, as well as impact of other algebraic restrictions on the properties of these operators on the general submanifold. In this paper, we consider the question that the structure operator F is parallel on the general submanifold of a Kaehler manifold and study its impact on the geometry of general submanifold.

2. Preliminaries

Let M be an n-dimensional smooth manifold immersed into an $n + k = 2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$ with Riemannian connection $\overline{\nabla}$ and the induced metric and connection on M be g and ∇, respectively. Then we have the following fundamental equations for the submanifold, namely

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad X, Y \in \mathfrak{X}(M), \tag{2.1}$$

$$\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \quad X \in \mathfrak{X}(M), N \in \Gamma(v), \tag{2.2}$$

where $\mathfrak{X}(M)$ is Lie-algebra of vector fields on M, $\Gamma(v)$ is the space of normal sections of the normal bundle v of M, h is the second fundamental form, A_N is the Weingarten map with respect to $N \in \Gamma(v)$, $\nabla^{\perp}$ is the connection in the normal bundle v. The Weingarten maps A_N are related to the second fundamental form h by

$$g(A_N X, Y) = g(h(X, Y), N), \quad X, Y \in \mathfrak{X}(M), N \in \Gamma(v).$$

For an n-dimensional submanifold M of an $n + k = 2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$, define:

$$JX = \phi(X) + F(X), \quad JN = \psi(N) + G(N),$$

where $X \in \mathfrak{X}(M)$ and $N \in \Gamma(v)$, and $\phi(X) = (JX)^T$ the tangential component of JX, $F(X) = (JX)^{\perp}$ the normal component of JX, $\psi(N) = (JN)^T$ and $G(N) = (JN)^{\perp}$, which define linear operators $\phi : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $F : \mathfrak{X}(M) \to \Gamma(v)$, $\psi : \Gamma(v) \to \mathfrak{X}(M)$ and $G : \Gamma(v) \to \Gamma(v)$, respectively. It is trivial implication of the definition that:

$$\phi^2(X) = -X - \psi(F(X)), \quad G^2(N) = -N - F(\psi(N)),$$

$$F(\phi(X)) = -G(F(X)), \quad \phi(\psi(N)) = -\psi(G(N)), \tag{2.3}$$

hold for $X \in \mathfrak{X}(M)$, $N \in \Gamma(v)$. Also

$$g(\phi(X), Y) = g(JX - F(X), Y) = g(JX, Y) = -g(X, JY) = -g(X, \phi(Y)) \tag{2.4}$$

similarly, we have

$$g(G(N_1), N_2) = -g(N_1, G(N_2)), \tag{2.5}$$

$$g(F(X), N) = -g(X, \psi(N)) \tag{2.6}$$

and

$$g(\psi(N), X) = -g(N, F(X)) \tag{2.7}$$

hold for $X, Y \in \mathfrak{X}(M)$ and $N, N_1, N_2 \in \Gamma(v)$.

If we define the covariant derivatives $(D_X F)(Y)$ and $(D_X \psi)(N)$ for the operators $F : \mathfrak{X}(M) \to \Gamma(v)$ and $\psi : \Gamma(v) \to \mathfrak{X}(M)$ by

$$(D_X F)Y = \nabla_X^{\perp} F(Y) - F(\nabla_X Y)$$

and

$$(D_X\psi)(N) = \nabla_X\psi(N) - \psi(\nabla_X^{\perp}N),$$

then we have the following:

Lemma 2.1 [4]**.** *The operators* ϕ, F, ψ *and* G *obey*

$$(\nabla_X\phi)(Y) = A_{F(Y)}X + \psi(h(X, Y)),$$

$$(D_XF)(Y) = G(h(X, Y)) - h(X, \phi(Y)),$$

$$(D_X\psi)(N) = A_{G(N)}X - \phi(A_NX)$$

and

$$(\nabla_X^{\perp}G)(N) = -F(A_NX) - h(X, \psi(N))$$

for $X, Y \in \mathfrak{X}(M)$ *and* $N \in \Gamma(v)$.

We define the operators

$$B : \psi \circ F : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

and

$$C : F \circ \psi : \Gamma(v) \to \Gamma(v),$$

then it is easy to see that they are symmetric operators. Also, using (2.3) we see that B commutes with ϕ that is $B \circ \phi = \phi \circ B$ and that G commutes with C that is $G \circ C = C \circ G$. As a result of this we get $trB \circ \phi = 0$ and $trG \circ C = 0$.

3. Submanifolds with Parallel F

In this section, we are interested in submanifolds with parallel F, that is,

$$\nabla_X^{\perp}FY = F(\nabla_XY),$$

where $X, Y \in \mathfrak{X}(M)$. Using Lemma 2.1, we immediately have

Lemma 3.1. *Let M be a submanifold of a Kaehler manifold* $\overline{M}$. *Then the operator F is parallel if and only if*

$$G(h(X, Y)) = h(X, \phi Y)$$

for $X, Y \in \mathfrak{X}(M)$.

Remark. Observe that if F is parallel, then by Lemma 3.1,

$$h(X, \phi Y) = h(\phi X, Y)$$

holds for $X, Y \in \mathfrak{X}(M)$. The operator C defined in previous section is symmetric and we have

$$(\nabla_X^{\perp} C)(N) = \nabla_X^{\perp} C(N) - C(\nabla_X^{\perp} N), \quad X \in \mathfrak{X}(M), N \in \Gamma(v).$$

The operator C is said to be *parallel* if $(\nabla_X^{\perp} C)(N) = 0$, $X \in \mathfrak{X}(M)$, $N \in \Gamma(v)$.

Theorem 3.1. *Let M be an n-dimensional submanifold of an* $(n + k) = 2m$ *-dimensional Kaehler manifold* $(\overline{M}, J, g)$. *If the operator F is parallel, then C is also parallel.*

Proof. We have for $X \in \mathfrak{X}(M)$, $N \in \Gamma(v)$ that

$$\nabla_X^{\perp}(CN) = \nabla_X^{\perp} F(\psi(N)) = (D_X F)(\psi(N)) + F(\nabla_X^{\perp} \psi(N))$$

$$= (D_X F)(\psi(N)) + F((D_X \psi)(N)) + \psi(\nabla_X^{\perp}(N)),$$

that is,

$$(\nabla_X^{\perp} C)(N) = (D_X F)(\psi(N)) + F((D_X \psi)(N)).$$

Using Lemma 2.1 in above equation, we get

$$(\nabla_X^{\perp} C)(N) = G(h(X, \psi(N)) - h(X, \phi(\psi N)) + F(A_{GN} X - \phi A_N X)). \quad (3.1)$$

Also, using Lemma 3.1, we get $g(h(X, \phi Y), N) = -g(h(X, Y), GN)$, that is, $g(\phi A_N X, Y) = g(A_{GN} X, Y)$, which gives

$$A_{GN} X = \phi A_N X. \quad (3.2)$$

Finally, using Lemma 3.1 together with equations (3.1) and (3.2), for parallel F, we get

$$(\nabla^{\perp}_{X}C)(N) = 0, \quad X \in \mathfrak{X}(M),\ N \in \Gamma(v)$$

that proves the theorem.

For $N \in \Gamma(v)$, if $C(N) = \lambda N$, $\lambda \in C^{\infty}(M)$, then λ is said to be an *eigenvalue* of C and N is called the *eigenvector* of C corresponding to eigenvalue λ. Using equation (3.2) we have the following:

Corollary 3.1. *Let M be an n-dimensional submanifold of an $(n + k) = 2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. If the operator F is parallel, then ψ is also parallel.*

Proof. If F is parallel, then we get equation (3.2). Using equation (3.2) together with Lemma 2.1, we get $(D_X\psi)(N) = 0$, that is, ψ is parallel.

Lemma 3.2. *Let M be an n-dimensional submanifold of an $(n + k) = 2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. If the operator F is parallel, then the eigenvalues of C are constants.*

Proof. Let $C(N) = \lambda N$, $N \in \Gamma(v)$, $\lambda \in C^{\infty}(M)$. Without loss of generality we can assume that N is a unit normal vector field. As F is parallel by Theorem 3.1, we have C is parallel and consequently

$$\begin{aligned} 0 &= (\nabla^{\perp}_{X}C)(N) \\ &= \nabla^{\perp}_{X}(CN) - C(\nabla^{\perp}_{X}N) \\ &= \nabla^{\perp}_{X}(\lambda N) - C(\nabla^{\perp}_{X}N) \\ &= X(\lambda)N + \lambda\nabla^{\perp}_{X}N - C(\nabla^{\perp}_{X}N). \end{aligned}$$

Taking inner product with $N \in \Gamma(v)$, we get

$$X(\lambda) = 0,$$

which proves that λ is a constant.

Let M be an n-dimensional submanifold of an $(n+k)=2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. Define

$$\| C \|^2 = \sum_{\alpha=1}^{k} g(C(N_\alpha), C(N_\alpha)),$$

for a local orthonormal frame $\{N_1, ..., N_\alpha\}$. Since C is symmetric we can choose an orthonormal frame $\{N_1, ..., N_\alpha\}$ that diagonalizes C. Then in light of Lemma 3.2, we have proved the following:

Lemma 3.3. *Let M be an n-dimensional submanifold of an $(n+k)=2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. If the operator F is parallel, then $\| C \|^2$ is a constant as well as the trC is also a constant and $trC = -\| \psi \|^2$.*

Using equation (2.3), we have $G^2 = -I - C$ and for a local orthonormal frame $\{N_1, ..., N_\alpha\}$ of normal vector fields, we have

$$\begin{aligned} \| G \|^2 &= \sum_{\alpha=1}^{k} g(G(N_\alpha), G(N_\alpha)) \\ &= \sum_{\alpha=1}^{k} g(N_\alpha + C(N_\alpha), N_\alpha) \\ &= k - \| \psi \|^2. \end{aligned} \tag{3.3}$$

Theorem 3.2. *Let M be an n-dimensional submanifold of an $(n+k)=2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. If the operator F is parallel, then $\| G \|^2$ is a constant.*

Proof. Suppose F is parallel. Then by equation (3.3)

$$\| G \|^2 + \| \psi \|^2 = k,$$

to prove $\| G \|^2$ is a constant, it is enough to show that $\| \psi \|^2$ is a constant. Since the operator C is symmetric, we have

$$X(\| \psi \|^2) = X\left(\sum_{\alpha=1}^{k} g(\psi(N_\alpha), \psi(N_\alpha))\right) = -X\left(\sum_{\alpha=1}^{k} g(C(N_\alpha), N_\alpha)\right)$$

$$= -\sum_{\alpha=1}^{k} \{g((\nabla_X^{\perp} C)(N_\alpha), N_\alpha) + 2g(C(N_\alpha), \nabla_X^{\perp} N_\alpha)\}.$$

Using Theorem 3.1 and the local orthonormal frame $\{N_1, ..., N_\alpha\}$ of normal vector fields that diagonalize C with $C(N_\alpha) = \lambda_\alpha N_\alpha$, we get together with Lemma 3.2 that

$$X(\| \psi \|^2) = -\sum_{\alpha=1}^{k} 2\lambda_\alpha g(N_\alpha, \nabla_X^{\perp} N_\alpha) = 0,$$

which proves that $\| \psi \|^2$ is a constant.

Theorem 3.3. *Let M be an n-dimensional submanifold of an $(n + k) = 2m$-dimensional Kaehler manifold $(\overline{M}, J, g)$. If the operator F is parallel, and $trC = 0$, then M is a complex submanifold.*

Proof. Suppose F is parallel and $trC = 0$. Then by Lemma 3.3, we have $\| \psi \|^2 = 0$ and that gives $\psi = 0$. This also gives $B = 0$ and consequently

$$0 = g(BX, X) = -g(FX, FX) = -\| FX \|^2, \quad X \in \mathfrak{X}(M),$$

that is, $FX = 0, X \in \mathfrak{X}(M)$. Thus the equations (2.3) and Lemma 2.1 prove that ϕ satisfies $\phi^2 = -I$ and $(\nabla_X \phi)(Y) = 0$. That is, M is a complex submanifold of the Kaehler manifold $\overline{M}$.

4. Examples

Example 4.1. Consider the Kaehler manifold $(R^4, J, \langle\, ,\rangle)$, where J is the complex structure defined by

$$J\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial x^2},\ J\left(\frac{\partial}{\partial x^2}\right) = -\frac{\partial}{\partial x^1},\ J\left(\frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial x^4},\ \text{and}\ J\left(\frac{\partial}{\partial x^4}\right) = -\frac{\partial}{\partial x^3}$$

$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ and $\frac{\partial}{\partial x^4}$ being coordinate vector fields on R^4 that is for each vector fields

$$X = f^1 \frac{\partial}{\partial x^1} + f^2 \frac{\partial}{\partial x^2} + f^3 \frac{\partial}{\partial x^3} + f^4 \frac{\partial}{\partial x^4} \in \mathfrak{X}(R^4),$$

$$JX = -f^2 \frac{\partial}{\partial x^1} + f^1 \frac{\partial}{\partial x^2} - f^4 \frac{\partial}{\partial x^3} + f^3 \frac{\partial}{\partial x^4}, \tag{4.1}$$

and $\langle\, ,\rangle$ is the Euclidean metric on R^4. We denote by $\overline{\nabla}$ the Euclidean connection on R^4. Take $M = R^3$ and the embedding $f : M \to R^4$

$$f(x, y, z) = (y, x, 0, z).$$

Then we find the local orthonormal frame $\{e_1, e_2, e_3, N\}$ of R^4, where

$$e_1 = \frac{\partial}{\partial x^2},\ e_2 = \frac{\partial}{\partial x^1},\ e_3 = \frac{\partial}{\partial x^4}\ \text{ and }\ N = \frac{\partial}{\partial x^3},$$

such that $\{e_1, e_2, e_3\}$ is local orthonormal frame on *M*. Let ∇ be the induced Riemannian connection on *M*. Then using properties of $\overline{\nabla}$, it is straight-foreword to check that

$$\nabla_{e_i} e_j = 0, \quad i, j = 1, 2, 3, \tag{4.2}$$

and using (4.1), we find that

$$F(e_1) = 0, \quad F(e_2) = 0 \ \text{ and } \ F(e_3) = -N, \tag{4.3}$$

and as *N* is parallel in the normal bundle, consequently using equations (4.1), (4.2) and (4.3), we get that

$$(D_X F)(Y) = 0, \quad X, Y \in \mathfrak{X}(M).$$

Thus *F* is parallel.

Next, we construct an example where *F* is not parallel, that is, *M* will not be a CR-submanifold.

Example 4.2. Consider 4-dimensional Euclidean space R^4 with Euclidean metric $\langle\, ,\rangle$. Then $(R^4, J, \langle\, ,\rangle)$ is a Kaehler manifold with the complex structure J defined by

$$J\left(\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial x^2},\ J\left(\frac{\partial}{\partial x^2}\right) = -\frac{\partial}{\partial x^1},\ J\left(\frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial x^4} \text{ and } J\left(\frac{\partial}{\partial x^4}\right) = -\frac{\partial}{\partial x^3},$$

where $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$, and $\frac{\partial}{\partial x^4}$ are the coordinate vector fields on R^4. It is easy to see that J is parallel with respect to the Euclidean connection $\overline{\nabla}$ on R^4, that is,

$$\overline{\nabla}_X JY = J\overline{\nabla}_X Y,$$

holds for $X, Y \in \mathfrak{X}(R^4)$ the Lie-algebra of smooth vector fields on R^4.

Now consider the product $M = S^1 \times S^1$ of two copies of the unit circle S^1 and define

$$f : M \to R^4$$

by

$$f(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) = (\cos\theta, \cos\varphi, \sin\theta, \sin\varphi),$$

where θ and φ are local coordinates of S^1 and S^1, respectively. Then it is straight forward to see that at $p = (\theta, \varphi) \in M$, differential df_p at $p \in M$ has the matrix respectively

$$df_p = \begin{bmatrix} -\sin\theta & 0 \\ 0 & -\sin\varphi \\ \cos\theta & 0 \\ 0 & \cos\varphi \end{bmatrix},$$

which has rank 2 at each $p \in M$; (as if $\sin\theta\sin\varphi = 0$, then $\cos\theta\cos\varphi \neq 0$ and vice-versa). Thus $f : M \to R^4$ is an immersion of M into R^4, that is,

M is a 2-dimensional submanifold of R^4. Choosing

$$e_1 = -\sin\theta\frac{\partial}{\partial x^1} + \cos\theta\frac{\partial}{\partial x^3}, \quad e_2 = -\sin\varphi\frac{\partial}{\partial x^2} + \cos\varphi\frac{\partial}{\partial x^4},$$

$$N_1 = \cos\theta\frac{\partial}{\partial x^1} + \sin\theta\frac{\partial}{\partial x^3}, \quad N_2 = \cos\varphi\frac{\partial}{\partial x^2} + \sin\varphi\frac{\partial}{\partial x^4}, \tag{4.4}$$

we get a local orthonormal frame $\{e_1, e_2, N_1, N_2\}$ of R^4 such that $\{e_1, e_2\}$ is a local orthonormal frame on M with respect to the induced metric g as a submanifold of R^4 and that $\{N_1, N_2\}$ is local field of normal to M.

Let $\bar{e}_1 = \frac{\partial}{\partial\theta}$ and $\bar{e}_2 = \frac{\partial}{\partial\varphi}$ be the vector fields on the first and second copies of S^1 in $M = S^1 \times S^1$. Then we have

$$e_1 = df_{(\theta,\varphi)}(\bar{e}_1)$$

and

$$e_2 = df_{(\theta,\varphi)}(\bar{e}_2).$$

Next, we compute the values of F at e_1 and e_2, respectively. Using

$$JX = \phi(X) + F(X), \quad X \in \mathfrak{X}(M)$$

and the equations (4.4), we get

$$\begin{aligned} Je_1 &= J\left(-\sin\theta\frac{\partial}{\partial x^1} + \cos\theta\frac{\partial}{\partial x^3}\right) \\ &= \left(-\sin\theta\frac{\partial}{\partial x^2} + \cos\theta\frac{\partial}{\partial x^4}\right) \in \mathfrak{X}(R^4). \end{aligned}$$

We can express it as

$$Je_1 = ae_1 + be_2 + cN_1 + dN_2,$$

where a, b, c and $d \in C^\infty(R^4)$. Then by (4.4)

$$\begin{aligned} Je_1 &= (-a\sin\theta + c\cos\theta)\frac{\partial}{\partial x^1} + (-b\sin\varphi + d\cos\varphi)\frac{\partial}{\partial x^2} \\ &\quad + (a\cos\theta + c\sin\theta)\frac{\partial}{\partial x^3} + (b\cos\varphi + d\sin\varphi)\frac{\partial}{\partial x^4}, \end{aligned}$$

equating the two values of Je_1, we conclude that

$$-a\sin\theta + c\cos\theta = 0,$$

$$a\cos\theta + c\sin\theta = 0,$$

$$-b\sin\varphi + d\cos\varphi = -\sin\theta$$

and

$$b\cos\varphi + d\sin\varphi = \cos\theta.$$

Solving these equations, we get

$$a = 0,\ b = \cos(\varphi - \theta),\ c = 0 \ \text{ and } \ d = \sin(\varphi - \theta),$$

that is,

$$Je_1 = \cos(\varphi - \theta)e_2 + \sin(\varphi - \theta)N_2,$$

thus using

$$Je_1 = \phi(e_1) + F(e_1),$$

we arrive at

$$F(e_1) = \sin(\theta - \varphi)N_2. \tag{4.5}$$

Similarly, using equation (4.4) we get

$$Je_2 = \sin\varphi\frac{\partial}{\partial x^1} - \cos\varphi\frac{\partial}{\partial x^3} \in \mathfrak{X}(R^4)$$

and consequently

$$Je_2 = -\cos(\varphi - \theta)e_1 + \sin(\varphi - \theta)N_1.$$

Thus we arrive at

$$F(e_2) = \sin(\varphi - \theta)N_1. \tag{4.6}$$

Now, we show that for this submanifold, F is not parallel. Since the immersion $f : M \to R^4$ is local embedding, we have

$$\overline{\nabla}_{e_1}e_1 = -\bar{e}_1(\sin\theta)\frac{\partial}{\partial x^1} + \bar{e}_1(\cos\theta)\frac{\partial}{\partial x^3}, \tag{4.7}$$

let ∇ be the Riemannian connection on M with respect to the induced metric. Then as

$$\overline{\nabla}_{e_1} e_1 = \overline{a} e_1 + \overline{b} e_2 + \overline{c} N_1 + \overline{d} N_2, \tag{4.8}$$

where $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d} \in C^{\infty}(R^4)$. Also using Gauss equation, we have

$$\overline{\nabla}_{e_1} e_1 = \nabla_{e_1} e_1 + h(e_1, e_1). \tag{4.9}$$

Inserting values of e_1, e_2, N_1 and N_2 into (4.8) and comparing with (4.7), we get

$$-\overline{a} \sin\theta + \overline{c} \cos\theta = -\overline{e}_1 \sin\theta,$$

$$\overline{a} \cos\theta + \overline{c} \sin\theta = \overline{e}_1 \cos\theta,$$

$$-\overline{b} \sin\varphi + \overline{d} \cos\varphi = 0$$

and

$$\overline{b} \cos\varphi + \overline{d} \sin\varphi = 0.$$

Solving these equations and substituting in (4.8), we get

$$\nabla_{e_1} e_1 = 0$$

and from (4.9), we get that

$$h(e_1, e_1) = -N_1.$$

Similarly, computing for $\overline{\nabla}_{e_1} e_2$, $\overline{\nabla}_{e_2} e_1$ and $\overline{\nabla}_{e_2} e_2$, we get

$$\nabla_{e_1} e_2 = 0, \nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = 0, h(e_1, e_2) = 0 \text{ and } h(e_1, e_2) = -N_2, \tag{4.10}$$

(this is consistent with $R(e_1, e_2)e_1 = 0$ as M is flat torus).

Also computing $\overline{\nabla}_{e_1} N_1$, $\overline{\nabla}_{e_2} N_1$, $\overline{\nabla}_{e_1} N_2$ and $\overline{\nabla}_{e_2} N_2$ together with equation (2.2) we conclude

$$\nabla^{\perp}_{e_1} N_1 = 0, \quad \nabla^{\perp}_{e_2} N_1 = 0, \quad \nabla^{\perp}_{e_1} N_2 = 0, \quad \nabla^{\perp}_{e_2} N_2 = 0. \tag{4.11}$$

Thus using equations (4.7), (4.10) and (4.11) we compute $(D_{e_1}F)(e_2)$ to arrive at

$$(D_{e_1}F)(e_2) = -\cos(\varphi - \theta)N_1.$$

Similarly, we have $(D_{e_1}F)(e_1) = -\cos(\theta - \varphi)N_2$, $(D_{e_2}F)(e_1) = \cos(\theta - \varphi)N_2$, $(D_{e_2}F)(e_2) = \cos(\varphi - \theta)N_1$. Since $\{e_1, e_2\}$ is a local orthonormal frame, we see that in general $(D_{e_i}F)(e_j) \neq 0$, $i, j = 1, 2$, that is, there are points where

$$(D_X F)(Y) \neq 0, \quad X, Y \in \mathfrak{X}(M),$$

that is, F is not parallel.

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