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ROUGHNESS OF SUBb-SEMIRINGS/IDEALS IN b-SEMIRINGS

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Abstract

In this paper, we discuss the roughness of subb-semirings, right ideals, left ideals and ideals in b-semirings.

1. Introduction and Preliminaries

The notion of rough sets was introduced by Pawlak [7] in 1982. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. The algebraic approach of rough sets was studied by some authors, for example, Iwinski [2], Kuroki [4-6], Biswas and Nanda [1] and Jun [3], etc.

Let S be a nonempty set and $*_1$ and $*_2$ be binary operations on S. $(S, *_1, *_2)$ is called a *b-semiring* if $(S, *_1)$ and $(S, *_2)$ are semigroups and for all $a, b, c \in S$,

$$a *_1 (b *_2 c) = (a *_1 b) *_2 (a *_1 c), (b *_2 c) *_1 a = (b *_1 a) *_2 (c *_1 a),$$

$$a *_2 (b *_1 c) = (a *_2 b) *_1 (a *_2 c) \text{ and } (b *_1 c) *_2 a = (b *_2 a) *_1 (c *_2 a).$$

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Let $(S, *_1, *_2)$ be a b-semiring. A nonempty subset T of S is called a *subb-semiring* if $(T, *_1, *_2)$ is a b-semiring. We have that T is a subb-semiring of S if and only if $a *_1 b \in T$ and $a *_2 b \in T$ for all $a, b \in T$. A nonempty subset R of S is called a *right ideal* of S if $a *_1 s \in R$ and $a *_2 s \in R$ for all $a \in R$ and $s \in S$. *Left ideals* of S are defined analogously. A nonempty subset I of S is called an *ideal* of S if I is both a right ideal and a left ideal of S. It is obvious that every right ideal, left ideal and ideal of S is a subb-semiring of S. An equivalence relation ρ on S is called a *congruence* on S if for all $a, b, c \in S$,

$$a \rho b \Rightarrow (a *_1 c) \rho (b *_1 c), (a *_2 c) \rho (b *_2 c), (c *_1 a) \rho (c *_1 b) \text{ and } (c *_2 a) \rho (c *_2 b).$$

In this paper, we discuss the roughness of subb-semirings, right ideals, left ideals and ideals in b-semirings.

2. Roughness of Subb-semirings, Left Ideals, Right Ideals and Ideals of b-semirings

Let S be a b-semiring and ρ be a congruence on S. Let $[a]_\rho$ denote the congruence class containing the element $a \in S$. A congruence ρ on S is said to be *complete* if $[a]_\rho *_1 [b]_\rho = [a *_1 b]_\rho$ and $[a]_\rho *_2 [b]_\rho = [a *_2 b]_\rho$ for all $a, b \in S$.

Let A be a nonempty subset of S and ρ be a congruence on S. The ρ-*lower approximation* and ρ-*upper approximation* of A are defined to be the sets

$$\underline{\rho}(A) = \{x \in S \mid [x]_\rho \subseteq A\} \text{ and } \overline{\rho}(A) = \{x \in S \mid [x]_\rho \cap A \neq \varnothing\},$$

respectively. Then $\underline{\rho}(A) \subseteq A \subseteq \overline{\rho}(A)$ and $\underline{\rho}(S) = S = \overline{\rho}(S)$.

Proposition 2.1. *Let ρ be a congruence on a b-semiring S and let A and B be nonempty subsets of S. Then the following statements are true*:

(i) $\overline{\rho}(A \cup B) = \overline{\rho}(A) \cup \overline{\rho}(B)$.

(ii) $\underline{\rho}(A \cap B) = \underline{\rho}(A) \cap \underline{\rho}(B)$.

(iii) $A \subseteq B$ *implies* $\overline{\rho}(A) \subseteq \overline{\rho}(B)$ *and* $\underline{\rho}(A) \subseteq \underline{\rho}(B)$.

(iv) $\overline{\rho}(A \cap B) \subseteq \overline{\rho}(A) \cap \overline{\rho}(B)$.

(v) $\underline{\rho}(A) \cup \underline{\rho}(B) \subseteq \underline{\rho}(A \cup B)$.

(vi) $\overline{\rho}(A) *_1 \overline{\rho}(B) \subseteq \overline{\rho}(A *_1 B)$ *and* $\overline{\rho}(A) *_2 \overline{\rho}(B) \subseteq \overline{\rho}(A *_2 B)$.

(vii) *If* ρ *is complete, then* $\underline{\rho}(A) *_1 \underline{\rho}(B) \subseteq \underline{\rho}(A *_1 B)$ *and* $\underline{\rho}(A) *_2 \underline{\rho}(B) \subseteq \underline{\rho}(A *_2 B)$.

Proof. (i) Note that

$$x \in \overline{\rho}(A \cup B) \Leftrightarrow [x]_\rho \cap (A \cup B) \neq \varnothing \Leftrightarrow ([x]_\rho \cap A) \cup ([x]_\rho \cap B) \neq \varnothing$$

$$\Leftrightarrow [x]_\rho \cap A \neq \varnothing \text{ or } [x]_\rho \cap B \neq \varnothing$$

$$\Leftrightarrow x \in \overline{\rho}(A) \text{ or } x \in \overline{\rho}(B) \Leftrightarrow x \in \overline{\rho}(A) \cup \overline{\rho}(B).$$

Therefore $\overline{\rho}(A \cup B) = \overline{\rho}(A) \cup \overline{\rho}(B)$.

(ii) Note that

$$x \in \underline{\rho}(A \cap B) \Leftrightarrow [x]_\rho \subseteq A \cap B \Leftrightarrow [x]_\rho \subseteq A \text{ and } [x]_\rho \subseteq B$$

$$\Leftrightarrow x \in \underline{\rho}(A) \text{ and } x \in \underline{\rho}(B) \Leftrightarrow x \in \underline{\rho}(A) \cap \underline{\rho}(B).$$

Then $\underline{\rho}(A \cap B) = \underline{\rho}(A) \cap \underline{\rho}(B)$.

(iii) Assume that $A \subseteq B$. Then $B = A \cup B$ and $A = A \cap B$. By (i) and (ii), we have

$$\overline{\rho}(A) \subseteq \overline{\rho}(A) \cup \overline{\rho}(B) = \overline{\rho}(A \cup B) = \overline{\rho}(B)$$

and

$$\underline{\rho}(A) = \underline{\rho}(A \cap B) = \underline{\rho}(A) \cap \underline{\rho}(B) \subseteq \underline{\rho}(B).$$

Therefore $\overline{\rho}(A) \subseteq \overline{\rho}(B)$ and $\underline{\rho}(A) \subseteq \underline{\rho}(B)$.

(iv) By (iii), we have $\overline{\rho}(A \cap B) \subseteq \overline{\rho}(A)$ and $\overline{\rho}(A \cap B) \subseteq \overline{\rho}(B)$. Thus $\overline{\rho}(A \cap B) \subseteq \overline{\rho}(A) \cap \overline{\rho}(B)$.

(v) By (iii), we have $\underline{\rho}(A) \subseteq \underline{\rho}(A \cup B)$ and $\underline{\rho}(B) \subseteq \underline{\rho}(A \cup B)$. Thus $\underline{\rho}(A) \cup \underline{\rho}(B) \subseteq \underline{\rho}(A \cup B)$.

(vi) Let $u \in \overline{\rho}(A) *_1 \overline{\rho}(B)$. Then $u = x *_1 y$ for some $x \in \overline{\rho}(A)$ and $y \in \overline{\rho}(B)$. Then there exist $a, b \in S$ such that $a \in [x]_\rho \cap A$ and $b \in [y]_\rho \cap B$. Thus $a *_1 b \in A *_1 B$ and $a *_1 b \in [x]_\rho *_1 [y]_\rho \subseteq [x *_1 y]_\rho$. Hence $a *_1 b \in [x *_1 y]_\rho \cap (A *_1 B)$. This implies $u = x *_1 y \in \overline{\rho}(A *_1 B)$. Therefore $\overline{\rho}(A) *_1 \overline{\rho}(B) \subseteq \overline{\rho}(A *_1 B)$. Similarly, $\overline{\rho}(A) *_2 \overline{\rho}(B) \subseteq \overline{\rho}(A *_2 B)$.

(vii) Assume that ρ is complete and let $u \in \underline{\rho}(A) *_1 \underline{\rho}(B)$. Then $u = x *_1 y$ for some $x \in \underline{\rho}(A)$ and $y \in \underline{\rho}(B)$. Then $[x]_\rho \subseteq A$ and $[y]_\rho \subseteq B$. Then $[x *_1 y]_\rho = [x]_\rho *_1 [y]_\rho \subseteq A *_1 B$. Thus $u = x *_1 y \in \underline{\rho}(A *_1 B)$. Therefore $\underline{\rho}(A) *_1 \underline{\rho}(B) \subseteq \underline{\rho}(A *_1 B)$. Similarly, $\underline{\rho}(A) *_2 \underline{\rho}(B) \subseteq \underline{\rho}(A *_2 B)$. □

Let ρ be a congruence on a b-semiring S. Then a nonempty subset A of S is called a ρ-*upper* (ρ-*lower*) *rough subb-semiring* of S if the ρ-upper (ρ-lower) approximation of A is a subb-semiring of S, a ρ-*upper* (ρ-*lower*) *rough right ideal* (*left ideal*, *ideal*) of S if the ρ-upper (ρ-lower) approximation of A is a right ideal (left ideal, ideal) of S.

Theorem 2.2. *Let ρ be a congruence on a b-semiring S. Then*

(i) *Every subb-semiring of S is a ρ-upper rough subb-semiring of S.*

(ii) *Every right ideal* (*left ideal*, *ideal*) *of S is a ρ-upper rough right ideal* (*left ideal*, *ideal*) *of S.*

Proof. (i) Let A be a subb-semiring of S. Then $A \neq \emptyset$, $A *_1 A \subseteq A$ and $A *_2 A \subseteq A$. Since $A \neq \emptyset$ and $A \subseteq \overline{\rho}(A)$, $\overline{\rho}(A) \neq \emptyset$. By (iii) and (vi) of Proposition 2.1, we have

$$\overline{\rho}(A) *_1 \overline{\rho}(A) \subseteq \overline{\rho}(A *_1 A) \subseteq \overline{\rho}(A) \text{ and } \overline{\rho}(A) *_2 \overline{\rho}(A) \subseteq \overline{\rho}(A *_2 A) \subseteq \overline{\rho}(A).$$

Hence $\overline{\rho}(A)$ is a subb-semiring of S, that is, A is a ρ-upper rough subb-semiring of S.

(ii) Let A be a right ideal of S. Thus $A \neq \emptyset$, $A *_1 S \subseteq A$ and $A *_2 S \subseteq A$. Since $A \neq \emptyset$ and $A \subseteq \overline{\rho}(A)$, $\overline{\rho}(A) \neq \emptyset$. By (iii) and (vi) of

Proposition 2.1, we have

$$\overline{\rho}(A) *_1 S = \overline{\rho}(A) *_1 \overline{\rho}(S) \subseteq \overline{\rho}(A *_1 S) \subseteq \overline{\rho}(A)$$

and

$$\overline{\rho}(A) *_2 S = \overline{\rho}(A) *_2 \overline{\rho}(S) \subseteq \overline{\rho}(A *_2 S) \subseteq \overline{\rho}(A).$$

Hence $\overline{\rho}(A)$ is a right ideal of S, that is, A is a ρ-upper rough right ideal of S. The other case can also be proved in a similar way. □

Theorem 2.3. *Let* ρ *be a congruence on a b-semiring* S. *Then*:

(i) *If* A *is a subb-semiring of* S *such that* $\underline{\rho}(A) \neq \emptyset$, *then* A *is a* ρ-*lower rough subb-semiring of* S.

(ii) *If* A *is a right ideal* (*left ideal*, *ideal*) *of* S *such that* $\underline{\rho}(A) \neq \emptyset$, *then* A *is a* ρ-*lower rough right ideal* (*left ideal*, *ideal*) *of* S.

Proof. (i) Let A be a subb-semiring of S. Applying (iii) and (vii) of Proposition 2.1, we have

$$\underline{\rho}(A) *_1 \underline{\rho}(A) \subseteq \underline{\rho}(A *_1 A) \subseteq \underline{\rho}(A) \text{ and } \underline{\rho}(A) *_2 \underline{\rho}(A) \subseteq \underline{\rho}(A *_2 A) \subseteq \underline{\rho}(A).$$

Therefore $\underline{\rho}(A)$ is a subb-semiring of S, that is, A is a ρ-lower rough subb-semiring of S.

(ii) Let A be a right ideal of S. By (iii) and (vii) of Proposition 2.1, we have

$$\underline{\rho}(A) *_1 S = \underline{\rho}(A) *_1 \underline{\rho}(S) \subseteq \underline{\rho}(A *_1 S) \subseteq \underline{\rho}(A)$$

and

$$\underline{\rho}(A) *_2 S = \underline{\rho}(A) *_2 \underline{\rho}(S) \subseteq \underline{\rho}(A *_2 S) \subseteq \underline{\rho}(A).$$

Thus $\underline{\rho}(A)$ is a right ideal of S, that is, A is a ρ-lower rough right ideal of S. The other case can also be proved in a similar way. □

3. Rough Sets in a Quotient b-semiring

Let $(S, *_1, *_2)$ be a b-semiring and ρ be a congruence on S. Define binary operations $\bullet_1$ and $\bullet_2$ on S/ρ by

$$[a]_\rho \bullet_1 [b]_\rho = [a *_1 b]_\rho \text{ and } [a]_\rho \bullet_2 [b]_\rho = [a *_2 b]_\rho.$$

Then $(S/\rho, \bullet_1, \bullet_2)$ is a b-semiring.

A b-semiring $(S/\rho, \bullet_1, \bullet_2)$ is called a *quotient b-semiring of a b-semiring S by a congruence* ρ.

Note that if ρ is a complete congruence on S, then $(S/\rho, *_1, *_2)$ is a quotient b-semiring of a b-semiring $(S, *_1, *_2)$ by a congruence ρ.

Let ρ be a congruence on a b-semiring S. The ρ-lower and ρ-upper approximations can be presented in an equivalent form as shown below:

$$\underline{\rho}(A)/\rho = \{[x]_\rho \in S/\rho \mid [x]_\rho \subseteq A\}$$

and

$$\overline{\rho}(A)/\rho = \{[x]_\rho \in S/\rho \mid [x]_\rho \cap A \neq \varnothing\}.$$

Now we discuss these sets as subsets of a quotient b-semiring S/ρ of S by ρ.

Theorem 3.1. *Let* ρ *be a complete congruence on a b-semiring S and A be a subb-semiring of S. Then the following statements hold*:

(i) $\overline{\rho}(A)/\rho$ *is a subb-semiring of* S/ρ.

(ii) *If* $\underline{\rho}(A) \neq \varnothing$, *then* $\underline{\rho}(A)/\rho$ *is a subb-semiring of* S/ρ.

Proof. (i) Since A is a subb-semiring of S and $A \subseteq \overline{\rho}(A)$, $\overline{\rho}(A)/\rho \neq \varnothing$. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\overline{\rho}(A)/\rho$. Then $[x]_\rho \cap A \neq \varnothing$ and $[y]_\rho \cap A \neq \varnothing$. Thus there exist $a, b \in S$ such that $a \in [x]_\rho \cap A$ and $b \in [y]_\rho \cap A$. So $a *_1 b \in [x]_\rho *_1 [y]_\rho$ and $a *_2 b \in [x]_\rho *_2 [y]_\rho$. Since A is a subb-semiring of S, $a *_1 b \in A$ and $a *_2 b \in A$. This implies that $a *_1 b \in ([x]_\rho *_1 [y]_\rho) \cap A$ and $a *_2 b \in ([x]_\rho *_2 [y]_\rho) \cap A$. So $[x]_\rho *_1 [y]_\rho \in \overline{\rho}(A)/\rho$ and $[x]_\rho *_2 [y]_\rho \in \overline{\rho}(A)/\rho$. Therefore $\overline{\rho}(A)/\rho$ is a subb-semiring of S/ρ.

(ii) Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\underline{\rho}(A)/\rho$. Then $[x]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Since A is a subb-semiring of S, we have $[x]_\rho *_1 [y]_\rho \subseteq A *_1 A \subseteq A$ and $[x]_\rho *_2 [y]_\rho \subseteq A *_2 A \subseteq A$. Thus $[x]_\rho *_1 [y]_\rho \in \underline{\rho}(A)/\rho$ and $[x]_\rho *_2 [y]_\rho \in \underline{\rho}(A)/\rho$. Hence $\underline{\rho}(A)/\rho$ is a subb-semiring of S/ρ. □

Theorem 3.2. *Let* ρ *be a complete congruence on a b-semiring* S. *If* A *is a right ideal* (*left ideal*, *ideal*) *of* S, *then* $\overline{\rho}(A)/\rho$ *is a right ideal* (*left ideal*, *ideal*) *of* S/ρ.

Proof. Since A is a right ideal of S and $A \subseteq \overline{\rho}(A)$, $\overline{\rho}(A)/\rho \neq \emptyset$. Let $[x]_\rho \in \overline{\rho}(A)/\rho$ and $[y]_\rho \in S/\rho$. Then $[x]_\rho \cap A \neq \emptyset$. So there exists $a \in [x]_\rho \cap A$. Thus $a \in [x]_\rho$ and $a \in A$. Let $b \in [y]_\rho$. Since A is a right ideal of S, we have $a *_1 b \in A *_1 S \subseteq A$ and $a *_2 b \in A *_2 S \subseteq A$. Thus $a *_1 b \in ([x]_\rho *_1 [y]_\rho) \cap A$ and $a *_2 b \in ([x]_\rho *_2 [y]_\rho) \cap A$. Then $[x]_\rho *_1 [y]_\rho \in \overline{\rho}(A)/\rho$ and $[x]_\rho *_2 [y]_\rho \in \overline{\rho}(A)/\rho$. Hence $\overline{\rho}(A)/\rho$ is a right ideal of S/ρ. The other case can also be proved in a similar way. □

Theorem 3.3. *Let* ρ *be a complete congruence on a b-semiring* S. *If* A *is a right ideal* (*left ideal*, *ideal*) *of* S *and* $\underline{\rho}(A) \neq \emptyset$, *then* $\underline{\rho}(A)/\rho$ *is a right ideal* (*left ideal*, *ideal*) *of* S/ρ.

Proof. Let $[x]_\rho \in \underline{\rho}(A)/\rho$ and $[y]_\rho \in S/\rho$. Then $[x]_\rho \subseteq A$. Since A is a right ideal of S, we have $[x]_\rho *_1 [y]_\rho \subseteq A *_1 S \subseteq A$ and $[x]_\rho *_2 [y]_\rho \subseteq A *_2 S \subseteq A$. Thus $[x]_\rho *_1 [y]_\rho \in \underline{\rho}(A)/\rho$ and $[x]_\rho *_2 [y]_\rho \in \underline{\rho}(A)/\rho$. Therefore $\underline{\rho}(A)/\rho$ is a right ideal of S/ρ. The other case can also be proved in a similar way. □

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References

[1] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bull. Polish Acad. Sci. Math. 42 (1994), 251-254.

[2] J. Iwinski, Algebraic approach to rough sets, Bull. Polish Acad. Sci. Math. 35 (1987), 673-683.

[3] Y. B. Jun, Roughness of gamma-subsemigroups/ideals in gamma-semigroups, Bull. Korean Math. Soc. 40 (2003), 531-536.

[4] N. Kuroki, Fuzzy congruences and fuzzy normal subgroups, Inform. Sci. 60 (1992), 247-259.

[5] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100 (1997), 139-163.

[6] N. Kuroki and P. P. Wang, The lower and upper approximations in a fuzzy group, Inform. Sci. 90 (1996), 203-220.

[7] Z. Pawlak, Rough sets, Internat. J. Comp. Inf. Sci. 11 (1982), 341-356.