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THE HYBRID APPROACH: LAGRANGIAN AND PROPER EQUALITY CONSTRAINTS (SCALARIZATION OF VECTOR OPTIMIZATION PROBLEMS (SVOP))

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Abstract

Traditional approaches were used to solve vector optimization problems (VOP). However, the traditional approaches have several limitations. Thus, researchers and practitioners use non-domination based techniques. This paper proposes a hybrid approach: scalarization of vector optimization problems which combines the characteristics of both Lagrangian and proper equality constraints characterizations. So the VOP can be solved regardless of convexity assumption. The study considers the qualitative and quantitative set of feasible parameters, the solvability set and stability set of the first kind. Followed by, two illustrative examples which are given to clarify the proposed approach.

1. Introduction

The vector optimization problems (VOP) exist in many fields; such as engineering design [6], antenna design [8], location science [1], statistics [2], management science [7], environmental analysis [9] and space exploration [20], etc.

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To solve vector optimization problems, traditional approaches were used, which are based on scalarizing the vector objectives into a single objective. Subsequently changing the problem formulation to a single objective optimization problem which has an optimal solution, and finally finds a new solution in a single run.

Determination of a single solution for VOP is performed using characteristics, such as, weighting, Lagrangian, constraints, proper equality constraints, generalized Tchebycheff norm and its extension problems sum method, weighted Tchebycheff and the hybrid problems [3-6, 12-14, 19], etc.

In multi-objective optimization, the efficient solution is not necessarily a single element. However, becoming a subset of the feasible space, and thereby requiring them to be applied as many times as the number of desired efficient solutions. However, drawbacks have accompanied these traditional approaches which encourage the researchers and practitioners to use non-domination based techniques to find a set of efficient points rather than just a single global optimal point [18]. Therefore, an analytical approach will be applied to find a set of efficient solutions.

El-sawy [5] mentioned several new SVOP which characterized the efficient solutions of corresponding VOP. This paper studies one of these scalarization problems which combines Lagrangian approach together with proper equality constraint approach. The deduced approach has the proprieties of Lagrangian and proper equality constraint approaches. Therefore, there is no need for efficient test and proper equality constraint which is generally applicable. Particularly, nonlinear problems can be solved without assuming any convexity or concavity [11]. It is much easier to compute optimal solutions of problems with equality constraints than those with inequality constraints. Furthermore, in many system design, control, or programming problems there are finite number of real variables and conditions specifically described by equalities and/or inequalities [10].

This paper is divided into seven main sections: Section 2 deals with problem formulation and some related definitions. While Section 3 presents the hybrid approach: Lagrangian and proper constraints.

Section 4 deals with the characterization of the efficient solution of VOP. Section 5 deals with the qualitative analysis of basic notions in parametric convex programming. And this section is divided into two subsections; (1) Characterization of the set of feasible parameters. (2) Characterization of the stability set of the first kind. Section 6 illustrates two examples to clarify the proposed hybrid scalarization. Section 7 concludes the results and indicates some future studies.

2. Problem Formulation and Some Related Definitions

2.1. Vector optimization problems (VOP)

The vector optimization problems (VOP) are defined as follows:

$$\min_{x \in M} F(x) = \{f_1(x), f_2(x), \ldots, f_m(x)\}, \tag{1}$$

where

$$M = \{x : x \in R^n : G(x) \leq 0\},\ F : R^n \to R^m,\ f_i : R^n \to R,\ i = 1, 2, \ldots, m$$

and

$$G : R^n \to R^l,\ G(x) = \{g_1(x), g_2(x), \ldots, g_l(x)\},\ g_i : R^n \to R,\ i = 1, 2, \ldots, l.$$

We mean by the convexity assumption that M is a convex set and $f_i : R^n \to R,\ i = 1, 2, \ldots, m$ are convex functions defined on M.

Definition 1 (Efficient solution)**.** A point x^* is said to be an *efficient solution* of VOP if there exists no other $x \in M$ such that $F(x) \leq F(x^*)$, i.e., $f_j(x) \leq f_j(x^*)$ for all $j = 1, 2, \ldots, m$ with strict inequality for at least one j. Efficient solution is also called *non-dominated solution*, *noninferior solution* and *Pareto optimal solution*. We denote the set of all efficient solutions of VOP by X^* [3].

Definition 2 (Stability)**.** Let $\omega_i(y) = \inf\{f_i(x) : G(x) \leq y, i = 1, 2, \ldots, m; y \in R^l\}$. Then VOP is said to be *stable* if $\omega_i(0)$ are finite and there exist scalar L_i such that $\frac{(\omega_i(0) - \omega_i(y))}{\| y \|} \leq L_i$, for all $y \neq 0$, $i = 1, 2, \ldots, m$ [3], where $\| \cdot \|$ is any norm of interest.

3. The Hybrid Approach: Lagrangian and Proper Constraint

The suggested hybrid approach combines the characteristics of both Lagrangian and proper equality constraint characterizations and it has the following form:

$$P_k(u, \alpha)\begin{cases} \min F_K(x, u) = \left\{ f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) \right\}, & \text{(2a)} \\ \text{subject to } M_e(\alpha) = \{x \in M : f_i(x) = \alpha_i,\ j = 1, 2, \ldots, m\}, & \text{(2b)} \\ u \in U_k = \{(u_1, u_2, \ldots, u_{K-1}, u_{K+1}, \ldots, u_m)^t \in R^{m-1} : u_i \geq 0 \\ \qquad\qquad \text{for each } i \neq K\}, \\ \alpha \in A_k = \{(\alpha_1, \alpha_2, \ldots, \alpha_m)^t \in R^m;\ P_k(u, \alpha) \text{ is feasible}\}. \end{cases}$$

The set A_k is called the *set of feasible parameters*. For a given $x^* \in M_e(\alpha)$ the symbol $P_k(u, \alpha)$ denotes the problem, where $\alpha_i = \alpha_i^* = f_i(x^*)$.

4. Characterization of the Efficient Solution of VOP

This section deals with the characterization of the efficient solution of VOP which is done in a similar manner as in [3] using $P_k(u^\circ, \alpha)$ SVOP.

Theorem 1. *x^* is an efficient solution for VOP iff x^* is an optimal solution of $P_k(u^\circ, \alpha)$ for any given $u^\circ > 0$ and for some $\alpha \in A_k$.*

Proof. For necessity: assume that x^* is an efficient solution and for any given $u^\circ > 0$, x^* does not solve $P_k(u^\circ, \alpha)$ for any $\alpha \in A_k$, where $\alpha^* = F(x^*)$. Let x° be an optimal solution of $P_k(u^\circ, \alpha)$. Then we have

$$f_K(x^\circ) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^\circ) < f_K(x^*) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^*),$$

that is,

$$f_K(x^\circ) - f_K(x^*) + \sum_{i \neq K}^{m} (u_i(f_i(x^\circ) - f_i(x^*))) < 0,$$

which implies that $f_K(x^\circ) - f_K(x^*) < 0$ and $u_i(f_i(x^\circ) - f_i(x^*)) < 0$, for all $i \neq k$, since $u^\circ > 0$, $f_i(x^\circ) \leq f_i(x^*)$, for all $i \neq k$ and $F(x^\circ) \leq F(x^*)$. Hence x^* is not an efficient solution, which contradicts the assumption.

For sufficiency: suppose that x^* solves $P_k(u^\circ, \alpha)$ for some $\alpha \in A_k$. It must also solve $P_k(u^\circ, \alpha^*)$, where $\alpha^* = F(x^*)$. Suppose x^* is not an efficient solution, which implies that there exists an $x^\circ \in M$ such that $F(x^\circ) \leq F(x^*)$. Hence for any $u^\circ > 0$, $f_K(x^\circ) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^\circ) < f_K(x^*)$ $+ \sum_{i \neq K}^{m} u_i^\circ f_i(x^*)$, which contradicts the fact that x^* solves $P_k(u^\circ, \alpha^*)$, since $x^\circ \in M$. Thus x^* must be efficient solution.

Remark 1. To generate an efficient solution for VOP, one needs to search among optimal solutions of $P_k(u^\circ, \alpha)$ for any $\alpha \in A_k$ and $u \in U_k$.

Theorem 2. *Assume that one of the following holds*:

(1) $P_k(u^\circ, \alpha)$ *is stable*, M *is a convex set and* $f_i(x)$, $i = 1, 2, \ldots, m$ *are convex on* R^n;

or

(2) *All* $f_i(x)$, $i = 1, 2, \ldots, m$ *and* $g_i(x)$, $i = 1, 2, \ldots, l$ *are faithfully convex on* R^n. *Then* $X^* = \varnothing$ *if* $\varphi_k(\alpha^*) = -\infty$, *where*

$$\varphi_k(\alpha) = \inf\left\{f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) : x \in M_e(\alpha^*),\ \alpha^* = F(x^*)\right\}.$$

Proof. Since $\varphi(\alpha^*) = -\infty$, the weak duality theorem requires that the dual of $P_k(u^*, \alpha^*)$ is also $-\infty$. That is,

$$\sup_{v}\left\{\inf_{x \in M}\left\{f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) + \sum_{i=1}^{m} v_i(f_i(x) - f_i(x^*))\right\}\right\} = -\infty.$$

Consequently, for any bounded $\hat{x} \in M$

$$\sup_{v}\left\{\inf_{x \in M}\left\{f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) + \sum_{i=1}^{m} v_i(f_i(x) - f_i(\hat{x}))\right\}\right\} = -\infty.$$

Suppose, on the contrary that there exists an efficient solution of VOP, say $\breve{x}$. From theorem, $\breve{x}$ solves $P_k(\breve{u}, \breve{\alpha})$, where $f_i(\breve{x}) = \breve{\alpha}_i$, for all $i = 1, 2, \ldots, m$, then

$$-\infty < f_K(\breve{x}) + \sum_{i \neq K}^{m} u_i f_i(\breve{x}) = \sup_{v}\left\{\inf_{x \in M}\left\{f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) + \sum_{i=1}^{m} v_i(f_i(x) - f_i(\breve{x}))\right\}\right\},$$

which is a contradiction. Thus $X^* = \varnothing$.

5. Qualitative Analysis of Basic Notions in Parametric Convex Programming

The basic notions in parametric convex programming which are defined and analyzed in [15, 16], will be redefined and analyzed qualitatively for $P_k(u, \alpha)$.

5.1. Characterization of the set of feasible parameters

Definition 3 (Set of feasible parameters)**.** The *set of feasible parameters* of problem $P_k(u, \alpha)$ is defined by $A_k = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^t \in R^m : M_e(\alpha) \neq \varnothing\}$, where $M_e(\alpha) = \{x \in M : f_j(x) = \alpha_j,\ j = 1, 2, \ldots, m\}$ and $M = \{x \in R^n : g_j(x) \leq 0,\ j = 1, 2, \ldots, l\}$.

Remark 2. The set A_k is nonempty, unbounded, if the set $M_e(\alpha)$ is unbounded.

Lemma 1. *If* $f_i(x)$, $i = 1, 2, \ldots, m$ *are linear and* g_i, $i = 1, 2, \ldots, l$ *are convex functions, then the set* A_k *is convex.*

Proof. Assume $\alpha^1, \alpha^2 \in A_k$. Then there exist $x^1, x^2 \in M_e(\alpha)$ such that

$$f_i(x^1) = \alpha_i^1, \quad i = 1, 2, \ldots, m, \quad g_j(x^1) \leq 0, \quad j = 1, 2, \ldots, l,$$

$$f_i(x^2) = \alpha_i^2, \quad i = 1, 2, \ldots, m, \quad g_j(x^2) \leq 0, \quad j = 1, 2, \ldots, l.$$

Therefore, for all $0 \leq \lambda \leq 1$, we have

$$(1-\lambda) f_i(x^1) + \lambda f_i(x^2) = (1-\lambda)\alpha_i^1 + \lambda\alpha_i^2, \quad i = 1, 2, \ldots, m$$

and

$$(1-\lambda) g_j(x^1) + \lambda g_j(x^2) \leq 0, \quad j = 1, 2, \ldots, l,$$

since $f_i(x)$, $i = 1, 2, \ldots, m$ are linear and g_j, $j = 1, 2, \ldots, l$ are convex, we have

$$f_i((1-\lambda)x^1 + \lambda x^2) = (1-\lambda)\alpha_i^1 + \lambda\alpha_i^2, \quad i = 1, 2, \ldots, m,$$

$$g_j((1-\lambda)x^1 + \lambda x^2) \leq 0, \quad j = 1, 2, \ldots, l.$$

From the convexity of $M_e(\alpha)$, we have $\{(1-\lambda)x^1 + \lambda x^2\} \in M_e(\alpha)$, then $\{(1-\lambda)\alpha^1 + \lambda\alpha^2\} \in A_k$, for all $0 \leq \lambda \leq 1$. Hence A_k is convex.

Lemma 2. *If* $M_e(\alpha^1) \cap M_e(\alpha^2) \neq \emptyset$ *and* $\alpha^1 = \alpha^2$ *for any* $\alpha^1, \alpha^2 \in A_k$, *then* $M_e(\alpha^1) = M_e(\alpha^2)$.

Proof. Let $x^* \in M_e(\alpha^1) \cap M_e(\alpha^2)$. Then $f_i(x^*) = \alpha_i^1$, $i = 1, 2, \ldots, m$ and $f_i(x^*) = \alpha_i^2$, $i = 1, 2, \ldots, m$, hence $\alpha^1 = \alpha^2$.

The previous lemma says, according to Theorem 1, the efficient solutions generated for certain $\alpha^j \in A_k$ cannot be generated again for $\alpha^i \in A_k$ if $\alpha^j \neq \alpha^i$.

5.2. Characterization of the stability set of the first kind

Definition 4 (Solvability set)**.** The *solvability set* for the proposed problem $P_k(u, \alpha)$ denoted by B_α is defined by

$$B_\alpha = \left\{(u, \alpha) \in (U_k \times A_k) : \underset{x \in M_e(\alpha)}{Min} \left\{ f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) \right\} \text{ exists} \right\},$$

$$B_\alpha = \{(u, \alpha) \in (U_k \times A_k) : X^*(u, \alpha) \neq \varnothing\},$$

where $X^*(u, \alpha)$ is defined by

$$X^*(u, \alpha) = \{x^* \in R^n : F_K(x^*, u)\}$$

$$= \left\{ f_K(x^*) + \sum_{i \neq K}^{m} u_i f_i(x^*) = \underset{x \in M_e(\alpha)}{Min} \left\{ f_K(x) + \sum_{i \neq K}^{m} u_i f_i(x) \right\} \right\}$$

and $X^* = \bigcup_{(u, \alpha) \in B_\alpha} X^*(u, \alpha)$.

Definition 5 (Stability set of the first kind)**.** Suppose that $(\overline{u}, \overline{\alpha}) \in B_\alpha$ with a corresponding efficient solution $\overline{x} \in X^*(\overline{u}, \overline{\alpha})$. Then the *stability set of the first kind* of $P_k(u, \alpha)$ corresponding to $\overline{x}$ denoted by $S_\alpha(\overline{x})$ is defined by

$$S_e(\overline{x}) = \{(u, \overline{\alpha}) \in B_\alpha : \overline{x} \in X^*(u, \alpha), \overline{\alpha} = F(\overline{x})\}.$$

Lemma 3. *If* $\alpha^1, \alpha^2 \in A_k$ *and* $\alpha^1 \neq \alpha^2$, *then* $X^*(u, \alpha^1) \cap X^*(u, \alpha^2) = \varnothing$.

Proof. Suppose that $X^*(u, \alpha^1) \cap X^*(u, \alpha^2) \neq \varnothing$ and $\alpha^1 \neq \alpha^2$. Let

$$x^* \in X^*(u, \alpha^1) \cap X^*(u, \alpha^2), x^* \in X^*(u, \alpha^1) \text{ and } x^* \in X^*(u, \alpha^2).$$

Let $S_e^1(x^*) = \{(u, \alpha^1) \in B_\alpha : x^* \in X^*(u, \alpha^1), \alpha^1 = F(x^*)\}$, and $S_e^2(x^*) = \{(u, \alpha^2) \in B_\alpha : x^* \in X^*(u, \alpha^2), \alpha^2 = F(x^*)\}$, which imply that $\alpha^1 = \alpha^2$.

To generate the stability set of the first kind for the proposed problem $P_k(u, \alpha)$. Assuming that the problem $P_k(u, \alpha)$ is stable and let $\overline{x} \in X^*(u, \alpha)$. Then there exist $\overline{\mu} \in R^m$ and $\overline{\gamma} \in R^l$, $\overline{\gamma} \geq 0$, such that $(\overline{x}, \overline{\mu}, \overline{\gamma})$ solves the following Kuhn-Tucker conditions (K-T.C.) for $P_k(u, \alpha)$ [3]:

$$\frac{\partial f_K(\overline{x})}{\partial x_v} + \sum_{i \neq K}^{m} (u_i + \overline{\mu}) \frac{\partial f_i(\overline{x})}{\partial x_v} + \sum_{j=1}^{l} \overline{\gamma}_j \frac{\partial g_j(\overline{x})}{\partial x_v} = 0, \quad v = 1, 2, \ldots, n,$$

$f_i(x) - \alpha_i = 0, \; \overline{\mu}_i \in R^m, \quad i = 1, 2, \ldots, m,$

$g_j(\overline{x}) = 0, \; \overline{\gamma}_j \geq 0, \text{ for } j \in J = \{1, 2, \ldots\} \subseteq \{1, 2, \ldots, l\},$

$g_j(\overline{x}) < 0, \; \overline{\gamma}_j = 0, \text{ for } j \notin J,$

$u_i > 0, \; i = 1, 2, \ldots, k-1, k+1, \ldots, m.$

$S_\alpha(\overline{x})$ can be determined by solving the previous K-T.C. for), $P_k(u, \alpha)$.

Theorem 3. *The set* $S_\alpha(\overline{x})$ *is convex, and* $S_\alpha(\overline{x}) \cup \{(0, \hat{\alpha})\}$ *is closed convex cone with vertex at* $(0, \hat{\alpha})$.

Proof. Let $(u^1, \overline{\alpha}), (u^2, \overline{\alpha}) \in S_\alpha(\overline{x})$. Then $f_K(\overline{x}) + \sum_{i \neq K}^{m} u_i^s f_i(\overline{x}) \leq f_K(x)$ $+ \sum_{i \neq K}^{m} u_i^s f_i(x)$, for all $x \in M_e(\overline{\alpha})$ and $s = 1, 2$, therefore

$$f_K(\overline{x}) + \sum_{i \neq K}^{m} (\eta_1 u_i^1 + \eta_2 u_i^2) f_i(\overline{x})$$

$$\leq f_K(x) + \sum_{i \neq K}^{m} (\eta_1 u_i^1 + \eta_2 u_i^2) f_i(x), \quad \text{for all } x \in M_e(\overline{\alpha}),$$

$\eta_1 \geq 0, \eta_2 \geq 0, \eta_1 + \eta_2 = 1$, then $(\eta_1 u_i^1 + \eta_2 u_i^2) \in S_\alpha(\overline{x})$, hence $S_\alpha(\overline{x})$ is convex and $S_\alpha(\overline{x}) \cup \{(0, \hat{\alpha})\}$ is convex cone.

To prove that $S_\alpha(\overline{x}) \cup \{(0, \hat{\alpha})\}$ is closed, let $u^n \in S_\alpha(\overline{x})$, for all $n = 1, 2, \ldots$, be a convergent sequence such that $\lim_{n \to \infty} u^p = u^*$. Then

$$f_K(\overline{x}) + \sum_{i \neq K}^{m} u_i^p f_i(\overline{x}) \leq f_K(x) + \sum_{i \neq K}^{m} u_i^p f_i(x),$$

for $p = 1, 2, 3, \ldots$ and for all $x \in M_e(\overline{\alpha})$. Now, we have

$$\lim_{n \to \infty} \left\{ f_K(\overline{x}) + \sum_{i \neq K}^{m} u_i^p f_i(\overline{x}) \right\} \leq \lim_{n \to \infty} \left\{ f_K(x) + \sum_{i \neq K}^{m} u_i^p f_i(x) \right\},$$

that is,

$$f_K(\bar{x}) + \sum_{i \neq K}^{m} \lim_{n \to \infty} \{u_i^p f_i(\bar{x})\} \leq f_K(x) + \sum_{i \neq K}^{m} \lim_{n \to \infty} \{u_i^p f_i(x)\}, \text{ for all } x \in M_e(\overline{\alpha}),$$

it follows that

$$f_K(\bar{x}) + \sum_{i \neq K}^{m} u_i^* f_i(\bar{x}) \leq f_K(x) + \sum_{i \neq K}^{m} u_i^* f_i(x), \text{ for all } x \in M_e(\overline{\alpha}),$$

therefore $u^* \in S_\alpha(\bar{x}) \cup \{(0, \hat{\alpha})\}$.

Theorem 4. *If interior of* $S_\alpha(x^1) \cap S_\alpha(x^2) \neq \varnothing$, *then* $S_\alpha(x^1) = S_\alpha(x^2)$.

Proof. Let $(u^\circ, \overline{\alpha}) \in \operatorname{int}\{S_\alpha(x^1) \cap S_\alpha(x^2)\}$. Then $f_K(x^1) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^1)$ $= f_K(x^2) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^2)$. Assume that $(u^1, \overline{\alpha}) \in S_\alpha(x), u^1 \neq u^\circ$. Then there exists $0 \leq \lambda \leq 1$ such that

$$\hat{u} = (1 - \lambda)u^1 + \lambda u^\circ, (\hat{u}, \overline{\alpha}) \in S_\alpha(x^2),$$

therefore

$$f_K(x^2) + \sum_{i \neq K}^{m} \hat{u}_i f_i(x^2) \leq f_K(x^1) + \sum_{i \neq K}^{m} \hat{u}_i f_i(x^1),$$

that is,

$$(1 - \lambda)\left\{f_K(x^2) + \sum_{i \neq K}^{m} u_i^1 f_i(x^2)\right\} + \lambda\left\{f_K(x^2) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^2)\right\}$$

$$\leq (1 - \lambda)\left\{f_K(x^1) + \sum_{i \neq K}^{m} u_i^1 f_i(x^1)\right\} + \lambda\left\{f_K(x^1) + \sum_{i \neq K}^{m} u_i^\circ f_i(x^1)\right\}.$$

Therefore,

$$\left\{f_K(x^2) + \sum_{i \neq K}^{m} u_i^1 f_i(x^2)\right\} < \left\{f_K(x^1) + \sum_{i \neq K}^{m} u_i^1 f_i(x^1)\right\} < \left\{f_K(x) + \sum_{i \neq K}^{m} u_i^1 f_i(x)\right\},$$

for all $x \in M_e(\overline{\alpha})$, then $(u^1, \overline{\alpha}) \in S_\alpha(\bar{x})$, hence $S_\alpha(x^1) \subseteq S_\alpha(x^2)$.

Using similar fashion, we can prove that $S_\alpha(x^2) \subseteq S_\alpha(x^1)$. Hence $S_\alpha(x^1) = S_\alpha(x^2)$.

Remark 3. Theorems 3 and 4 are true for nondifferentiable VOP [17].

6. Numerical Results

This section discusses two examples; Example 1 is related to convex assumption, while Example 2 is associated to non-convex assumption.

6.1. Example 1

6.1.1. Analytical approach

Consider a VOP

$$\min F(x) = \{f_1(x),\ f_2(x),\ f_3(x)\} \tag{3}$$

subject to

$$g_1(x) = x_1 + 2x_2 - 12 \leq 0, \tag{4a}$$

$$g_2(x) = -x_1 + x_2 - 4 \leq 0, \tag{4b}$$

$$g_3(x) = -x_1 \leq 0, \tag{4c}$$

$$g_4(x) = -x_2 \leq 0, \tag{4d}$$

where $f_1(x) = (x_1 - 1)^2 + (x_2 - 1)^2$, $f_2(x) = (x_1 - 2)^2 + (x_2 - 3.5)^2$ and $f_3(x) = (x_1 - 3)^2 + (x_2 - 2)^2$.

The proposed scalarization to a given VOP is

$$\min\{(x_1 - 1)^2 + (x_2 - 1)^2 + u_1\{(x_1 - 2)^2 + (x_2 - 3.5)^2\} + u_2\{(x_1 - 3)^2 + (x_2 - 2)^2\}\}$$

subject to

$$(x_1 - 1)^2 + (x_2 - 1)^2 = \alpha_1,$$

$$(x_1 - 2)^2 + (x_2 - 3.5)^2 = \alpha_2,$$

$$(x_1 - 3)^2 + (x_2 - 2)^2 = \alpha_3 \text{ and } (4),$$

where $u_i \geq 0, i = 1, 2$ and α_i, $i = 1, 2, 3$ are chosen such that the feasible set is nonempty, i.e., α_i, $i = 1, 2, 3,$ are less than or equal to the minimum distances between the three points $(1, 1)^t$, $(2, 3.5)^t$ and $(3, 2)^t$ and the line $x_1 + 2x_2 - 12 = 0,$ which lies between $(12, 0)^t$ and $\left(\frac{4}{3}, \frac{16}{3}\right)^t$ and the line $-x_1 + x_2 - 4 = 0,$ which lies between $(0, 4)^t$ and $\left(\frac{4}{3}, \frac{16}{3}\right)^t$.

Applying the K-T. necessary conditions, yields

$$2x_1(1 + u_1 + u_2) - 2(1 + 2u_1 + 3u_2) + 2x_1(\mu_1 + \mu_2 + \mu_3)$$

$$- 2(\mu_1 + 2\mu_2 + 3\mu_3) + \gamma_1 - \gamma_2 - \gamma_3 = 0, \tag{5a}$$

$$2x_2(1 + u_1 + u_2) - 2(1 + 3.5u_1 + 2u_2) + 2x_2(\mu_1 + \mu_2 + \mu_3)$$

$$- 2(\mu_1 + 3.5\mu_2 + 2\mu_3) + 2\gamma_1 + \gamma_2 - \gamma_4 = 0, \tag{5b}$$

$$\mu_1((x_1 - 1)^2 + (x_2 - 1)^2 - \alpha_1) = 0, \tag{5c}$$

$$\mu_2((x_1 - 2)^2 + (x_2 - 3.5)^2 - \alpha_2) = 0, \tag{5d}$$

$$\mu_3((x_1 - 3)^2 + (x_2 - 2)^2 - \alpha_3) = 0, \tag{5e}$$

$$\gamma_1(x_1 + 2x_2 - 12) = 0, \tag{5f}$$

$$\gamma_2(-x_1 + x_2 - 4) = 0, \tag{5g}$$

$$\gamma_3 x_1 = 0, \tag{5h}$$

$$\gamma_4 x_2 = 0, \tag{5i}$$

and $x_1 + 2x_2 - 12 \leq 0, -x_1 + x_2 - 4 \leq 0, x_1 \geq 0, x_2 \geq 0,$ where μ_i, $i = 1, 2, 3$ are arbitrary and $\gamma_j \geq 0,$ $j = 1, 2, 3, 4.$ For this specific problem, none of the constraint in (4) can be binded at the optimal point, since the corresponding solution of (5) which satisfies all the nonnegativity requirements for u_i, $i = 1, 2$ and γ_j, $j = 1, 2, 3, 4.$ For example, if $x_1 = x_2 = 0,$ then from (5c) and (5d) $\gamma_1 = \gamma_2 = \mu_1 = \mu_2 = \mu_3 = 0.$ Consequently, from (5a) and (5b), we have

$$-2(1 + 2u_1 + 3u_2) - \gamma_3 = 0,$$

$$-2(1 + 3u_1 + 2u_2) - \gamma_4 = 0,$$

which mean that u_i, $i = 1, 2$ have nonnegative values. When $g_1(x)$ or $g_2(x)$ is binding, for example, let $x_1 = 12$ and $x_2 = 0$, which imply that $\gamma_1 = \gamma_3 = 0$ and $\mu_1 = \mu_2 = \mu_3 = 0$. Substituting these values in (5a) and (5b), we get

$$2(11 + 10u_1 + 9u_2) + \gamma_1 = 0, \tag{6a}$$

$$-2(1 + 3.5u_1 + 2u_2) + 2\gamma_1 - \gamma_4 = 0. \tag{6b}$$

Subtracting (6a) from (6b), we get $\gamma_4 = -(46 + 47u_1 + 4u_2) < 0$, which violates $\gamma_4 \geq 0$. Performing the similar analyses to the other constraint in (5), we can conclude that none of the constraint are binding and the only possible solution to (5) is $\gamma_j = 0$, $j = 1, 2, 3, 4$ and

$$x_1^* = v_1 + 2v_2 + 3v_3, \tag{7a}$$

$$x_2^* = v_1 + 3.5v_2 + 2v_3, \tag{7b}$$

where $v_1 = \frac{1}{c}$, $v_2 = \frac{u_1}{c}$ and $v_3 = \frac{u_2}{c}$ and $c = 1 + u_1 + u_2 \geq 1$. If we define

$$x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x^2 = \begin{pmatrix} 2 \\ 3.5 \end{pmatrix}, x^3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \text{ then } x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = v_1 x^1 + v_2 x^2 + v_3 x^3. \tag{8}$$

Since $f_i = 0$, $i = 1, 2, 3$ are convex and x defined above is unique solution of system (5), x is also a unique solution of $P_k(u, \alpha)$ for a given $u \in U_\alpha$ and α_i, $i = 1, 2, 3$.

Consequently, x is also an efficient solution of VOP (3)-(4), the efficient set of problems is given by

$$X^* = \left\{ x : x \in R^2,\ x = v_1 x^1 + v_2 x^2 + v_3 x^3,\ v_1 = \frac{1}{c},\ v_2 = \frac{u_1}{c} \text{ and } v_3 = \frac{u_2}{c},\ c = 1 + u_1 + u_2,\ u_i > 0,\ i = 1, 2 \right\},$$

see Figure 1.

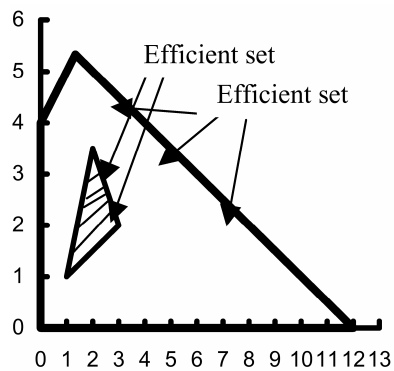

Figure 1. Efficient solution set of a given VOP in decision space.

6.1.2. Numerical approach

If we choose $u_i \geq 0$, $i = 1, 2$ and for suitable choices of α_i, $i = 1, 2, 3$ with the aid of any package (such as LINGO) for solving non-linear programming problem, we get a discrete efficient point, see Figure 2.

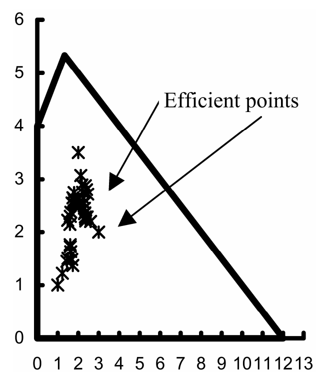

Figure 2. Some discrete efficient points of a given VOP in efficient set.

6.2. Example 2

Consider a VOP

$$\min F(x) = \{f_1(x), f_2(x)\}$$

subject to

$$g(x) = -x \leq 0,$$

where

$$f_1(x) = (x-3)^2 + 1 \quad \text{and} \quad f_2(x) = \begin{cases} -x^2 + 6x, & \text{for } 0 \le x \le 3 + \sqrt{10}, \\ -1, & \text{for } x > 3 + \sqrt{10} \end{cases}$$

solution. The function $f_1(x)$ is convex, while $f_2(x)$ is non-convex on R, see Figure 3.

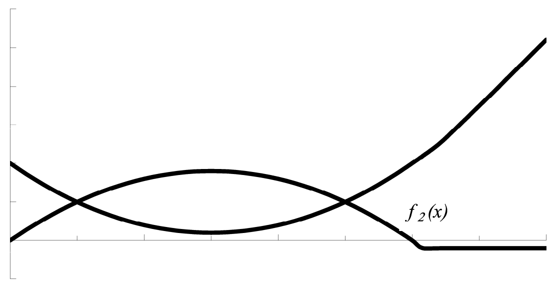

Figure 3. Convex function $f_1(x)$ and non-convex function $f_2(x)$.

The efficient set of problems is given by: $X^* = \{x \in R : 0 \le x \le 3 + \sqrt{10}\}$. The proposed scalarization to the given VOP is:

For $0 \le x \le 3 + \sqrt{10}$

$$\min f_1(x) + uf_2(x)$$

subject to

$$(x-3)^2 + 1 = \alpha_1;$$

$$-x^2 + 6x = \alpha_2;$$

$$g_1(x) = x - (3 + \sqrt{10}) \le 0;$$

$$g_2(x) = -x \le 0.$$

If we choose $u_i \ge 0$, $i = 1, 2$ and for suitable choices of α_i, $i = 1, 2$ with the aid of any package for solving non-linear programming problem, we get a discrete efficient point

$$x \in X^* = \{x \in R : 0 \le x \le 3 + \sqrt{10}\}.$$

For $x > 3 + \sqrt{10}$, the function $f_2(x) = 1$, and the VOP reduced to non-linear programming without feasible solution.

Figure 4 illustrates the efficient solution set of a given VOP in an objective space.

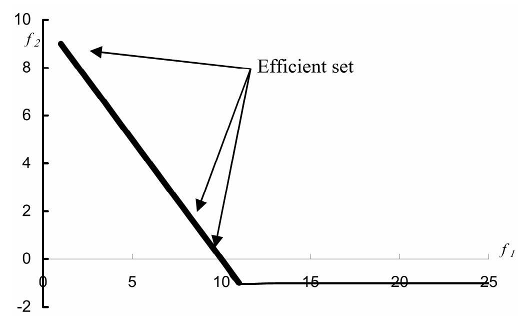

Figure 4. Efficient solution set of a given VOP in objective space. (None convex efficient set of Example 2.)

Although the function $f_2(x)$ is not convex; the proposed approach can solve it successfully. This means that it handles the duality gap.

Remark 4. Practically; in Examples 1 and 2; it is enough to use one equality constraint from the functions f_i, $i = 1, 2, \ldots, m$; because we use all these functions in the objective function.

7. Conclusion

Traditional approaches were used to solve vector optimization problems. However, due to the traditional approaches limitations, several analytic approaches were proposed by researchers, such as scalarizations of vector optimization problems. In this paper, we studied one of these scalarization problems which combines Lagrangian approach together with proper equality constraint approach. The deduced approach has the proprieties of Lagrangian and proper equality constraint approaches. Therefore, there is no need for efficient test and proper equality constraint which is generally applicable.

The main result of the study is that solving the scalarization problem which combines Lagrangian approach together with proper equality constraint approach is more beneficial than using single approach; since blending two or more approaches to solve the scalarization problem combines the strength of each approach regardless of the function convexity, which means that the proposed approach can handle the duality gap.

For further studies it is suggested to apply stability set for the second, third, and fourth kinds. In addition, the relation between other scalarization problems should also be studied.

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