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A CLASS OF HARMONIC MULTIVALENT FUNCTIONS AND A COEFFICIENT INEQUALITY

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Abstract

A coefficient inequality for a new class of harmonic multivalent functions is derived and is found to be a sufficient condition for harmonic multivalent functions to belong to this class. This condition is shown to be also necessary for a subclass of the new class introduced here. Certain other properties of the subclass are also obtained.

1. Introduction

In a seminal paper on harmonic functions Clunie and Sheil-Small [2] introduced and investigated a class H of harmonic functions $f = h + \overline{g}$

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that are univalent and sense preserving in the unit disc $\Delta = \{z : |z| < 1\}$ with the normalization

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

This work opened up a series of studies of various subclasses of the class H in which coefficient inequalities which provided sufficient conditions for harmonic functions to be elements of the subclasses were obtained.

Recently Ahuja and Jahangiri [1] defined the class $H_p(n)$ $(p, n \in N = \{1, 2, \ldots\})$ consisting of all p-valent harmonic functions $f = h + \overline{g}$ that are sense preserving in Δ and h and g of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{2}$$

A subclass $G_H(\gamma)$ of H consisting of functions $f = h + \overline{g} \in H$ that satisfy the condition

$$\operatorname{Re}\left\{(1 + e^{i\alpha}) \frac{z \dfrac{\partial}{\partial \theta}(f(z))}{\dfrac{\partial z}{\partial \theta} f(z)} - e^{i\alpha}\right\} \geq \gamma, \quad 0 \leq \gamma < 1,$$

$0 \leq r < 1$ and α, θ real, was introduced and studied in [4]. We further let $G_{\overline{H}}(\gamma)$ [4] denote the subclass of $G_H(\gamma)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n.$$

The differential operator D^m was introduced by Sălăgean [5]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [3] defined the modified Sălăgean operator for harmonic functions.

For multivalent harmonic functions $f = h + \overline{g}$ given by (2), the modified Sălăgean operator has been defined in [6] and is given by

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}, \quad p > m, \tag{3}$$

where for $p > m$,

$$D^m h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{k+p-1}{p}\right)^m a_{k+p-1} z^{k+p-1},$$

$$D^m g(z) = \sum_{k=1}^{\infty} \left(\frac{k+p-1}{p}\right)^m b_{k+p-1} z^{k+p-1}.$$

In this paper, we introduce a new class $G_H(m, n, p, \gamma)$ of Sălăgean-type harmonic multivalent functions based on the modified Sălăgean operator $D^m f$. For $0 \leq \gamma < 1$, $m \in N$, $n \in N_0$, $m > n$ and $z \in \Delta$, $G_H(m, n, p, \gamma)$ denotes the family of multivalent harmonic functions f of the form (2) such that

$$\operatorname{Re}\left\{(1 + e^{i\alpha}) \frac{D^m f(z)}{D^n f(z)} - e^{i\alpha}\right\} \geq \gamma, \tag{4}$$

where $D^m f$ is defined by (3).

Let $\overline{G_H}(m, n, p, \gamma)$ denote the subclass of $G_H(m, n, p, \gamma)$ consisting of harmonic functions $f_m = h + \overline{g}_m$ such that h and g_m are of the form

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \tag{5}$$

where $a_{k+p-1}, b_{k+p-1} \geq 0$, $|b_p| < 1$.

We note that

1. $\overline{G_H}(1, 0, 1, \gamma)$ coincides with the class $G_{\overline{H}}(\gamma)$ [4].

2. $\overline{G_H}(n+1, n, 1, \gamma)$ coincides with the class $RS_H(n, \gamma)$ [9].

3. $\overline{G_H}(m, n, 1, \gamma)$ coincides with the class $G_{\overline{H}}(m, n, \gamma)$ [7].

4. When $\alpha = 0$, $\overline{G_H}(m, n, 1, \gamma)$ coincides with the class $\overline{S_H}(m, n, (1+\gamma)/2)$ [8].

Here we obtain coefficient inequality which gives a sufficient condition for a function f to be in $G_H(m, n, p, \gamma)$. This coefficient inequality is indeed a necessary condition for f to be in $G_H(m, n, p, \gamma)$. As a consequence of this inequality, extreme points, distortion bounds are obtained for the class $\overline{G_H}(m, n, p, \gamma)$.

2. Main Results

We derive the coefficient inequality which gives a sufficient condition for functions in $G_H(m, n, p, \gamma)$.

Theorem 2.1. *Let* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (2). *Furthermore, let*

$$\sum_{k=1}^{\infty}\{\psi(m, n, p, \gamma)|a_{k+p-1}| + \theta(m, n, p, \gamma)|b_{k+p-1}|\} \leq 2, \tag{6}$$

where

$$\psi(m, n, p, \gamma) = \frac{2\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n(1+\gamma)}{1-\gamma}$$

and

$$\theta(m, n, p, \gamma) = \frac{2\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n(1+\gamma)}{1-\gamma}$$

$a_p = 1$, $0 \leq \gamma < 1$, $m \in N$, $n \in N_0$, *and* $m > n$. *Then f is sense preserving, harmonic univalent in* Δ *and* $f \in G_H(m, n, p, \gamma)$.

Proof. If $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$= 1 - \left|\frac{\sum_{k=1}^{\infty} b_{k+p-1}(z_1^{k+p-1} - z_2^{k+p-1})}{(z_1 - z_2)\sum_{k=2}^{\infty} a_{k+p-1}(z_1^{k+p-1} - z_2^{k+p-1})}\right|$$

$$\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\left[2\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n (1+\gamma)\right] |b_{k+p-1}|}{1-\gamma}}{1 - \sum_{k=2}^{\infty} \frac{\left[2\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n (1+\gamma)\right] |a_{k+p-1}|}{1-\gamma}} \geq 0$$

which proves univalence. Note that f is sense preserving in Δ. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} (k+p-1)|a_{k+p-1}||z|^{k+p-2}$$

$$> 1 - \sum_{k=2}^{\infty} \frac{2\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n (1+\gamma)}{1-\gamma} |a_{k+p-1}|$$

$$\geq \sum_{k=1}^{\infty} \frac{2\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n (1+\gamma)}{1-\gamma} |b_{k+p-1}|$$

$$\geq \sum_{k=1}^{\infty} (k+p-1)|b_{k+p-1}||z|^{k+p-2} \geq |g'(z)|.$$

According to the condition (4) we only need to show that if (6) holds, then

$$\operatorname{Re}\left\{\frac{(1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z)}{D^n f(z)}\right\} = \operatorname{Re}\frac{A(z)}{B(z)} \geq \gamma,$$

where $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $0 \leq r < 1$ and $0 \leq \gamma < 1$.

Note that $A(z) = (1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z)$ and $B(z) = D^n f(z)$. Using the fact that $\operatorname{Re} w \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{7}$$

Substituting for $A(z)$ and $B(z)$ in (7) we obtain

$$|\,(1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z) + (1-\gamma)D^n f(z)\,|$$

$$-\,|\,(1+\gamma)D^n f(z) - ((1+e^{i\alpha})D^m f(z) - e^{i\alpha}D^n f(z))\,|$$

$$= \Bigg|(1+e^{i\alpha})\Bigg[z^p + \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^m a_{k+p-1}z^{k+p-1}$$

$$+ (-1)^m \sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^m \overline{b_{k+p-1}z^{k+p-1}}\Bigg]$$

$$+ (1-\gamma-e^{i\alpha})\Bigg[z^p + \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^n a_{k+p-1}z^{k+p-1}$$

$$+ (-1)^n \sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^n \overline{b_{k+p-1}z^{k+p-1}}\Bigg]\Bigg|$$

$$-\Bigg|(1+\gamma+e^{i\alpha})\Bigg[z^p + \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^n a_{k+p-1}z^{k+p-1}$$

$$+ (-1)^n \sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^n \overline{b_{k+p-1}z^{k+p-1}}\Bigg]$$

$$-(1+e^{i\alpha})\Bigg[z^p + \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^m a_{k+p-1}z^{k+p-1}$$

$$+ (-1)^m \sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^m \overline{b_{k+p-1}z^{k+p-1}}\Bigg]\Bigg|$$

$$= \Bigg|(2-\gamma)z^p + \sum_{k=2}^{\infty}\Bigg\{\Bigg[\left(\frac{k+p-1}{p}\right)^m + (1-\gamma)\left(\frac{k+p-1}{p}\right)^n\Bigg]$$

$$+ e^{i\alpha}\left[\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n\right]\Bigg\} a_{k+p-1} z^{k+p-1}$$

$$- (-1)^n \sum_{k=1}^{\infty} \left\{\left[\left(\frac{k+p-1}{p}\right)^n (\gamma - 1) - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^m\right]\right.$$

$$\left. + e^{i\alpha}\left[\left(\frac{k+p-1}{p}\right)^n - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^m\right]\right\} \overline{b_{k+p-1} z^{k+p-1}}\Bigg|$$

$$- \Bigg| \gamma z^p - \sum_{k=2}^{\infty} \left\{\left[\left(\frac{k+p-1}{p}\right)^m - (1+\gamma)\left(\frac{k+p-1}{p}\right)^n\right]\right.$$

$$\left. + e^{i\alpha}\left[\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n\right]\right\} a_{k+p-1} z^{k+p-1}$$

$$+ (-1)^n \sum_{k=1}^{\infty} \left\{\left[(1+\gamma)\left(\frac{k+p-1}{p}\right)^n - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^m\right]\right.$$

$$\left. + e^{i\alpha}\left[\left(\frac{k+p-1}{p}\right)^n - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^m\right]\right\} \overline{b_{k+p-1} z^{k+p-1}}\Bigg|$$

$$\geq (2-\gamma)|\, z^p \,| - \sum_{k=2}^{\infty} \left[2\left(\frac{k+p-1}{p}\right)^m - \gamma\left(\frac{k+p-1}{p}\right)^n\right] |\, a_{k+p-1} \,||\, z \,|^{k+p-1}$$

$$- \sum_{k=1}^{\infty} \left|\left[\gamma\left(\frac{k+p-1}{p}\right)^n - (-1)^{m-n} 2\left(\frac{k+p-1}{p}\right)^m\right]\right| |\, b_{k+p-1} \,||\, z \,|^{k+p-1}$$

$$- \gamma|\, z^p \,| - \sum_{k=2}^{\infty} \left[2\left(\frac{k+p-1}{p}\right)^m - (2+\gamma)\left(\frac{k+p-1}{p}\right)^n\right] |\, a_{k+p-1} \,||\, z \,|^{k+p-1}$$

$$- \sum_{k=1}^{\infty} \left|\left[\left(\frac{k+p-1}{p}\right)^n (2+\gamma) - (-1)^{m-n} 2\left(\frac{k+p-1}{p}\right)^m\right]\right| |\, b_{k+p-1} \,||\, z \,|^{k+p-1}$$

$$= \begin{cases} 2(1-\gamma)|z^p| - 2\sum_{k=2}^{\infty}\left[2\left(\frac{k+p-1}{p}\right)^m - \gamma\left(\frac{k+p-1}{p}\right)^n \right. \\ \left. -\left(\frac{k+p-1}{p}\right)^n\right]|a_{k+p-1}||z|^{k+p-1} - 2\sum_{k=1}^{\infty}\left[2\left(\frac{k+p-1}{p}\right)^m \right. & m-n \text{ is odd} \\ \left. +\left(\frac{k+p-1}{p}\right)^n(1+\gamma)\right]|b_{k+p-1}||z|^{k+p-1} \\ 2(1-\gamma)|z^p| - 2\sum_{k=2}^{\infty}\left[2\left(\frac{k+p-1}{p}\right)^m - \gamma\left(\frac{k+p-1}{p}\right)^n \right. \\ \left. -\left(\frac{k+p-1}{p}\right)^n\right]|a_{k+p-1}||z|^{k+p-1} - 2\sum_{k=1}^{\infty}\left[2\left(\frac{k+p-1}{p}\right)^m \right. & m-n \text{ is even} \\ \left. -\left(\frac{k+p-1}{p}\right)^n(1+\gamma)\right]|b_{k+p-1}||z|^{k+p-1} \end{cases}$$

$$= 2(1-\gamma)|z^p|\left\{1 - \frac{\sum_{k=2}^{\infty} 2\left(\frac{k+p-1}{p}\right)^m - \gamma\left(\frac{k+p-1}{p}\right)^n - \left(\frac{k+p-1}{p}\right)^n}{1-\gamma}|a_{k+p-1}||z|^{k+p-2}\right.$$

$$\left. - \sum_{k=1}^{\infty} \frac{2\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n(1+\gamma)}{1-\gamma}|b_{k+p-1}||z|^{k+p-1}\right\}$$

$$> 2(1-\gamma)\left\{1 - \left[\sum_{k=2}^{\infty} \frac{2\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n(1+\gamma)}{1-\gamma}|a_{k+p-1}|\right.\right.$$

$$\left.\left. + \sum_{k=1}^{\infty} \frac{2\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n(1+\gamma)}{1-\gamma}|b_{k+p-1}|\right]\right\}.$$

This last expression is non-negative by (6) and so the proof is complete. The harmonic univalent functions

$$f(z) = z^p + \sum_{k=2}^{\infty} \frac{1}{\psi(m, n, p, \gamma)} x_k z^{k+p-1} + \sum_{k=1}^{\infty} \frac{1}{\theta(m, n, p, \gamma)} \overline{y_k z^{k+p-1}}, \quad (8)$$

where $m \in N,\ n \in N_0,\ m > n$ and $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1,$ show that the coefficient bound given by (6) is sharp. The functions of the form (8) are in $G_H(m, n, p, \gamma)$ because

$$\sum_{k=1}^{\infty} \{\psi(m, n, p, \gamma) |a_{k+p-1}| + \theta(m, n, p, \alpha) |b_{p+k-1}|\}$$

$$= 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.$$

In the following theorem it is shown that the condition (6) is also necessary for functions $f_m = h + \overline{g_m}$, where h and g_m are of the form (5).

Theorem 2.2. *Let* $f_m = h + \overline{g_m}$ *be given by* (5). *Then* $f_m \in \overline{G_H}(m, n, p, \gamma)$ *if and only if*

$$\sum_{k=1}^{\infty} [\psi(m, n, p, \gamma) a_{k+p-1} + \theta(m, n, p, \alpha) b_{p+k-1}] \leq 2(1-\gamma), \tag{9}$$

where $a_p = 1,\ 0 \leq \alpha < 1,\ m \in N,\ n \in N_0$ *and* $m > n$.

Proof. Since $\overline{G_H}(m, n, p, \gamma) \subset G_H(m, n, p, \gamma),$ we only need to prove the "only if " part of the theorem. For functions f_m of the form (5), we note that the condition $\operatorname{Re}\left\{(1 + e^{i\alpha}) \frac{D^m f_m(z)}{D^n f_m(z)} - e^{i\alpha}\right\} \geq \gamma$ is equivalent to

$$\operatorname{Re}\left\{\left\{(1-\gamma) z^p - \sum_{k=2}^{\infty} \left\{\left[\left(\frac{k+p-1}{p}\right)^m - \gamma\left(\frac{k+p-1}{p}\right)^n\right]\right.\right.\right.$$

$$\left.\left. + e^{i\alpha}\left[\left(\frac{k+p-1}{p}\right)^m - \left(\frac{k+p-1}{p}\right)^n\right]\right]\right\} a_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{2m-1} \sum_{k=1}^{\infty} \left\{\left[\left(\frac{k+p-1}{p}\right)^m - \gamma(-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n\right]\right.$$

$$+ e^{i\alpha}\left[\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n\right]\Bigg\} b_{k+p-1}\bar{z}^{k+p-1}\Bigg\} \Bigg/$$

$$\Bigg\{ z^p - \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^n a_{k+p-1}z^{k+p-1}$$

$$+ (-1)^{m+n-1}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^n b_{k+p-1}\bar{z}^{k+p-1}\Bigg\}\Bigg\} \geq 0. \tag{10}$$

The above required condition (10) must hold for all values of z in Δ. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\Bigg[1 - \gamma - \sum_{k=2}^{\infty}\left[2\left(\frac{k+p-1}{p}\right)^m - (1+\gamma)\left(\frac{k+p-1}{p}\right)^n\right] a_k r^{k-1}$$

$$- \sum_{k=1}^{\infty}\left[2\left(\frac{k+p-1}{p}\right)^m - (-1)^{m-n}\left(\frac{k+p-1}{p}\right)^n (1+\gamma)\right] b_{k+p-1}r^{k-1}\Bigg] \Bigg/$$

$$\left[1 - \sum_{k=2}^{\infty}\left(\frac{k+p-1}{p}\right)^n a_{k+p-1}r^{k-1} - (-1)^{m+n}\sum_{k=1}^{\infty}\left(\frac{k+p-1}{p}\right)^n b_{k+p-1}r^{k+p-1}\right] \geq 0. \tag{11}$$

If the condition (9) does not hold, then the numerator in (11) is negative for r sufficiently close to 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (11) is negative. This contradicts the required condition for $f_m \in \overline{G}_H(m, n, p, \gamma)$ and the proof is complete.

Next we determine the extreme points of the closed convex hull of $\overline{G}_H(m, n, p, \gamma)$ denoted by $clco\,\overline{G}_H(m, n, p, \gamma)$.

Theorem 2.3. *Let* f_m *be given by* (5). *Then* $f_m \in \overline{G}_H(m, n, p, \gamma)$ *if and only if*

$$f_m(z) = \sum_{k=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + y_{k+p-1}g_{m_{k+p-1}}(z)\right],$$

where

$$h_p(z) = z^p,\ h_{k+p-1}(z) = z^p - \frac{1}{\psi(m,\ n,\ p,\ \gamma)} z^{k+p-1} \quad (k = 2,\ 3,\ \ldots)$$

and

$$g_{m_{k+p-1}}(z) = z^p + (-1)^{m-1} \frac{1}{\theta(m,\ n,\ p,\ \gamma)} \bar{z}^{k+p-1} \quad (k = 1,\ 2,\ \ldots)$$

$x_{k+p-1} \geq 0$, $y_{k+p-1} \geq 0$, $x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}$. *In particular, the extreme points of* $\overline{G}_H(m,\ n,\ p,\ \gamma)$ *are* $\{h_{p+k-1}\}$ *and* $\{g_{m_{k+p-1}}\}$.

Proof. Suppose

$$f_m(z) = \sum_{k=1}^{\infty} [x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z)]$$

$$= \sum_{k=1}^{\infty} (x_{k+p-1} y_{k+p-1}(z)) z^p - \sum_{k=2}^{\infty} \frac{1}{\psi(m,\ n,\ p,\ \gamma)} x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{m-1} \sum_{k=1}^{\infty} \frac{1}{\theta(m,\ n,\ p,\ \gamma)} y_{k+p-1} \bar{z}^{k+p-1}.$$

Then

$$\sum_{k=2}^{\infty} \psi(m,\ n,\ p,\ \gamma) \left(\frac{1}{\psi(m,\ n,\ p,\ \gamma)} x_{k+p-1} \right)$$

$$+ \sum_{k=1}^{\infty} \theta(m,\ n,\ p,\ \gamma) \left(\frac{1}{\theta(m,\ n,\ p,\ \gamma)} y_{k+p-1} \right)$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} + \sum_{k=1}^{\infty} y_{k+p-1} = 1 - x_p \leq 1$$

and so $f_m(z) \in clco\, \overline{G}_H(m,\ n,\ p,\ \gamma)$.

Conversely, suppose $f_m(z) \in clco\, \overline{G}_H(m,\ n,\ p,\ \gamma)$.

Letting

$$x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1},$$

set

$$x_{k+p-1} = \psi(m, n, p, \gamma) a_{k+p-1} \; (k = 2, 3, \ldots)$$

and

$$y_{k+p-1} = \theta(m, n, p, \gamma) b_{k+p-1} \; (k = 1, 2, \ldots).$$

We obtain the required representation, since

$$f_m(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} \bar{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1}{\psi(m, n, p, \gamma)} x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{m-1} \sum_{k=1}^{\infty} \frac{1}{\theta(m, n, p, \gamma)} y_{k+p-1} \bar{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} [z^p - h_{k+p-1}(z)] x_{k+p-1} - \sum_{k=1}^{\infty} [z^p - g_{m_{k+p-1}}(z)] y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p + \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z)$$

$$+ \sum_{k=1}^{\infty} y_{k+p-1} g_{m_{k+p-1}}(z)$$

$$= \left[\sum_{k=1}^{\infty} x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z)\right].$$

The following theorem gives the distortion bounds for functions in $\overline{G}_H(m, n, p, \gamma)$ which yields a covering result for this class.

Theorem 2.4. *Let* $f_m \in \overline{G}_H(m, n, p, \gamma)$. *Then for* $|z| = r < 1$, *we have*

$$|f_m(z)| \leq (1 + b_p)r^p + \{\phi(m, n, p, \gamma) - \Omega(m, n, p, \gamma)b_p\}r^{p+1}$$

and

$$|f_m(z)| \geq (1 - b_p)r^p - \{\phi(m, n, p, \gamma) - \Omega(m, n, p, \gamma)b_p\}r^{p+1},$$

where

$$\phi(m, n, p, \gamma) = \frac{1-\gamma}{\left(\frac{p+1}{p}\right)^n \left\{\left(\frac{p+1}{p}\right)^{m-n+1} - (1+\gamma)\right\}},$$

$$\Omega(m, n, p, \gamma) = \frac{\left(\frac{p+1}{p}\right) - (-1)^{m-n}(1+\gamma)}{\left(\frac{p+1}{p}\right)^n \left\{\left(\frac{p+1}{p}\right)^{m-n+1} - (1+\gamma)\right\}}.$$

Proof. We prove the right hand side inequality for $|f_m|$. The proof for the left hand inequality can be done using similar arguments. Let $f_m \in \overline{G}_H(m, n, p, \gamma)$. Taking the absolute value of f_m then by Theorem 2.2, we obtain

$$|f_m(z)|$$

$$= \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1}z^{k+p-1} + (-1)^{m-1}\sum_{k=1}^{\infty} b_{k+p-1}z^{k+p-1} \right|$$

$$\leq r^p + \sum_{k=2}^{\infty} a_{k+p-1}r^{k+p-1} + \sum_{k=1}^{\infty} b_{k+p-1}r^{k+p-1}$$

$$= (1 + b_p)r^p + \phi(m, n, p, \gamma)\sum_{k=2}^{\infty} \frac{1}{\phi(m, n, p, \gamma)}(a_{k+p-1} + b_{k+p-1})r^{p+1}$$

$$\leq (1 + b_p)r^p + \phi(m, n, p, \gamma)r^{p+1}\left[\sum_{k=2}^{\infty} \psi(m, n, p, \gamma)a_{k+p-1} + \theta(m, n, p, \gamma)b_{k+p-1}\right]$$

$$\leq (1 + b_p)r^p + \{\phi(m, n, p, \gamma) - \Omega(m, n, p, \gamma)b_p\}r^{p+1}.$$

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.1. *Let* $f_m \in \overline{G}_H(m, n, p, \gamma)$. *Then for* $|z| = r < 1$, *we have*

$$\{w : |w| < 1 - b_p - [\phi(m, n, p, \gamma) - \Omega(m, n, p, \gamma)b_p] \subset f_m(\Delta)\}.$$

Remark 2.1. The results of this paper for $p = 1$ coincide with the results in [7].

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