Advances in Differential Equations and Control Processes
Volume 2, Number 2, November 2008, Pages 135-151
Published online: March 1, 2009
This paper is available online at http://www.pphmj.com

BOUNDARY CONTROL FOR COOPERATIVE SYSTEMS INVOLVING PARABOLIC OPERATORS WITH AN INFINITE NUMBER OF VARIABLES

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Abstract

The object of this paper is the study of optimal control for $n \times n$ cooperative parabolic systems through Neumann conditions. We first prove the existence of solutions for these systems and then we discuss the optimal control of boundary type for these systems. Our considered systems involve parabolic operators with an infinite number of variables and also with variable coefficients.

Introduction

Some optimal control problems for systems governed by parabolic operators are introduced in [5, 12, 21]. These systems in the form

$$\begin{cases} \dfrac{\partial}{\partial t} y + A(t) y = f & \text{in } Q = R^{\infty} \times (0, T) \\ \dfrac{\partial y}{\partial v_A} = 0 & \text{on } \sum = \Gamma \times (0, T), \end{cases}$$

2000 Mathematics Subject Classification: 49J20.

Keywords and phrases: cooperative parabolic systems, spaces with an infinite number of variables, existence and uniqueness of solutions, boundary control problems.

Received January 1, 2009

where $A(t)$ is a second order self-adjoint elliptic operator with an infinite number of variables. The *boundary control problems* for such systems governed by both elliptic and hyperbolic type operators defined on spaces with an infinite number of variables are also discussed in [4, 6].

The corresponding distributed control problems are discussed, for example in [10, 18, 19].

Some problems for non cooperative systems are presented in [22, 24].

Some applications for boundary control problems are introduced, for example in [1, 11, 14] and for distributed control problems in [2, 16, 20, 23].

Using the theory of Lions [17], Brezanskii [3] and Gali et al. [8, 10], Serag [21] studied the optimal control of distributed type for $n \times n$ cooperative systems involving parabolic operators with an infinite number of variables. Here, we consider the problem with control in the boundary through Neumann conditions. We first prove the existence and uniqueness of the state for $n \times n$ cooperative parabolic systems involving parabolic operators with an infinite number of variables; then we find the set of equations and inequalities that characterize the boundary control for these systems. In section II, we introduce the problem through homogeneous Neumann conditions and in section III, we study the problem with non-homogeneous Neumann conditions.

I. Function spaces on R^∞

In this paper, we shall consider spaces of functions of infinitely many variables (see [3-5]). For this purpose, we shall introduce the infinite product $R^\infty = R^1 \times R^1 \times \cdots$, with elements $(x = (x_n)_{n=1}^\infty \in R^\infty, x_n \in R^1)$, and we denote by $d\rho(x)$ the product of measures $d\rho(x) = p_1(x_1)dx_1 \otimes p_2(x_2)dx_2 \otimes \cdots$, defined on the σ-hall of cylindrical sets in R^∞ generated by the finite dimensional Borel sets, where $(p_k(t))_{k=1}^\infty$ is a sequence of weights such that

$$0 \prec p_k(t) \in C^\infty(R^1), \quad \int_{R^1} p_k(t)dt = 1.$$

With respect to this measure and on R^{∞}, with sufficiently smooth boundary Γ, we construct the space $L^2(R^{\infty}, d\rho(x))$ of functions $u(x)$ which are measurable such that

$$\| u \|_{L^2(R^{\infty},\, d\rho(x))} = \left(\int_{R^{\infty}} | u |^2 d\rho(x) \right)^{1/2} \prec \infty.$$

We shall set $L^2(R^{\infty}, d\rho(x)) = L^2(R^{\infty})$.

$L^2(R^{\infty})$ is a Hilbert space for the scalar product

$$(u, v)_{L^2(R^{\infty})} = \int_{R^{\infty}} u(x) v(x) d\rho(x)$$

associated to the above norm. For functions which are continuously differentiable up to the boundary Γ of R^{∞} and which vanish in a neighborhood of ∞, we introduce the scalar product

$$(u, v) = \sum_{| \alpha | \leq 1} (D^{\alpha} u, D^{\alpha} v)_{L^2(R^{\infty})}, \tag{1}$$

where D^{α} is defined by

$$D^{\alpha} = \frac{\partial^{| \alpha |}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n} \cdots}, \quad | \alpha | = \sum_{i=1}^{\infty} \alpha_i$$

and the differentiation in the sense of generalized function, and after the completion, we obtain the Sobolev space $W^1(R^{\infty})$. This space forms a *Hilbert space* endowed with the scalar product defined in (1). The space $W^1(R^{\infty})$ forms a positive space. We can construct negative space $W^{-1}(R^{\infty})$ with respect to the zero space $L^2(R^{\infty})$ and then we have the following imbedding

$$W^1(R^{\infty}) \subseteq L^2(R^{\infty}) \subseteq W^{-1}(R^{\infty}).$$

Let $L^2(0, T; W^1(R^{\infty}))$ denote the space of measurable function $t \to f(t)$ on open interval $(0, T)$ and the variable t denotes the time.

We assume that $t \in (0, T)$, $T < \infty$ with Lebesgue measure dt on $(0, T)$ such that

$$\| f(t) \|_{L^2(0,\, T;\, W^1(R^\infty))} = \left(\int_{(0, T)} \| f(t) \|^2_{W^1(R^\infty)} dt \right)^{1/2} \prec \infty$$

endowed with the scalar product

$$(f(t),\, g(t))_{L^2(0,\, T;\, W^1(R^\infty))} = \int_{(0, T)} (f(t),\, g(t))_{W^1(R^\infty)} dt,$$

which is a Hilbert space.

Analogously, we can define the *spaces* $L^2(0, T; L^2(R^\infty)) = L^2(Q)$ and $L^2(0, T; W^{-1}(R^\infty))$, then we have a chain in the form

$$L^2(0, T; W^1(R^\infty)) \subseteq L^2(0, T; L^2(R^\infty)) \subseteq L^2(0, T; W^{-1}(R^\infty)),$$

where $Q = R^\infty \times (0, T)$ with boundary $\Sigma = \Gamma \times (0, T)$.

By Cartesian product, we have the following chain:

$$(L^2(0, T; W^1(R^\infty)))^n \subseteq (L^2(0, T; L^2(R^\infty)))^n \subseteq (L^2(0, T; W^{-1}(R^\infty)))^n.$$

II. Boundary Control for $n \times n$ Cooperative Homogeneous Neumann Systems involving Parabolic Operators with an Infinite Number of Variables

In this section, we find the necessary and sufficient condition for the control to be optimal for the following $n \times n$ cooperative homogeneous Neumann systems involving parabolic operators with an infinite number of variables

$$\begin{cases} \dfrac{\partial y_i(x)}{\partial t} + (A(t))\, y_i = \displaystyle\sum_{j=1}^{n} h_{ij}(x)\, y_j + f_i(x, t) \text{ in } Q \\ \left.\dfrac{\partial y_i(x)}{\partial \nu}\right|_{\Sigma} = 0, \quad 1 \le i \le n, \\ y_i(x, 0) = y_{0,\, i}(x), \;\; x \in R^\infty,\; y_{0,\, i}(x) \in L^2(R^\infty), \end{cases} \tag{N}$$

where

$$(A(t))y(x) = -\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k(x_k, t)}} \frac{\partial^2}{\partial x_k^2} \sqrt{p_k(x_k, t)} y(x) + q(x, t)y(x)$$

$$= -\sum_{k=1}^{\infty} D_k^2 y(x) + q(x, t)y(x), \tag{2}$$

$$D_k^2 y(x) = \frac{1}{\sqrt{p_k(x_k, t)}} \frac{\partial^2}{\partial x_k^2} \sqrt{p_k(x_k, t)} y(x),$$

$q(x, t)$ is a real valued function in x which is bounded and measurable on R^∞ such that $q(x, t) \geq c$, $0 < c \leq 1$ and $F = (f_1, f_2, \ldots, f_n)$ is a given function.

We assume that $h_{ij}(x)$ are bounded functions such that

$$h_{ij}(x) \succ 0 \text{ for all } i \neq j, \text{ for all } x, \tag{3}$$

$$h_{ij} = h_{ji} \text{ for all } 1 \leq i, j \leq n. \tag{4}$$

System (N) is called *cooperative* if (3) holds [7].

We have (see [21])

Theorem 1. *For a given* $F = (f_1, f_2, \ldots, f_n) \in (L^2(0, T; W^{-1}(R^\infty)))^n$, *there exists a unique solution* $Y = (y_1, y_2, \ldots, y_n) \in (L^2(0, T; W^1(R^\infty)))^n$ *for system* (N).

Proof. We define on $(L^2(0, T; W^1(R^\infty)))^n$ for each t a continuous bilinear form $\pi(t; Y, \Phi) : (L^2(0, T; W^1(R^\infty)))^n \times (L^2(0, T; W^1(R^\infty)))^n \to R$ by

$$\pi(t; Y, \Phi) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} \int_{R^\infty} D_k y_i(x) D_k \phi_i(x) d\rho(x)$$

$$+ \sum_{i=1}^{n} \int_{R^\infty} q(x, t) y_i(x) \phi_i(x) d\rho(x)$$

$$- \sum_{i, j=1}^{n} \int_{R^\infty} h_{ij}(x) y_j(x) \phi_i(x) d\rho(x). \tag{5}$$

From (5), we have

$$\pi(t;\ Y,\ Y) = \sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty} |\ D_k y_i(x)\ |^2 d\rho(x) + \sum_{i=1}^{n}\int_{R^\infty} q(x,\ t)|\ y_i(x)\ |^2 d\rho(x)$$

$$- \sum_{i,\ j=1}^{n}\int_{R^\infty} h_{ij}(x) y_i(x) y_j(x) d\rho(x),$$

then

$$\pi(t;\ Y,\ Y) + \sum_{i,\ j=1}^{n}\int_{R^\infty} h_{ij}(x) y_i(x) y_j(x) d\rho(x)$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty} |\ D_k y_i(x)\ |^2 d\rho(x) + \sum_{i=1}^{n}\int_{R^\infty} q(x,\ t)|\ y_i(x)\ |^2 d\rho(x),$$

hence

$$\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty} |\ D_k y_i(x)\ |^2 d\rho(x) + \sum_{i=1}^{n}\int_{R^\infty} q(x,\ t)|\ y_i(x)\ |^2 d\rho(x)$$

$$= \pi(t;\ Y,\ Y) + \sum_{i=1}^{n}\int_{R^\infty} h_{ii}(x)|\ y_i(x)\ |^2 d\rho(x) + \sum_{i \neq j}^{n}\int_{R^\infty} h_{ij}(x) y_i(x) y_j(x) d\rho(x).$$

From (3) and (4), we deduce that

$$\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty} |\ D_k y_i(x)\ |^2 d\rho(x) + \sum_{i=1}^{n}\int_{R^\infty} q(x,\ t)|\ y_i(x)\ |^2 d\rho(x)$$

$$\leq \pi(t;\ Y,\ Y) + c_1 \sum_{i=1}^{n}\int_{R^\infty} |\ y_i(x)\ |^2 d\rho(x) + 2c_1 \sum_{i \prec j}^{n}\int_{R^\infty} y_i(x) y_j(x) d\rho(x)$$

$$= \pi(t;\ Y,\ Y) + c_1 \left(\sum_{i=1}^{n} \|\ y_i(x)\ \|_{L^2(R^\infty)}\right)^2$$

$$= \pi(t;\ Y,\ Y) + c_1 \sum_{i=1}^{n} \|\ y_i(x)\ \|_{L^2(R^\infty)}^2, \tag{6}$$

since

$$\sum_{i=1}^{n}\sum_{k=1}^{\infty}\int_{R^\infty} | D_k y_i(x) |^2 d\rho(x) + \sum_{i=1}^{n}\int_{R^\infty} q(x, t) | y_i(x) |^2 d\rho(x)$$

$$\geq \sum_{i=1}^{n}\sum_{k=1}^{\infty} \| D_k y_i(x) \|^2_{L^2(R^\infty)} + c\sum_{i=1}^{n} \| y_i(x) \|^2_{L^2(R^\infty)}$$

$$= \sum_{k=1}^{\infty} \| D_k Y(x) \|^2_{(L^2(R^\infty))^n} + c\| Y(x) \|^2_{(L^2(R^\infty))^n}$$

$$+ c\sum_{k=1}^{\infty} \| D_k Y(x) \|^2_{(L^2(R^\infty))^n} - c\sum_{k=1}^{\infty} \| D_k Y(x) \|^2_{(L^2(R^\infty))^n}$$

$$= c\left(\| Y(x) \|^2_{(L^2(R^\infty))^n} + \sum_{k=1}^{\infty} \| D_k Y(x) \|^2_{(L^2(R^\infty))^n} \right)$$

$$+ (1 - c)\sum_{k=1}^{\infty} \| D_k Y(x) \|^2_{(L^2(R^\infty))^n}$$

$$= c\| Y(x) \|^2_{(W^1(R^\infty))^n} + (1 - c)\sum_{k=1}^{\infty} \| D_k Y(x) \|^2_{(L^2(R^\infty))^n} \geq c\| Y(x) \|^2_{(W^1(R^\infty))^n},$$

then from (6), we obtain

$$\pi(t; Y, Y) + c_1\| Y(x) \|^2_{(L^2(R^\infty))^n} \geq c\| Y(x) \|^2_{(W^1(R^\infty))^n}. \tag{7}$$

Now, let $\Phi \to L(\Phi)$ be a continuous linear form defined on $(L^2(0, T; W^1(R^\infty)))^n$ by

$$L(\Phi) = \sum_{i=1}^{n}\int_{Q} f_i(x, t)\phi_i(x) d\rho(x) dt,$$

then by Lax-Milgram lemma, there exists a unique solution $Y = (y_1, y_2, ..., y_n) \in (L^2(0, T; W^1(R^\infty)))^n$ such that $\pi(t; Y, \Phi) = L(\Phi)$ for all $\Phi \in (L^2(0, T; W^1(R^\infty)))^n$ and hence for system (N).

Formulation of control problem for system (N)

The space $(L^2(\Sigma))^n$ being the space of controls. For a control $u = (u_1, u_2, ..., u_n) \in (L^2(\Sigma))^n$, the state $Y(u) = (y_1(u), y_2(u), ..., y_n(u)) \in (L^2(0, T; W^1(R^\infty)))^n$ of the system is given by the solution of

$$\begin{cases} \dfrac{\partial y_i(x)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) y_i(u) = \sum_{j=1}^{n} h_{ij}(x) y_j(u) + f_i \text{ in } Q \\ \left.\dfrac{\partial y_i(u)}{\partial \nu}\right|_{\Sigma} = u_i, \quad y_i(x, 0, u) = y_{0,i}(x),\ x \in R^\infty,\ 1 \le i \le n. \end{cases} \tag{8}$$

The observation equation is given by $Z(u) = (z_1(u), z_2(u), ..., z_n(u)) = Y(u) = (y_1(u), y_2(u), ..., y_n(u))$.

For a given $Z_d = (z_{d1}, z_{d2}, ..., z_{dn}) \in (L^2(Q))^n$, the cost function is given by

$$J(v) = \sum_{i=1}^{n} \| y_i(v) - z_{di} \|^2_{L^2(Q)} + M \sum_{i=1}^{n} \| v_i \|^2_{L^2(\Sigma)}, \tag{9}$$

where M is a positive constant.

The control problem then is to find $\inf J(v)$ over a closed convex subset U_{ad} of $(L^2(\Sigma))^n$.

Since the cost function (9) can be written as (see [17]):

$$J(v) = a(v, v) - 2L(v) + \| Y(v) - Z_d \|^2_{(L^2(Q))^n},$$

where $a(v, v)$ is a continuous coercive bilinear form and $L(v)$ is a continuous linear form on $(L^2(0, T; W^1(R^\infty)))^n$. Then using the general theory of Lions [17], there exists a unique optimal control $u \in U_{ad}$ such that $J(u) = \inf J(v)$ for all $v \in U_{ad}$. Moreover, we have the following theorem which gives the necessary and sufficient conditions of optimality:

Theorem 2. *Assume that* (7) *holds, the cost function is given by* (9). *A necessary and sufficient condition for* $u = (u_1, u_2, \ldots, u_n) \in (L^2(\Sigma))^n$ *to be an optimal control is that the following equations and inequalities are satisfied*:

$$\begin{cases} \dfrac{-\partial p_i(u)}{\partial t} + \left(-\displaystyle\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_i(u) - \displaystyle\sum_{j=1}^{n} h_{ij}(x) p_j(u) = y_i(u) - z_{di} \text{ in } Q \\ \left.\dfrac{\partial p_i(u)}{\partial \nu}\right|_{\Sigma} = 0, \quad p_i(x, T, u) = 0, \; x \in R^{\infty} \text{ for all } 1 \le i \le n \end{cases} \tag{10}$$

$$(P(u) + Mu, v - u)_{(L^2(\Sigma))^n} \ge 0 \quad \forall v = (v_1, v_2, \ldots, v_n) \in U_{ad} \tag{11}$$

together with (8), *where* $P(u) = (p_1(u), p_2(u), \ldots, p_n(u))$ *is the adjoint state.*

Proof. Since

$$(P, BY)_{(L^2(Q))^n} = \sum_{i=1}^{n} \int_{(0, T)} \left(p_i, \frac{\partial y_i(u)}{\partial t} - \sum_{k=1}^{\infty} D_k^2 y_i + q(x, t) y_i - \sum_{j=1}^{n} h_{ij} y_j\right)_{L^2(R^{\infty})} dt.$$

Using Green's formula

$$\begin{aligned} &(P, BY)_{(L^2(Q))^n} \\ &= \sum_{i=1}^{n} \int_{(0, T)} \Bigg[\left(\frac{-\partial p_i(u)}{\partial t}, y_i(u)\right)_{L^2(R^{\infty})} + \left(-\sum_{k=1}^{\infty} D_k^2 p_i, y_i(u)\right)_{L^2(R^{\infty})} \\ &\quad + (q(x, t) p_i, y_i(u))_{L^2(R^{\infty})} - \left(\sum_{j=1}^{n} h_{ij} p_j, y_i(u)\right)_{L^2(R^{\infty})} - (p_i(u), u)_{L^2(\Gamma)}\Bigg] dt. \end{aligned}$$

Hence from (4), we have

$$\begin{aligned} B^* P(u) &= B^* \{p_1(u), p_2(u), \ldots, p_n(u)\} \\ &= \left\{\frac{-\partial p_1(u)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_1(u) - \sum_{j=1}^{n} h_{1j}(x) p_j(u),\right. \end{aligned}$$

$$\frac{-\partial p_2(u)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_2(u) - \sum_{j=1}^{n} h_{2j}(x) p_j(u), \ldots$$

$$\left.\frac{-\partial p_n(u)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_n(u) - \sum_{j=1}^{n} h_{nj}(x) p_j(u)\right\}.$$

Then the adjoint equation $\frac{\partial P(u)}{\partial t} + B^* P(u) = Y(u) - Z_d$ can be written as the first equation in (10).

The optimal control $u = (u_1, u_2, \ldots, u_n) \in (L^2(\Sigma))^n$ is characterized by (see [17])

$$\sum_{i=1}^{n} J'(u)(v_i - u_i) \geq 0 \quad \forall v = (v_1, v_2, \ldots, v_n) \in U_{ad}$$

which is equivalent to

$$\sum_{i=1}^{n} (y_i(u) - z_{di},\, y_i(v) - y_i(u))_{L^2(Q)} + M(u_i,\, v_i - u_i)_{L^2(\Sigma)} \geq 0.$$

This inequality can be written as

$$\sum_{i=1}^{n} \int_{(0,\, T)} (y_i(u) - z_{di},\, y_i(v) - y_i(u))_{L^2(R^\infty)} dt + M(u_i,\, v_i - u_i)_{L^2(\Sigma)} \geq 0$$

from (10)

$$\sum_{i=1}^{n} \int_{(0,\, T)} \left(\frac{-\partial p_i}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_i \right.$$

$$\left. - \sum_{j=1}^{n} h_{ij} p_j,\, y_i(v) - y_i(u)\right)_{L^2(R^\infty)} dt + M(u_i,\, v_i - u_i)_{L^2(\Sigma)} \geq 0.$$

Using Green's formula, we obtain

$$\sum_{i=1}^{n} \int_{(0,\, T)} \left(p_i(u), \left(\frac{\partial}{\partial t} - \sum_{k=1}^{\infty} D_k^2(I) + q(x, t) - \sum_{j=1}^{n} h_{ij}\right) y_i(v) - y_i(u)\right)_{L^2(R^\infty)} dt$$

$$- \int_{(0,\,T)} \left(\frac{\partial p_i(u)}{\partial \nu},\, y_i(v) - y_i(u) \right)_{L^2(\Gamma)} dt$$

$$+ \int_{(0,\,T)} \left(p_i(u),\, \frac{\partial(y_i(v) - y_i(u))}{\partial \nu} \right)_{L^2(\Gamma)} dt + M(u_i,\, v_i - u_i)_{L^2(\Sigma)} \geq 0.$$

Using (8)

$$\sum_{i=1}^{n} \int_{(0,\,T)} (p_i(u),\, v_i - u_i)_{L^2(\Gamma)} dt + M(u_i,\, v_i - u_i)_{L^2(\Sigma)} \geq 0$$

which is equivalent to

$$(P(u) + Mu,\, v - u)_{(L^2(\Sigma))^n} \geq 0.$$

III. Non-homogeneous Neumann Problems

In this section, we study the boundary control for the following $n \times n$ cooperative systems through non-homogeneous Neumann conditions:

$$\begin{cases} \dfrac{\partial y_i(x)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x,\,t) \right) y_i = \sum_{j=1}^{n} h_{ij}(x) y_j + f_i(x,\,t) \text{ in } Q \\ \dfrac{\partial y_i(x)}{\partial \nu}\Big|_{\Sigma} = g_i, \quad 1 \leq i \leq n, \\ y_i(x,\,0) = y_{0,\,i}(x), \;\; x \in R^{\infty},\; y_{0,\,i}(x) \in L^2(R^{\infty}), \end{cases} \tag{D}$$

where $g_i \in W^{-1/2}(\Sigma)$ for all $1 \leq i \leq n$.

With the same bilinear form defined in (5), we shall prove the following theorem which gives the existence and uniqueness of the state for system (D).

Theorem 3. *For a given* $F = (f_1,\, f_2,\, \ldots,\, f_n) \in (L^2(0,\,T;\, W^{-1}(R^{\infty})))^n$, *there exists a unique solution* $Y = (y_1,\, y_2,\, \ldots,\, y_n) \in (L^2(0,\,T;\, W^1(R^{\infty})))^n$ *for system* (D).

Proof. Let $\Phi \to L(\Phi)$ be a continuous linear form defined on $(L^2(0, T; W^1(R^\infty)))^n$ by

$$L(\Phi) = \sum_{i=1}^{n} \int_Q f_i(x, t)\phi_i d\rho dt + \int_\Sigma g_i \phi_i d\Sigma, \tag{12}$$

for all $\Phi = (\phi_1, \phi_2, ..., \phi_n) \in (L^2(0, T; W^1(R^\infty)))^n$, $g = (g_1, g_2, ..., g_n) \in (W^{-1/2}(\Sigma))^n$.

Since (7) holds, by Lax-Milgram lemma, there exists a unique element $Y = (y_1, y_2, ..., y_n) \in (L^2(0, T; W^1(R^\infty)))^n$ such that

$$\pi(t; Y, \Phi) = L(\Phi) \text{ for all } \Phi \in (L^2(0, T; W^1(R^\infty)))^n. \tag{13}$$

Hence Y is a solution of

$$\left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) y_i = \sum_{j=1}^{n} h_{ij}(x) y_j + f_i(x, t) \text{ in } Q \text{ for all } 1 \le i \le n,$$

this equation satisfies the Neumann condition. Multiplying both sides by $\Phi = (\phi_1, \phi_2, ..., \phi_n) \in (L^2(0, T; W^1(R^\infty)))^n$ and integrating over Q, we get

$$\sum_{i=1}^{n} \sum_{k=1}^{\infty} \int_Q (-D_k^2(I) + q(x, t)) y_i \phi_i d\rho dt$$

$$- \sum_{i,j=1}^{n} \int_Q h_{ij}(x) y_j \phi_i d\rho dt = \sum_{i=1}^{n} \int_Q f_i \phi_i d\rho dt.$$

From (13)

$$\sum_{i=1}^{n} \sum_{k=1}^{\infty} \int_Q (-D_k^2(I) + q(x, t)) y_i \phi_i d\rho dt$$

$$- \sum_{i,j=1}^{n} \int_Q h_{ij}(x) y_j \phi_i d\rho dt = \pi(t; Y, \Phi) - \int_\Sigma g_i \phi_i d\Sigma.$$

Using Green's formula, we obtain

$$\pi(t;\, Y,\, \Phi) - \sum_{i=1}^{n} \int_{\Sigma} \frac{\partial y_i}{\partial \nu} \phi_i d\,\Sigma = \pi(t;\, Y,\, \Phi) - \int_{\Sigma} g_i \phi_i d\,\Sigma.$$

Hence

$$g_i \Big|_{\Sigma} = \frac{\partial y_i}{\partial \nu}\Big|_{\Sigma} \quad \forall i = 1,\, 2,\, \ldots,\, n.$$

Formulation of control problem for system (D)

The space $(W^{-1/2}(\Sigma))^n$ is the space of controls. The state $Y(u) = (y_1(u),\, y_2(u),\, \ldots,\, y_n(u)) \in (L^2(0,\, T;\, W^1(R^{\infty})))^n$ of the system is given by the solution of

$$\begin{cases} \dfrac{\partial y_i(u)}{\partial t} + \left(-\displaystyle\sum_{k=1}^{\infty} D_k^2(I) + q(x,\, t)\right) y_i(u) = \displaystyle\sum_{j=1}^{n} h_{ij}(x) y_j(u) + f_i \text{ in } Q \\ \dfrac{\partial y_i(u)}{\partial \nu}\Big|_{\Sigma} = g_i + u_i,\; y_i(x,\, 0,\, u) = y_{0,i}(x),\;\; x \in R^{\infty},\, 1 \le i \le n. \end{cases} \tag{14}$$

The observation equation is given by $Z(u) = (z_1(u),\, z_2(u),\, \ldots,\, z_n(u)) = Y(u) = (y_1(u),\, y_2(u),\, \ldots,\, y_n(u))$.

For a given $Z_d = (z_{d1},\, z_{d2},\, \ldots,\, z_{dn}) \in (L^2(Q))^n$, the cost function is given by

$$J(v) = \sum_{i=1}^{n} \| y_i(v) - z_{di} \|_{L^2(Q)}^2 + M \sum_{i=1}^{n} (v_i,\, v_i)_{(W^{-1/2}(\Sigma))^n}, \tag{15}$$

where M is a positive constant.

The control problem then is to find $\inf J(v)$ over a closed convex subset U_{ad} of $(W^{-1/2}(\Sigma))^n$. Then as in Section II, there exists a unique optimal control $u \in U_{ad}$ such that $J(u) = \inf J(v)$ for all $v \in U_{ad}$. Moreover, we have

Theorem 4. *The necessary and sufficient condition for* $u = (u_1, u_2, \ldots, u_n) \in (W^{-1/2}(\Sigma))^n$ *to be an optimal control is that the following equations and inequalities are satisfied* (10) *and*

$$(P(u), v - u)_{(L^2(\Sigma))^n} + M(u, v - u)_{(W^{-1/2}(\Sigma))^n} \geq 0$$

$$\forall v = (v_1, v_2, \ldots, v_n) \in U_{ad} \tag{16}$$

together with (14).

Proof: As in Section II, the adjoint equation is given by

$$\frac{-\partial p_i(u)}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_i(u) - \sum_{j=1}^{n} h_{ij}(x) p_j(u) = y_i(u) - z_{di} \text{ in } Q.$$

The optimal control $u = (u_1, u_2, \ldots, u_n) \in (W^{-1/2}(\Sigma))^n$ is characterized by (see [17])

$$\sum_{i=1}^{n} J'(u)(v_i - u_i) \geq 0 \quad \forall v = (v_1, v_2, \ldots, v_n) \in U_{ad},$$

which is equivalent to

$$\sum_{i=1}^{n} (y_i(u) - z_{di}, y_i(v) - y_i(u))_{L^2(Q)} + M(u_i, v_i - u_i)_{W^{-1/2}(\Sigma)} \geq 0,$$

this inequality can be written as

$$\sum_{i=1}^{n} \int_{(0, T)} (y_i(u) - z_{di}, y_i(v) - y_i(u))_{L^2(R^\infty)} dt + M(u_i, v_i - u_i)_{W^{-1/2}(\Sigma)} \geq 0,$$

from (10)

$$\sum_{i=1}^{n} \int_{(0, T)} \left(\frac{-\partial p_i}{\partial t} + \left(-\sum_{k=1}^{\infty} D_k^2(I) + q(x, t)\right) p_i \right.$$

$$\left. - \sum_{j=1}^{n} h_{ij} p_j, y_i(v) - y_i(u) \right)_{L^2(R^\infty)} dt + M(u_i, v_i - u_i)_{W^{-1/2}(\Sigma)} \geq 0.$$

Using Green's formula, we obtain

$$\sum_{i=1}^{n}\int_{(0,T)}\left(p_i(u), \left(\frac{\partial}{\partial t}-\sum_{k=1}^{\infty}D_k^2(I)+q(x,t)-\sum_{j=1}^{n}h_{ij}\right)y_i(v)-y_i(u)\right)_{L^2(R^\infty)}dt$$

$$-\int_{(0,T)}\left(\frac{\partial p_i(u)}{\partial \nu},\ y_i(v)-y_i(u)\right)_{L^2(\Gamma)}dt$$

$$+\int_{(0,T)}\left(p_i(u),\ \frac{\partial(y_i(v)-y_i(u))}{\partial \nu}\right)_{L^2(\Gamma)}dt + M(u_i,\ v_i-u_i)_{W^{-1/2}(\Sigma)} \geq 0.$$

Using (14)

$$\sum_{i=1}^{n}\int_{(0,T)}(p_i(u),\ v_i-u_i)_{L^2(\Gamma)}dt + M(u_i,\ v_i-u_i)_{W^{-1/2}(\Sigma)} \geq 0,$$

which is equivalent to

$$(P(u),\ v-u)_{(L^2(\Sigma))^n} + M(u,\ v-u)_{(W^{-1/2}(\Sigma))^n} \geq 0,$$

therefore, we have proved.

Acknowledgement

The author would like to express her gratitude to Professor H. M. Serag, Mathematics Department, Faculty of Science, Al-Azhar University for suggesting the problem and critically reading the manuscript.

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