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DISCRETE FOURIER TRANSFORMS AND PLANE ROTATIONS

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Abstract

We establish a relationship between compositions of rotations with an angle that is an integer multiple of $2\pi/n$ about centers $z_0, z_1, ..., z_{n-1}$ and the discrete Fourier transform of $(z_0, z_1, ..., z_{n-1}) \in \mathbb{C}^n$. Applications of a geometrical flavor are discussed, as well as a connection with quasi-random subsets of $\mathbb{Z}_n$.

1. Introduction

Let n be a positive integer and let $\omega = \exp(2\pi i/n)$. The discrete Fourier transform D_n is the linear endomorphism of $\mathbb{C}^n$ which maps an n-tuple $Z = (z_0, ..., z_{n-1}) \in \mathbb{C}^n$ into the n-tuple $\hat{Z} = D_n(Z) = (\hat{z}_0, ..., \hat{z}_{n-1}) \in \mathbb{C}^n$, where

$$\hat{z}_k = \sum_{r=0}^{n-1} z_r \omega^{-kr} \tag{1}$$

for $k = 0, 1, ..., n-1$. This is an invertible endomorphism. To invert it,

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one can use the relation

$$z_k = \frac{1}{n}\sum_{r=0}^{n-1} \hat{z}_r \omega^{kr} \tag{2}$$

for $k = 0, 1, \ldots, n-1$. Alternatively, the discrete Fourier transform can be viewed as a linear endomorphism of the finite dimensional Hilbert space $L^2(\mathbb{Z}/n\mathbb{Z})$ of complex functions defined on the 'discrete circle' $\mathbb{Z}/n\mathbb{Z}$. An excellent presentation of the basic properties of the discrete Fourier transform can be found, for example, in the Chapter 2 of [10].

The discrete Fourier transform or, in its computationally-efficient version, the fast Fourier transform [2], [3] has numerous applications in areas such as fast multiplication of large integers [9], numerical methods in partial differential equations [6], numerical methods in difference equations [4], numerical optimization of integral equations [8], data compression, digital signal processing [7], spectral analysis, and many others.

In the present paper we will derive, inspired by the Problem B4 in the 2004 William Lowell Putnam Mathematical Competition, an interesting connection between the discrete Fourier transforms and plane geometry. It is known that the rotations together with the translations form (the group of) all orientation-preserving isometries, or rigid motions, of the Euclidean plane [5]. By using complex numbers we can write the equation of a plane rotation around the center z_0, with (counterclockwise) angle θ as

$$w - z_0 = e^{i\theta}(z - z_0) = e^{i\theta}z + (1 - e^{i\theta})z_0. \tag{3}$$

By using (3), a well known, but very important property of plane rotations can be derived:

Theorem 1. *If* $R_0, R_1, \ldots, R_{n-1}$ *are n plane rotations with angles* $\theta_0, \theta_1, \ldots, \theta_{n-1}$, *respectively, then the composition* $R_{n-1} \circ R_{n-2} \circ \cdots \circ R_0$ *is either a rotation of angle* $\theta_0 + \theta_1 + \cdots + \theta_{n-1}$, *if* $\theta_0 + \theta_1 + \cdots + \theta_{n-1} \notin 2\pi\mathbb{Z}$, *or a translation, if* $\theta_0 + \theta_1 + \cdots + \theta_{n-1} \in 2\pi\mathbb{Z}$.

2. Formulating the Main Problem

We will now use Theorem 1 to formulate our main problem. Let $z_0, z_1, \ldots, z_{n-1}$ be n complex numbers, not necessarily distinct. Let us pick an integer k such that $0 \leq k \leq n-1$, and let

$$R_0^k,\ R_1^k,\ \ldots,\ R_{n-1}^k$$

be the rotations with $2k\pi/n$ around the centers $z_0, z_1, \ldots, z_{n-1}$, respectively. According to Theorem 1, the composition

$$R_{n-1}^k \circ R_{n-2}^k \circ \cdots \circ R_0^k \tag{4}$$

is a plane translation. Let t_k be the complex number representing the translation vector associated to (4). We will be interested in the following problem:

Problem 2. Find a meaningful algebraic relation between the n-tuple of centers $(z_0, z_1, \ldots, z_{n-1})$ and the n-tuple of translation vectors $(t_0, t_1, \ldots, t_{n-1})$.

Problem B4 in the 2004 William Lowell Putnam Mathematical Competition gives, with the above notations, $z_m = m+1$ for $0 \leq m \leq n-1$, and asks for t_1.

3. Discrete Fourier Transform and the Answer to the Main Problem

In what follows we will show that in the transition from $(z_0, z_1, \ldots, z_{n-1})$ to $(t_0, t_1, \ldots, t_{n-1})$, the discrete Fourier transform plays an important role. Let $\alpha = \omega^k = e^{2k\pi i/n}$, and let $z \in \mathbb{C}$ be arbitrary. Let w_0 be the complex point obtained after a rotation of z with $2k\pi/n$ around z_0. Then, by (3),

$$w_0 = \alpha z + (1-\alpha)z_0.$$

Next, we rotate w_0 with $2k\pi/n$ around z_1 and we will get w_1, given by

$w_1 = \alpha w_0 + (1-\alpha) z_1$, or

$$w_1 = \alpha^2 z + \alpha(1-\alpha) z_0 + (1-\alpha) z_1.$$

Further, rotating w_1 with $2k\pi/n$ around z_2 will lead to $w_2 = \alpha w_1 + (1-\alpha) z_2$, or

$$w_2 = \alpha^3 z + \alpha^2(1-\alpha) z_0 + \alpha(1-\alpha) z_1 + (1-\alpha) z_2.$$

By induction, it can be easily shown that, for $0 \le r \le n-1$, the complex point w_r obtained out of z after a composition of rotations with $2k\pi/n$ around $z_0, z_1, \ldots, z_r$ (in this order) is given by

$$w_r = \alpha^{r+1} z + \alpha^r (1-\alpha) z_0 + \alpha^{r-1}(1-\alpha) z_1 + \cdots + (1-\alpha) z_r.$$

Going through all the centers $z_0, \ldots, z_{n-1}$ corresponds to $r = n-1$, so that the composition $R_{n-1}^k \circ R_{n-2}^k \circ \cdots \circ R_0^k$ will map z into

$$w_{n-1} = \alpha^n z + \alpha^{n-1}(1-\alpha) z_0 + \alpha^{n-2}(1-\alpha) z_1 + \cdots + (1-\alpha) z_{n-1}. \tag{5}$$

Since $\alpha^n = 1$ and $\alpha = \omega^k$, (5) represents a translation in the complex plane, with a vector represented by

$$t_k = (1-\alpha)(\alpha^{n-1} z_0 + \alpha^{n-2} z_1 + \cdots + z_{n-1}) = (1-\omega^k) \sum_{r=0}^{n-1} z_r \omega^{k(n-r-1)}.$$

By using the above relation together with $\omega^n = 1$, and in conjunction with (1), we can rewrite t_k in terms of the discrete Fourier transform $\hat{Z} = D_n(Z) = (\hat{z}_0, \ldots, \hat{z}_{n-1})$ of the vector of centers of rotation, $Z = (z_0, z_1, \ldots, z_{n-1})$:

$$t_k = (1-\omega^k)\omega^{-k} \sum_{r=0}^{n-1} z_r \omega^{-kr} = (\omega^{-k} - 1)\hat{z}_k. \tag{6}$$

Thus, we can state our main result:

Theorem 3. *Let* $z_0, z_1, \ldots, z_{n-1}$ *be complex numbers. For all* k *and* r *with* $0 \le k$, $r \le n-1$, *let* R_r^k *be the rotation around* z_r *with* $2k\pi/n$.

Then for each k, the composition $R_{n-1}^k \circ R_{n-2}^k \circ \cdots \circ R_0^k$ is a translation by a vector represented by the complex number $t_k = (\omega^{-k} - 1)\hat{z}_k$, where $\omega = \exp(2\pi i/n)$ and $\hat{Z} = (\hat{z}_0, ..., \hat{z}_{n-1})$ is the discrete Fourier transform of $Z = (z_0, ..., z_{n-1})$.

The following examples illustrate Theorem 3 in some concrete cases.

1. The case of n centers of rotation, equally spaced. Consider the special case $z_r = r + 1$ for $0 \le r \le n - 1$. Then (6) becomes

$$t_k = (\omega^{-k} - 1)\sum_{r=0}^{n-1}(r + 1)\omega^{-kr} = (1 - \omega^k)\sum_{r=1}^{n} r\omega^{-kr}. \tag{7}$$

From (7), and the identity

$$\sum_{r=1}^{n} rX^r = \frac{X^n(nX^2 - nX - X) + X}{(X - 1)^2}$$

in which we set $X = \omega^{-k}$ with $k \ne 0$, we get

$$t_k = (1 - \omega^k)\sum_{r=1}^{n} r\omega^{-kr} = (1 - \omega^k)\frac{n\omega^{-2k} - n\omega^{-k}}{(\omega^{-k} - 1)^2} = n \tag{8}$$

for all $k \ne 0$. Note that the case $k = 0$ always gives $t_0 = 0$. $t_1 = n$ is the answer to Problem B4 in the 2004 Putnam Exam.

2. Centers of rotation are the vertices of a regular n-gon. Let us now assume that the centers of rotation are the vertices of the standard regular n-gon inscribed in the circle $|z| = 1$, that is,

$$z_r = \omega^r \tag{9}$$

for $r = 0, 1, 2, ..., n - 1$. Then

$$\hat{z}_k = \sum_{r=0}^{n-1}\omega^{r(1-k)} = n\delta_{1k} = \begin{cases} 0, & \text{if } k \ne 1 \\ n, & \text{if } k = 1 \end{cases}. \tag{10}$$

Thus, from (6) and (10) we get

$$t_k = n(\omega^{-k} - 1)\delta_{1k}.$$

This has a nice geometrical meaning. Thus, for a fixed $k \neq 1$, if we apply to any complex point z, a sequence of rotations with angles $2k\pi/n$ about the centers $z_0, z_1, \ldots, z_{n-1}$ (in this order) given by (9), we will get back to z. In the case $k = 1$, however, the net effect of the sequence of rotations with angles $2\pi/n$ each about the centers $z_0, z_1, \ldots, z_{n-1}$, will be a translation of the form

$$w = z + n(\omega^{-1} - 1) = z - 2ni\sin\left(\frac{\pi}{n}\right)\left(\cos\left(\frac{\pi}{n}\right) - i\sin\left(\frac{\pi}{n}\right)\right) \tag{11}$$

which is a point at a distance $2n\sin\left(\frac{\pi}{n}\right)$ from z. Note that when $n \to \infty$, the translation difference $w - z$, with w given by (11), approaches $-2\pi i$.

3. Quasi-random subsets of $\mathbb{Z}_n$. In this class of examples we will have only two distinct centers of rotation, 0 and 1. We can generate sequences $z_0, z_1, \ldots, z_{n-1}$ of zeros and ones by using characteristic functions $\chi_S : \mathbb{Z}_n \to \{0, 1\}$ of subsets of $S \subset \mathbb{Z}_n$: $z_r = \chi_S(r)$ for $r = 0, 1, \ldots, n-1$. In [1], a list of equivalent definitions of quasi-randomness for subsets of $S \subset \mathbb{Z}_n$ are provided. One of them is the 'EXP' criterion, which identifies quasi-randomness by using incomplete exponential sums over S with nontrivial additive characters. These need to be 'small' in order for S to be quasi-random: thus, according to the 'EXP' criterion, quasi-random subsets are precisely those satisfying

$$\sum_{r \in \mathbb{Z}_n} \chi_S(r)\exp\left(\frac{2\pi irk}{n}\right) = o(n) \tag{12}$$

for $k = 1, 2, \ldots, n-1$. In the case of quasi-random subsets $\varnothing \neq S \neq \mathbb{Z}_n$ and sequences of centers of rotation $z_r \in \{0, 1\}$ defined by $z_r = 1$ if $r \in S$ and $z_r = 0$ if $r \notin S$, (12) in conjunction with Theorem 3 implies that the translation vectors t_k are all 'small' when compared to n:

$$t_k = (\omega^{-k} - 1)\hat{z}_k = (\omega^{-k} - 1)\sum_{r=0}^{n-1} \chi_S(r)\exp\left(-\frac{2\pi ikr}{n}\right) = o(n)$$

for all $k = 0, 1, \ldots, n-1$.

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