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# EXISTENCE OF POSITIVE SOLUTIONS OF SOME QUASILINEAR ELLIPTIC EQUATION

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## Abstract

In this paper we consider the following problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)u(x)|u|^{p-2}(1-|u|^{\gamma}), & x \in \Omega, \\ \partial u/\partial n = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega$ is a smooth bounded domain in $\Re^N$, $1 < p < N$, $\lambda$ is real parameter and $a(x)$ changes sign. We show that a continuum of positive solutions bifurcates out from the principal eigenvalue $\lambda_1$ of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda a(x)u(x)|u|^{p-2}, & x \in \Omega, \\ \partial u/\partial n = 0, & x \in \partial\Omega. \end{cases}$$

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## 1. Introduction

In this paper we consider the following problem:

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2} + f(\lambda, x, u), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \Re^N$ is a smooth bounded domain, $p > 1$, $\lambda \in \Re$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian operator and $a(x)$ may change sign on $\Omega$. Here we say a function $a(x)$ changes sign if the measures of the sets $\{x \in \Omega; a(x) > 0\}$ and $\{x \in \Omega; a(x) < 0\}$ are both positive.

We are mainly concerned with the existence of positive solutions to (1.1) for $\lambda$ in certain range.

A host of literature exists for this type of problems when $p = 2$ and $f(\lambda, x, u) = \pm\lambda a(x) u(1 - u)$ (see [7]). In this setting, (1.1) is a reaction diffusion equation, where the real parameter $\lambda > 0$ corresponds to the reciprocal of the diffusion coefficient and the unknown function $u$ represents a relative frequency.

The bifurcation problem of type (1.1) has received extensive attention recently, and we refer to [1, 4] and [5, 6] for details.

The study of existence of positive solutions of the $p$-Laplacian sees great increase in number of papers published. We mention [9, 2, 3] to name a few. Loosely speaking, most references mentioned use variational methods, and as such, only the case where (in essence) $a(x) < 0$, $\lambda > 0$ and $f(|\lambda|, x, |u|) > 0$ was studied thoroughly, and their methods break down when $a(x)$ changes sign.

We show, however, that, when $a(x)$ changes sign, the variational method proves the existence of a positive solution for a special range of $\lambda$ in the case $p$-Laplacian.

Our method relies on the eigencurve theory developed in [2, 3]. It turns out that the sign of the integral $\int_\Omega a$ plays an important role for the range of $\lambda$ for which (1.1) has a positive solution.

In the next section we prove our main results via a series of theorems.

## 2. Some Existence Results

We study the existence of positive solutions and bifurcation of the problem

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2}(1 + |u|^{\gamma}), & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

We first introduce some basic assumptions and notations which we will need in this paper.

We assume first that $p > 1$, $a(x)$ is a smooth weight function which changes sign on $\Omega$. We study the influence of the function $a(x)$ on the existence of positive solutions of (2.1).

Consider the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2} + \mu u |u|^{p-2}, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where we treat the eigenvalue $\mu$ associated with a positive eigenfunction as a function of $\lambda$.

By taking

$$S_\lambda = \left\{ \int_\Omega |\nabla u|^p - \lambda \int_\Omega a|u|^p;\ u \in W^{1,p}(\Omega), \int_\Omega |u|^p = 1 \right\}$$

it can be shown that $\mu(\lambda) = \inf S_\lambda$ and that an eigenfunction corresponding to $\mu(\lambda)$ does not change sign on $\Omega$. Thus, clearly, $\lambda$ is a principal eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda a(x) u |u|^{p-2}, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

if and only if $\mu(\lambda) = 0$.

Now we can establish the following proposition.

**Proposition 1.** *Assume that* $a \in L^{\infty}(\Omega)$. *Then* $\mu(\lambda)$ *is a continuous and concave function of* $\lambda$ *and* $\mu(0) = 0$. *If* $a(x) > 0$, *then* $\mu(\lambda)$ *is decreasing, and if* $a(x) < 0$, *then* $\mu(\lambda)$ *is increasing. Assume, now, that* $a(x)$ *changes sign in* $\Omega$:

(i) *If* $\int_{\Omega} a(x)dx < 0$, *then there exists a unique* $\lambda_1^+ > 0$ *such that* $\mu(\lambda_1^+) = 0$ *and* $\mu(\lambda) > 0$ *for* $\lambda \in (0, \lambda_1^+)$.

(ii) $\int_{\Omega} a(x)dx = 0$, *then* $\mu(0) = 0$ *and* $\mu(\lambda) < 0$ *if* $\lambda \neq 0$.

(iii) *If* $\int_{\Omega} a(x)dx > 0$, *then there exists a unique* $\lambda_1^- < 0$ *such that* $\mu(\lambda_1^-) = 0$ *and* $\mu(\lambda) > 0$ *for* $\lambda \in (\lambda_1^-(0), 0)$ (*see* [2, 3, 8]).

**Remark 1.** It follows from this proposition that when $a(x)$ changes sign and $\int_{\Omega} a(x) < 0$, the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda a(x)u(x)|u|^{p-2}, & x \in \Omega, \\ \partial u/\partial n = 0, & x \in \partial\Omega, \end{cases}$$

has a positive eigenvalue $\lambda_1^+$ associated with a positive eigenfunction.

**Remark 2.** It is easy to see that if $\lambda_1^-$ and $\lambda_1^+$ exist, then $\mu(\lambda) > 0$ for all $\lambda \in (\lambda_1^-, \lambda_1^+)$. With these constructions, we have

**Theorem 1.** *Assume that* $a(x)$ *changes sign and* $\int_{\Omega} a(x) \neq 0$. *Then for any* $\lambda$ *strictly between* $\lambda_1^-$ *and* $\lambda_1^+$, *the relation*

$$\|u\|_{\lambda} := \left(\int_{\Omega} (|\nabla u|^p - \lambda a|u|^p)\right)^{\frac{1}{p}}$$

*defines an equivalent norm on* $W^{1,p}(\Omega)$.

**Proof.** We first prove that $\| u \|_\lambda$ is a norm and then show the equivalence property. It is easy to see that $\| u \|_\lambda$ is induced by the bilinear form

$$\langle u, v\rangle_\lambda = \int_\Omega (| \nabla u |^{p-2} \nabla u \nabla v - \lambda a | u |^{p-2} uv).$$

Since $\lambda$ is strictly between $\lambda_1^-$ and $\lambda_1^+$, it follows that

$$\langle u, u\rangle = \int_\Omega (| \nabla u |^p - \lambda a | u |^p) \geq \mu(\lambda) \int_\Omega | u |^p > 0$$

for all $u \in W^{1,p}(\Omega) - 0$ and so $\langle u, v\rangle_\lambda$ induces a norm of the form $\| u \|_\lambda$.

To see the equivalence of the norms, suppose the contrary. Then there exists $u_n \in W^{1,p}(\Omega)$ such that $\| u_n \| = 1$ and

$$\int_\Omega (| \nabla u_n |^p - \lambda a | u_n |^p) dx \to 0.$$

The variational characterization of $\mu(\lambda)$ gives

$$\| u_n \|_\lambda \geq \mu(\lambda) \int_\Omega | u_n |^p.$$

Since $\lambda_1^- < \lambda < \lambda_1^+$, it follows that $\mu(\lambda) > 0$ so $u_n \to 0$ in $L^p(\Omega)$. This implies $\int_\Omega a | u_n |^p \to 0$ and since

$$\lim_{n\to\infty} \left[ \int_\Omega (| \nabla u_n |^p - \lambda a | u_n |^p) \right] = 0,$$

we have $\int_\Omega | \nabla u_n |^p \to 0$. This contradicts the fact that $\| u_n \| = 1$ and so the theorem is proved. □

Now we can state our main results.

We define the functional $I_\lambda$ and $J$ on the space $W^{1,p}(\Omega)$ by

$$I_\lambda(u) = \frac{1}{p} \int_\Omega (| \nabla u |^p - \lambda a | u |^p) \text{ and } J(u) = \frac{1}{p+\gamma} \int_\Omega a | u |^{p+\gamma}.$$

Note that $I_\lambda(u) + \lambda J(u)$ is the natural energy functional associated with the problem (2.1), that is bounded neither above nor below.

We set

$$M = \{u \in W^{1,p}(\Omega); (\lambda J'(u), u) = -1\}$$

$$= \left\{u \in W^{1,p}(\Omega); \lambda \int_\Omega a|u|^{p+\gamma} = -1\right\}.$$

Since $a(x)$ is a sign changing function, there exists an open subset $B$ of $\Omega$ such that $a(x) < 0$ on $B$. Then taking $u \in W^{1,p}(\Omega)$ with $\operatorname{Supp} u \subseteq B$, we get $M \neq \varnothing$.

Moreover, as $L^{p+\gamma}(\Omega)$ may be embedded compactly in $W^{1,p}(\Omega)$, $M$ is weakly closed in $W^{1,p}(\Omega)$.

Now using the homogeneity of (2.1), a solution of the equation (2.1) can also be obtained by solving a constrained minimization problem for the functional

$$F_\lambda(u) = \int_\Omega (|\nabla u|^p - \lambda a|u|^p) = \frac{1}{p}\|u\|_\lambda^p$$

on the $W^{1,p}(\Omega)$, restricted to the set $M$.

It is easy to see that $F_\lambda$ is sequentially weakly lower semicontinuous and Theorem 1 shows that $F_\lambda$ is coercive. It follows (see [10, Theorem 1.2]) that $F_\lambda$ is bounded from below on $M$ and attains its infimum on $M$.

Suppose that $F_\lambda$ assumes its infimum at $u_\lambda \in M$. Then $|u_\lambda| \in M$ and $F_\lambda(u_\lambda) = F_\lambda(|u_\lambda|)$. Thus we may assume that $u_\lambda \geq 0$ on $\Omega$.

By the Lagrange multiplier rule there exists a parameter $u \in \Re$ such that

$$(F'_\lambda(u), v) = \mu(J'(u), v)$$

for all $v \in W^{1,p}(\Omega)$, i.e.,

$$\int_\Omega (| \nabla u_\lambda |^{p-2} \nabla u_\lambda \nabla v - \lambda a | u_\lambda |^{p-2} u_\lambda v) + \mu\lambda \int_\Omega a | u_\lambda |^{p+\gamma-2} u_\lambda v,$$

for all $v \in W^{1,p}(\Omega)$.

Setting $v = u_\lambda$, above gives

$$\int_\Omega (| \nabla u_\lambda |^p - \lambda a | u_\lambda |^p) + \mu\lambda \int_\Omega a | u_\lambda |^{p+\gamma} dx = 0,$$

i.e.,

$$\| u_\lambda \|_\lambda^p = -\mu\lambda \int_\Omega a | u_\lambda |^{p+\gamma} = \mu(p+\gamma).$$

Since $u_\lambda \in M$ cannot vanish identically, $\| u_\lambda \|_\lambda > 0$ and so $\mu > 0$. Scaling with a suitable power of $\mu$, we obtain a weak solution $u = \mu^{\gamma^{-1}} u_\lambda \in W^{1,p}(\Omega)$ of (2.1) in the sense that

$$\mu^{-\frac{p-1}{\gamma}} \left[ \int_\Omega (| \nabla u |^{p-2} \nabla u \nabla v - \lambda a | u |^{p-2} uv) \right] + \mu\mu^{-\frac{p+\gamma-1}{\gamma}} \lambda \int_\Omega a | u |^{p+\gamma-2} uv = 0,$$

i.e.,

$$\int_\Omega (| \nabla u |^{p-2} \nabla u \nabla v - \lambda a | u |^{p-2} uv + \lambda a | u |^{p+\gamma-2} uv) = 0$$

for all $v \in W^{1,p}(\Omega)$.

It follows from standard regularity arguments that $u \in C^2(\Omega)$ is a classical solution satisfying the appropriate boundary condition, and finally by the maximum principle $u > 0$ on $\Omega$.

So we have proved the following.

**Theorem 2.** *Assume that* $a(x)$ *changes sign and* $\int_\Omega a(x) \neq 0$. *Then for any nonzero* $\lambda$ *strictly between* $\lambda_1^-$ *and* $\lambda_1^+$, *the problem* (2.1) *has a positive solution*, *provided that* $\lambda \neq 0$.

**Remark 3.** Using a similar argument, Theorem 2 can also be obtained for the case, where $f(\lambda, x, u) = -\lambda a(x)|u|^{p+\gamma-2}$ by considering the same functional $I_\lambda$ constrained to the set $M = \{u \in W^{1,p}(\Omega); (\lambda J'(u), u) = 1\}$.

In this case the Lagrange multiplier $\mu < 0$ and the change of variable $(-\mu)^{1/\gamma} u_\lambda$ is required.

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