Far East Journal of Dynamical Systems
Volume 11, Issue 1, 2009, Pages 1-16
Published online: February 23, 2009
This paper is available online at http://www.pphmj.com

ESTIMATES OF LARGE DEVIATIONS IN DYNAMICAL SYSTEMS BY A NON-ADDITIVE THERMODYNAMIC FORMALISM

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Abstract

In this article we obtain bounds for large deviations processes in dynamical systems, in which non-additive sequences are considered. More specifically, for sequences $\Phi = \{\varphi_n : X \to \mathbf{R}\}$ and dynamical maps $f : X \to X$ such that $\varphi_{n+1} - \varphi_n \circ f$ converges uniformly we estimate the growing rate of $\nu\left(x : \frac{1}{n}\varphi_n(x) \in I\right)$, where ν is a finite Borel measure on X and I is a real interval. We consider also a particular case in which the process description easily follows from Ellis theorem.

1. Introduction

By large deviations processes involving dynamical systems is understood the probability that some dynamical quantity is in a given set

2000 Mathematics Subject Classification: 37C45, 60F10.

Keywords and phrases: large deviations, non-additive sequences, thermodynamic formalism.

Support of this work by the Consejo Nacional de Investigaciones Científicas y Técnicas, Agencia Nacional de Promoción Científica y Tecnológica and Universidad Nacional de La Plata of Argentina is greatly appreciated. F.V. is a member of CONICET.

Received October 2, 2006

after a large number of iterations of the dynamics. An interesting situation is the following: Let X be a compact metric space, $f : X \to X$ be a continuous map and ν be finite Borel measure on X. For a potential φ let us consider the statistical sum $S_n(\varphi)(x) := \sum_{i=0}^{n-1} \varphi(f^i(x))$. Assuming that $\frac{1}{n} S_n(\varphi)$ converges ν – a.e. in X to a constant $\overline{\psi}$ the large deviation problem concerns with the estimation, within an accepted margin of error $\delta > 0$, of $\left| \frac{1}{n} S_n(\varphi) - \overline{\psi} \right|$. More precisely, for $E_n = \left\{ x : \frac{1}{n} S_n(\varphi)(x) \in I \right\}$, where $I = (\overline{\psi} - \delta, \overline{\psi} + \delta)$ or $I = (\overline{\psi}, \infty) \cup (-\infty, -\overline{\psi})$, i.e., $\nu(E_n) \to 0$, $n \to \infty$, we wish to know if $\nu(E_n)$ has exponential convergence.

Sequences like $\{S_n(\varphi)\}$ have the property of additivity: $S_{n+m}(\varphi)(x) = S_n(\varphi)(x) + S_m(\varphi)(f^n(x))$. Estimates of large deviations for these kinds of sequences were studied by Young [14]. In (Kifer [7]) Kifer gave a more general approach than in (Young [14]) unifying results of large deviations in stochastic process and dynamical systems. However in the framework of dynamics an additive case was treated.

A natural extension in this context is to consider more general sequences, in particular sequences for which the additivity property has been relaxed. The interest on the non-additive sequences is their relationship with Dimension Theory. Indeed, a non-additive version of the thermodynamic formalism, due to Barreira [1], allows to compute the dimension of a great class of Cantor like-sets, which appears from geometric constructions, like Moran ones.

We consider sequences of continuous functions $\{\varphi_n : X \to \mathbf{R}\}$ such that $\varphi_{n+1} - \varphi_n \circ f$ converges uniformly to ψ in X. For any $x \in X$ the value $\frac{1}{n} \varphi_n(x)$ is called the *time average* for the sequence. We also assume that the time average is convergent ν – a.e. in X.

In order to obtain a large deviation process description we must find the so-called rate function. For E_n, now defined $E_n = \left\{x : \frac{1}{n}\varphi_n(x) \in I\right\}$, the rate function is a semi-continuous map $g : \mathbf{R} \to [0, \infty]$ such that:

- If I is a closed subset of $\mathbf{R}$, then $\lim_{n\to\infty} \sup \frac{1}{n} \log \nu(E_n) \leq -\inf\{g(s) : s \in I\}$,

- If I is an open subset of $\mathbf{R}$, then $\lim_{n\to\infty} \inf \frac{1}{n} \log \nu(E_n) \geq -\inf\{g(s) : s \in I\}$.

One of the objectives of this article is to relate the large deviations processes to *multifractal analysis*. We have shown in an already published article (Mesón and Vericat [8]) that, for the *multifractal decomposition* $X = \bigcup_{s\in[-\infty,\, +\infty]} K_s \cup Y$ with $K_s = \left\{x : \lim_{n\to\infty} \frac{1}{n}\varphi_n(x) = s\right\}$ and Y is the set in which the limit does not exist, the rate function $g(s)$ is a characteristic dimension of K_s in the sense of Pesin [9] theory. In this work we prove that $g(s)$ is essentially the Legendre transform of the topological non-additive pressure of the sequence $\{\varphi_n\}$.

A related issue we consider here is some description of periodic points. This could be taken as a particular case of the problem of large deviations. Actually, in this case we have additive sequences $\{S_n(\varphi)\}$ with particular conditions imposed on the potential φ and on the dynamics. However the solution of the large deviations problem is here derived, by the application of Ellis theorem, in a more direct way than Young approach (Young [14]).

2. Definitions and Previous Results

A. Non-additive thermodynamic formalism

As we commented in the introduction a non-additive version of the Ruelle thermodynamic formalism was presented by Barreira [1]. In that context general sequences of potentials $\Phi = \{\varphi_n : X \to \mathbf{R}\}$ are considered. Let us recall that, in particular, a sequence Φ will be *additive* if $\varphi_{n+m}(x) = \varphi_n(x) + \varphi_m(f^n(x))$, for every x, n, m. This condition is fulfilled, for instance, by the sequence $\Phi = \{S_n(\varphi)\}$.

Let $f : X \to X$ be a continuous map and (X, d) be a compact metrizable space. Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be a finite covering of X and a string be defined as a sequence

$L = \{\ell_0, \ell_1, \dots, \ell_{n-1}\}$ such that $\{U_{\ell_0}, \dots, U_{\ell_{n-1}}\} \subset \mathcal{U}$, $\ell_i \in \{1, 2, \dots, N\}$.

The length of the string

$L = \{\ell_0, \ell_1, \dots, \ell_{n-1}\}$ is denoted by $n(L) = n$. Let call $W_n(\mathcal{U})$ the set of the strings L with length n for the covering $\mathcal{U}$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}(U_{\ell_1}) \cap \cdots \cap f^{-n+1}(U_{\ell_{n-1}}), \tag{1}$$

if $Z \subset X$ it says that $\Pi = \{L = (\ell_0, \ell_1, \dots, \ell_{n-1})\}$ covers Z if

$$Z \subset \bigcup_{L \in \Pi} X(L). \tag{2}$$

For a general sequence Φ and a covering $\mathcal{U}$ it is defined the number

$$\nu_n(\Phi, \mathcal{U}) := \sup\{|\varphi_n(x) - \varphi_n(y)| : x, y \in L, \text{ for some } L \in W_n(\mathcal{U})\}. \tag{3}$$

For the definition is assumed that $\limsup_{\Delta(\mathcal{U}) \to 0} \limsup_{n \to \infty} \frac{\nu_n(\Phi, \mathcal{U})}{n} = 0$ ($\Delta(\mathcal{U}) =$ diameter of $\mathcal{U}$).

If L is a string let $\varphi(L) := \begin{cases} \sup_{X(L)} \varphi_n, & \text{if } X(L) \neq \varnothing, \\ -\infty, & \text{if } X(L) = \varnothing. \end{cases}$

For any real number s:

$$M(\Phi, s, Z, \mathcal{U}) = \inf_{\Gamma} \sum_{L \in \Pi} \exp(-sn(L) + \varphi(L)), \tag{4}$$

where the infimum is taken over all the collections of strings $\Pi \subset W_n(\mathcal{U})$ which covers Z.

Let

$$M(\mathcal{U}, Z, s) = \lim_{n\to\infty} M(\mathcal{U}, Z, s, n). \tag{5}$$

There is a unique number $\bar{s}$ such that $M(\mathcal{U}, Z, \cdot)$ jumps from $+\infty$ to 0, now let

$$P(\Phi, Z, \mathcal{U}) = \bar{s} = \sup\{s : M(\mathcal{U}, Z, s) = +\infty\} = \inf\{s : M(\mathcal{U}, Z, s) = 0\}. \tag{6}$$

Finally the number

$$P(\Phi, Z) = \lim_{\Delta(\mathcal{U})\to 0} P(\Phi, Z, \mathcal{U}) \tag{7}$$

is the *topological pressure* of Φ restricted to Z. The limit does exist (Barreira [1]).

The non-additive version of the classical variational principle is established in the following manner: let $\mathcal{M}_f(X)$ be the set of f-invariant probability measures on X, for any $Z \subset X$ with $\mathcal{M}_f(Z)$ is denoted the set of measures $\mu \in \mathcal{M}_f(X)$ with $\mu(Z) = 1$. Let $x \in X$ be a generic point for a measure $\mu \in \mathcal{M}_f(X)$, i.e., the sequence $\mu_{x,n} := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}$, where δ_x is the unit measure in x, is weakly convergent (with $n \to \infty$) to μ. With $V(x)$ is denoted the sets of weak limit points of the sequence $\{\mu_{x,n}\}$, it holds that $V(x) \neq \varnothing$, and $V(x) \subset \mathcal{M}_f(X)$ (Barreira [1]). Let $\mathcal{L}(Z) = \{x \in Z : V(x) \cap \mathcal{M}_f(Z) \neq \varnothing\}$.

Theorem (Barreira [1]). *Let* $\Phi = \{\varphi_n : X \to \mathbf{R}\}$ *be a sequence of continuous potentials, if there exists a continuous map* $\psi : X \to \mathbf{R}$ *such that* $\varphi_{n+1} - \varphi_n \circ f$ *converges uniformly to* ψ *in Z. Then*

$$P_{\mathcal{L}(Z)}(\Phi) = \sup_{\mu\in\mathcal{M}_f(X)}\left\{h_\mu(f) + \int \psi d\mu\right\}.$$

We shall consider term ψ as the "limit of the family Φ".

This Theorem generalizes the variational principle of Pesin and Pitskel [10] for noncompact sets. For compact sets and $\Phi = \{S_n(\varphi)\}$ the traditional variational principle (Walters [15]), (Ruelle [11]) is obtained. In this case $P(\Phi)$ is directly denoted by $P(\varphi)$.

Remark. If Z is compact and f-invariant, then $\mathcal{L}(Z) = Z$.

The Bowen topological entropy definition (Bowen [3]), a special case of characteristic dimensions, is recovered from the topological pressure in case of $\Phi \equiv 0$. It is denoted by $h_{top}(f, Z)$.

B. Multifractal analysis

Let a sequence $\Phi = \{\varphi_n\}$ and dynamics $f : X \to X$, let us consider the sets $K_s = \left\{x : \lim\limits_{n\to\infty} \frac{1}{n} \varphi_n(x) = s\right\}$. In this way a partition $X = \bigcup\limits_{s\in[-\infty,+\infty]} K_s \cup Y$ is obtained. If Y is the set of points for which $\lim\limits_{n\to\infty} \frac{1}{n} \varphi_n(x)$ does exist, then Y is known as the "irregular part". When $X \setminus Y$ is negligible in some sense, the partition of X in the level sets K_s is called a *multifractal decomposition*. A function G defined on the sets K_s which have in addition some special properties, is called a *characteristic dimension* in the sense of Pesin [9]. Examples of characteristic dimensions are the Hausdorff dimension and the topological entropy, mentioned above. A *multifractal spectrum*, specified by the sequence Φ and the function G, or simply a (Φ, G)*-spectrum* is the map $F(s) = G(K_s)$. This function is which captures the main information about the multifractal structure.

We recall main facts about the multifractal formalism developed in (Takens and Verbitski [13]). This is done in the additive case $\varphi_n = S_n(\varphi)$ and with the dynamics satisfying the conditions of specification and expansiveness and the potential belonging to a special class:

The *specification property* for a map $f : X \to X$ intuitively says that, for specified orbit segments, a periodic orbit approximating the trajectory can be found. This condition ensures abundance of periodic points and it is a concept introduced by Bowen in (Bowen [2]). This condition is indeed stronger than the existence of Markov partitions (Takens and Verbitski [13]), (Schmeling [12]). Its formal definition is:

A homeomorphism $f : X \to X$ has the *specification property* if: $I_1, \ldots, I_k$ is a finite disjoint collection of integer intervals $I_1, \ldots, I_k$, then for $\varepsilon > 0$, there is an integer $M(\varepsilon)$ and a function $\Phi : I = \bigcup I_i \to X$, such that:

(i) dist $(I_i, I_j) > M(\varepsilon)$ (Euclidean distance),

(ii) $f^{n_1 - n_2}(\Phi(n_1)) = \Phi(n_2)$,

(iii) $d(f^n(x), \Phi(n)) < \varepsilon$, for some $x : f^m(x) = x$, with $m \geq M(\varepsilon)$ + length(I) and for every $n \in I$.

A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $\delta > 0$, such that $d(f^n(x), f^n(y)) < \delta$, for any integer n implies $x = y$.

We consider the following metric in X, which is derived from the original one d:

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : i = 0, 1, \ldots, n-1\}, \tag{8}$$

and we denote by $B_{n,\varepsilon}(x)$ the ball of centre x and radius ε in the metric d_n.

A potential φ belongs to the class $\nu_f(X)$ if this condition is fulfilled: there are constants ε, $K > 0$ in such a way that

$$d_n(x, y) < \varepsilon \Rightarrow |S_n(\varphi)(x) - S_n(\varphi)(y)| < K. \tag{9}$$

Let $T(q) := P(q\varphi)$, with $q \in \mathbf{R}$, here $P(\varphi)$ is the usual topological pressure, which is obtained in the Barreira definition considering $\Phi = \{S_n(\varphi)\}$. The function $T(q)$ is convex, non increasing and continuously differentiable with $T'(q) = \int_X \varphi d\mu_q$, where μ_q is the Gibbs state (see below) for the potential $\varphi_q := q\varphi$.

Let $\mathcal{E}(s) = \inf_{q \in \mathbf{R}} \{s\alpha + T(q)\}$ the Legendre transform $T(q)$, so $\mathcal{E}(s(q)) = T(q) + qs(q)$ $(s(q) := T'(q))$. Let $\underline{s} = \lim_{q \to +\infty} s(q) = \inf_{q \in \mathbf{R}} \{s(q)\}$, $\overline{s} = \lim_{q \to -\infty} s(q) = \sup_{q \in \mathbf{R}} \{s(q)\}$. Then $K_s = \emptyset$, if $s \in (\underline{s}, \overline{s})$, so that the domain of definition of $\mathcal{E}(s)$ is the range of $T'(q)$. From the variational relationship between $\mathcal{E}(s)$ and $T(q)$ is derived that $\mathcal{E}(s)$ is concave, continuously differentiable in $(\underline{s}, \overline{s})$ and $\mathcal{E}'(s) = q$, such that $s = T'(q)$.

C. Gibbs states

A measure μ on X is called a *Gibbs measure*, or a *Gibbs state* associate to the potential φ, if for sufficiently small $\varepsilon > 0$, there are constants A_ε, $B_\varepsilon > 0$, such that for any $x \in M$ and for any positive integer n:

$$A_\varepsilon(\exp(S_n(\varphi)(x)) - nP(\varphi)) \leq \mu(B_{n,\varepsilon}(x)) \leq B_\varepsilon(\exp(S_n(\varphi)(x)) - nP(\varphi)).$$

Let $P_n(f) = \{x : f^n(x) = x\}$, set for a potential φ:

$$\mu_{\varphi, n} = \frac{1}{N(f, \varphi, n)} \sum_{x \in P_n(f)} \exp(S_n(\varphi)(x)) \delta_x, \tag{10}$$

where

$$N(f, \varphi, n) = \sum_{x \in P_n(f)} \exp(S_n(\varphi)(x))$$

and δ_x is the unit measure at x.

Let μ_φ be the weak limit of the sequence $\{\mu_{\varphi, n}\}$, i.e.,

$$\lim_{n \to \infty} \int g(x) d\mu_{\varphi, n} = \int g(x) d\mu_\varphi,$$

for every continuous $g(x)$ (Katok and Hasselblatt [5] and Ruelle [11]).

Theorem (Katok and Hasselblatt [5] and Ruelle [11]). *Let f be an expansive homeomorphism which have the specification property and φ be a potential belonging to the class $\nu_f(X)$. Then μ_φ is a Gibbs state associated to φ. Besides μ_φ is unique and also it is ergodic.*

If φ belongs to the class $\nu_f(X)$ and f is an expansive homeomorphism with the specification property, then the topological pressure associated to φ, f can be computed in the way:

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log\left(\sum_{x \in P_n(f)} \exp(S_n(\varphi)(x)) \right) \tag{11}$$

(Katok and Hasselblatt [5]), (Proposition 20.3.3).

Because f is expansive the set of periodic points $P_n(f)$ is n, δ-separated with δ the expansiveness constant (Katok and Hasselblatt [5]).

3. Estimates of Large Deviations

For our study of large deviation processes we work with sequences $\Phi = \{\varphi_n\}$ for which the hypothesis of the non-additive variational principle are satisfied, i.e., they must have a limit and

$$\limsup_{\Delta(\mathcal{U})\to 0} \limsup_{n\to\infty} \frac{\nu_n(\Phi, \mathcal{U})}{n} = 0$$

(c.f. Eq. (3) and below).

Let us denote

$$\mathcal{M}_f^0(X) = \{\nu \in \mathcal{M}_f(X) : \nu(B_{\varepsilon, n}(x)) \le C \exp(-nh), \text{ for some } C, \varepsilon > 0\}.$$

Here h is the usual topological entropy of f. The dynamics are continuous maps $f : X \to X$ with X a compact metric space.

Theorem 1. *Let* $\Phi = \{\varphi_n\}$ *be a sequence which fulfills the above conditions with a limit* ψ *in* X, *and let* $\nu \in \mathcal{M}_f^0(X)$. *If* $E_n = \left\{x : \frac{1}{n}\varphi_n(x) \ge s\right\}$, *then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu(E_n) \le \sup\left\{h_\mu(f) - h : \int \psi d\mu \ge s\right\}.$$

Proof. We must find a measure m such that $\int \psi d\mu \geq s$ and $\lim_{n\to\infty} \sup \frac{1}{n} \log m(E_n) \leq h_\mu(f) - h$. Let see the following:

$$\varphi_n - S_n(\psi) = \sum_{i=1}^{n-1} [(\varphi_{i+1} - \varphi_i \circ f - \psi) \circ f^{n-i-1} + \varphi_1 \circ f^{n-1}],$$

therefore

$$\left| \frac{1}{n} \varphi_n(x) - S_n(\psi)(x) \right| \leq \frac{1}{n} \left(\sum_{i=1}^{n-1} \| \varphi_{i+1} - \varphi_i \circ f - \psi \|_\infty + \| \varphi_1 \|_\infty \right),$$

with

$$\| \varphi \|_\infty := \sup_{x \in X} \{ | \varphi(x) | \}.$$

Since $\varphi_{n+1} - \varphi_n \circ f$ converges uniformly to ψ in X, for any $\delta > 0$ there exists a natural N such that

$$\| \varphi_{i+1} - \varphi_i \circ f - \psi \| < \delta \text{ and } \| \varphi_1 \|_\infty < n\delta, \text{ for any } n \geq N.$$

So that

$$\left| \frac{1}{n} \varphi_n(x) - S_n(\psi)(x) \right| \leq \frac{1}{n} \varphi \left[(n - N)\delta + \sum_{i=1}^{n-1} \| \varphi_{i+1} - \varphi_i \circ f - \psi \|_\infty \right] \leq 2\delta.$$

Now we proceed to find the measure μ : let A_n be a maximal (n, ε)-separated contained in E_n. The set A_n is finite because X is compact. For any $x \in X$, set $\mu_n := \frac{1}{n} \sum_{x \in A_n} \left[\frac{1}{B_n} \sum_{i=0}^{n-1} \delta_{f^i}(x) \right]$, where δ_x is the point measure in x and $B_n = card(A_n) \times \exp(nh)$.

Let μ be a weak limit of μ_n, we have $\int \psi d\mu_n = \frac{1}{n} S_n(\psi)(x)$, because μ_n is a linear combination of averages of point measures $\delta_{f^i}(x)$. Besides

$$-2\delta \leq \frac{1}{n} \varphi_n(x) - \frac{1}{n} S_n(\psi)(x) \leq 2\delta. \tag{12}$$

Now, since $E_n \subset \bigcup_{x \in A_n} B_{\varepsilon, n}(x)$, we have $\log \nu(E_n) \leq \sum_{x \in A_n} \nu(B_{\varepsilon, n}(x))$ $\leq C \times card(A_n) \times \exp(nh)$. Following the proof of the classical variational principle (Walters [15]) can be established that $h_\mu(f) \geq \log card(A_n)$. Therefore

$$\lim_{n\to\infty} \sup \frac{1}{n} \log \nu(E_n) \leq \lim_{n\to\infty} \sup \frac{1}{n} \sum_{x \in A_n} \nu(B_{\varepsilon, n}(x))$$

$$\leq \lim_{n\to\infty} \sup \frac{1}{n} B_n \leq h_\mu(f) - h,$$

where the third inequality is due to $\nu \in \mathcal{M}_f^0(X)$.

It remains to prove that $\int \psi d\mu \geq s$, this is immediately seen from $\int \psi d\mu_n = \frac{1}{n} S_n(\psi)(x) \geq s - 2\delta$, with δ arbitrary small. Because μ is defined as a weak limit of μ_n we have $\int \psi d\mu \geq s$. □

Consequently with Theorem 1 the rate function for the large deviation process considered in this above result is

$$g(s) = -\sup\left\{ h_\mu(f) - h : \int \psi d\mu = s \right\}.$$

Next we give a multifractal description of this function. In (Mesón and Vericat [8]) was proved that $\sup\left\{ h_\mu(f) : \int \psi d\mu = s \right\} = G(K_s) := h_{top}(f, K_s)$, with

$$K_s = \left\{ x : \lim_{n\to\infty} \frac{1}{n} \varphi_n(x) = s \right\}.$$

We set $T_\Phi(q) = P(q\Phi)$, this function is convex (Mesón and Vericat [8]), and so its Legendre transform $T_\Phi^*(s) = \inf_{q \in \mathbf{R}} \{P(q\Phi) - qs\}$ is defined.

Proposition 2. *Let us consider a dynamical system in which the entropy map* $\mu \to h_\mu(f)$ *is upper-semi-continuous, for any* $\mu \in \mathcal{M}_f(X)$

(*this space endowed with the weak topology*). *If* $T_{\Phi}^{*}(s)$ *is the Legendre transform of the map* $T_{\Phi}(q)$, *then* $H_{\Phi}(s) = \sup\left\{h_{\mu}(f) : \int \psi d\mu = s\right\} = T_{\Phi}^{*}(s)$.

Proof. We have, using the non-additive variational principle,

$$H_{\Phi}(s) = \sup_{\mu \in \mathcal{M}_f(X)}\left\{h_{\mu}(f) + q\int \psi d\mu : \int \psi d\mu = s\right\} - qs$$

$$\leq \sup_{\mu \in \mathcal{M}_f(X)}\left\{h_{\mu}(f) + s\int \psi d\mu\right\} - qs = P(q\Phi) - qs \leq T_{\Phi}^{*}(s).$$

For proving the other inequality is used that the entropy is upper semi-continuous. This condition is equivalent that $T_{\Phi}(q)$ is of class C^1. Proofs of this fact for the additive case can be seen in (Katok and Hasselblatt [5], Ruelle [11] and Keller [6]), the non-additive version is obtained following similar lines of demonstration.

Let $\overline{\delta}$ be a positive number chosen such that $\inf\{P(q\Phi) - qs - \overline{\delta}\} = 0$, by the part 3 of Theorem 4 in (Mesón and Vericat [8]), there exists an ergodic measure μ in $\mathcal{M}_f(X)$ such that $\int \psi d\mu = s$ and $h_{\mu}(f) \geq \overline{\delta}$. In the part 4 of the demonstration of the above mentioned theorem is established that the $H_{\Phi}(s)$ attains its supremum on ergodic measures. Therefore we have $H_{\Phi}(s) \geq \overline{\delta}$.

In the proof of the parts of the theorem in which relies Proposition 2 is where was used the differentiability of the pressure function $T_{\Phi}(q)$. □

4. Large Deviations of Periodic Points

This section is devoted to describe periodic points as a particular case of Theorem 1 in the sense that the sequences considered are $\{S_n(\varphi)\}$ with particular conditions imposed on the potential and the dynamics. They are:

- φ is a potential belonging to the class $\nu_f(X)$,

- f is a homeomorphism with specification and expansiveness (see Subsection 2.B).

The interesting point in this situation is the possibility of getting a large deviation description from a direct application of a result of Theoretical Probability (Ellis [4]). More precisely we estimate the

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\left[\nu_n\left\{x \in P_n(f) : \frac{1}{n}S_n(\varphi)(x) \geq s\right\}\right],$$

where

$$\nu_n := \frac{1}{card\, P_n(f)}\sum_{x\in P_n(f)}\delta_x, \tag{13}$$

with δ_x the point mass measure in x. This is known as the Bowen measure. It equidistributes the periodic points (Bowen [2]).

Let Z_n be measurable functions defined on probability spaces (Ω_n, ν_n) and let denote by E_n expectation with respect to ν_n. If $C_n(q) := \frac{1}{n}\log E_n(\exp(qZ_n))$ converges to a function $C(q)$ convex and differentiable for every q, then Ellis theorem (Ellis [4]) claims that, for any closed subset $F \subset \mathbf{R}$ and for any open subset $U \subset \mathbf{R}$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\nu_n\left[\frac{Z_n}{n} \in F\right] \leq -\inf_{t\in F} I(t),$$

$$\liminf_{n\to\infty}\frac{1}{n}\log\nu_n\left[\frac{Z_n}{n} \in U\right] \geq -\inf_{t\in U} I(t),$$

where $I(t) = \sup_{t\in\mathbf{R}}\{qt - C(q)\}$.

In our case we take $Z_n(x) = S_n(\varphi)(x)$, and

$$E_n(\exp(qZ_n)) = \frac{1}{card\, P_n(f)}\sum_{x\in P_n(f)}\exp(S_n(q\varphi)(x)).$$

The probabilities ν_n will be the measures $\mu_{0,n}$ (c.f. Eq. (10) with $\varphi \equiv 0$), whose weak limit is the Bowen measure (Bowen [2]).

To show that the Ellis theorem is applicable for our purposes let us note (recall the results of Takens-Verbitski formalism in Subsection 2.B and Eq. (11)) that:

$$\lim_{n\to\infty} C_n(q) = P(q\varphi) - h = T(q) - T(0),$$

where $T(q)$ is a convex differentiable function.

Now the description of the large deviation process is obtained in:

Theorem 3. *If* $\varphi \in \nu_f(X)$ *with equilibrium state* $\mu_\varphi \neq \mu_{\max}$ (*the measure of maximal entropy*), *then*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mu_{0,n}\left[\left\{x \in P_n(f) : \frac{1}{n} S_n(\varphi)(x) \geq s\right\}\right] \leq -H(s),$$

$$\lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log \mu_{0,n}\left[\left\{x \in P_n(f) : \frac{1}{n} S_n(\varphi)(x) \geq s\right\}\right] \geq -H(s),$$

where

$$H(s) := \begin{cases} T(0) - \mathcal{E}(s), & \textit{if } s \leq s(0), \\ T(0) - \mathcal{E}(s(0)), & \textit{if } s > s(0). \end{cases}$$

Proof. By the above proposition the hypothesis of the Ellis theorem is fulfilled with $C_n(q) := \frac{1}{n} \log\left[\frac{1}{N_n(f)} \sum_{x\in P_n(f)} \exp(S_n(q\varphi)(x))\right]$, $C(q) = T(q) - T(0)$ and $I(t) = \sup_{t\in\mathbf{R}}\{qt - C(q)\} = \sup_{t\in\mathbf{R}}\{qt - T(q) + T(0)\}$. We consider, for any s, the sets $F = [-s, \infty)$, $U = (-s, \infty)$.

Thus $I(-t) = \sup_{t\in\mathbf{R}}\{-qt - T(q) + T(0)\} = T(0) - \mathcal{E}(-t)$, and recall that $\mathcal{E}(s)$ is concave, continuously differentiable and has a maximum in $s(0) = -T'(0)$. By using these properties

$$\inf_{s\in F} I(s) = \inf_{s\in U} I(s) = \begin{cases} \mathcal{E}(s), & \text{if } s \leq s(0), \\ \mathcal{E}(s(0)), & \text{if } s > s(0). \end{cases}$$

Now applying the Ellis theorem

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mu_{0,n}\left[\left\{x\in P_n(f):\frac{1}{n}S_n(\varphi)(x)\in F\right\}\right]\le -\inf_{s\in F} I(s),$$

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mu_{0,n}\left[\left\{x\in P_n(f):\frac{1}{n}S_n(\varphi)(x)\in U\right\}\right]\ge -\inf_{s\in U} I(s).$$

Therefore

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\mu_{0,n}\left[\left\{x\in P_n(f):\frac{1}{n}S_n(\varphi)(x)\ge s\right\}\right]\le -H(s),$$

$$\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log\mu_{0,n}\left[\left\{x\in P_n(f):\frac{1}{n}S_n(\varphi)(x)> s\right\}\right]\ge -H(s). \qquad \square$$

Remark. The result of the above theorem gives large deviations from an interval $I=[s,\infty)$. For intervals $I=[-s,s]$ a similar demonstration leads to obtain the same rate function.

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