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DERIVED LENGTHS OF SYMMETRIC AND SKEW SYMMETRIC ELEMENTS IN GROUP ALGEBRAS

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Abstract

Let G be a nilpotent p-abelian group with cyclic derived subgroup, where p is an odd prime, and let F be a field of characteristic p. In this paper we consider the group algebra FG with the natural involution, and show that the Lie derived length of the set of symmetric elements of FG coincides with the Lie derived length of FG. Furthermore, we prove that the same is true for the Lie derived length of the set of skew symmetric elements and for the derived length of the set of symmetric units of FG as well.

1. Introduction

Let A be an algebra with an involution $*$. An element $x \in A$ is called *symmetric* (*skew symmetric*) with respect to $*$, if $x^* = x$ $(x^* = -x)$. Denote by A^+ and A^- the set of symmetric and skew symmetric elements

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of A, respectively. Amitsur [1] proved that an algebra satisfies a polynomial identity if and only if the set of symmetric elements of the algebra satisfies a polynomial identity. We remark that the polynomial identity which is satisfied by the algebra is not necessarily the same as the one which is satisfied by the symmetric elements.

Let FG be a group algebra of a group G over a field F of characteristic p, and let $*$ be the natural involution of FG, that is the involution induced by the map sending each group element to its inverse. The investigation of Lie-identities with respect to the symmetric (or skew symmetric) elements are intensively studied area of the theory of group algebras. Broche and Polcino [5, 6] gave necessary and sufficient conditions for FG^+ and FG^- to be commutative (satisfying the Lie identity $[x, y] = 0$). Giambruno and Sehgal [7] and Lee [12] proved that if G is a group not containing a copy of the quaternion group Q_8 of order 8 and $\mathrm{char}(F) \neq 2$, then FG^+ is Lie nilpotent if and only if FG is Lie nilpotent, that is $\mathrm{char}(F) = p$ and G is a nilpotent p-abelian group (the derived subgroup of G is a finite p-group). Lee [12] also described the groups G containing a copy of Q_8 such that FG^+ is Lie nilpotent in the case $\mathrm{char}(F) \neq 2$. Hence we can see that FG^+ satisfies a Lie-identity does not imply that the whole algebra FG satisfies the Lie-identity. In [8] the groups G were classified for which FG^- ($\mathrm{char}(F) \neq 2$) is Lie nilpotent. The characterization of all groups G such that FG^+ ($\mathrm{char}(F) \neq 2$) is Lie n-Engel can be found in [13].

The symmetric elements play an important role to investigate the group of units $U(FG)$ of the group algebra FG. Let us denote the set of symmetric units in $U(FG)$ by $U^+(FG)$. A classification of groups G such that $U^+(FG)$ satisfies a group identity can be found in the papers of Giambruno et al. [9], Sehgal and Valenti [15] and Bovdi [4]. We know from [5] that if F is a field of characteristic $p \neq 2$ and G is a nonabelian torsion group, then $U^+(FG)$ is an abelian group if and only if FG^+ is a commutative ring. Lee [14] proved that if F is a field of characteristic

$p \neq 2$ and G is a torsion group, then $U^{+}(FG)$ is nilpotent if and only if FG^{+} is Lie nilpotent.

For a subset S of the group algebra FG define the Lie derived series of S by induction as follows: let $\delta^{[0]}(S) = S$ and let $\delta^{[n+1]}(S)$ be the additive subgroup generated by the Lie commutators $[x, y] = xy - yx$ with $x, y \in \delta^{[n]}(S)$. The subset S is said to be *Lie solvable* if there exists a natural number n such that $\delta^{[n]}(S) = 0$, and the smallest such n is called the *Lie derived length* of S which will be denoted by $\mathrm{dl}_L(S)$. In other words, S is Lie solvable of derived length n if and only if S satisfies the polynomial identity $[x_1, x_2, ..., x_{2^n}]^{\circ} = 0$, where the Lie commutator $[x_1, x_2, ..., x_{2^n}]^{\circ}$ is defined inductively to be

$$[[x_1, x_2, ..., x_{2^{n-1}}]^{\circ}, [x_{2^{n-1}+1}, x_{2^{n-1}+2}, ..., x_{2^n}]^{\circ}]$$

with $[x_1, x_2]^{\circ} = [x_1, x_2]$. Actually, every notion introduced above can be assigned with a suitable group theoretical ones. Given subset S of a group G we denote by (S, S) the subgroup generated by all group commutators $(g, h) = g^{-1}h^{-1}gh$ with $g, h \in S$. Let $\delta_0(S) = S$ and $\delta_{n+1}(S) = (\delta_n(S), \delta_n(S))$. The subset S will be said to be *solvable* if $\delta_n(S) = \{1\}$ for some n; the smallest such n will be denoted by $\mathrm{dl}(S)$ and called the *derived length* of S. Evidently, if S is solvable of derived length n, then S satisfies the group identity $(x_1, x_2, ..., x_{2^n})^{\circ} = 1$ and n is the least such natural number. (The group commutator $(x_1, x_2, ..., x_{2^n})^{\circ}$ is defined as above replacing the Lie commutators by group commutators.)

We do not know too much about the exact value of the derived lengths of group algebras, even less about the derived lengths of a set of group algebras. This paper is devoted to study the Lie derived length of the set of symmetric and skew symmetric elements and the derived length of symmetric units. As we mentioned above a group G is called *p-abelian* for a prime p if the derived subgroup G' of G is a finite p-group. Although it is not described yet when the set of symmetric or skew symmetric elements is Lie solvable, but we know that if G is a p-abelian group and $\mathrm{char}(F) = p$, then FG is strong Lie solvable, so every subset of FG is Lie solvable and $U(FG)$ is a solvable group. We show here that the Lie derived length of both FG^+ and FG^- and the derived length of $U^+(FG)$ are equal to $\lceil \log_2(|G'|+1) \rceil$, and this value is not else than the Lie derived length of the group algebra FG.

Our main result is:

Theorem. *Let G be a nilpotent p-abelian group with cyclic derived subgroup, where p is an odd prime, and let F be a field of characteristic p. Then*

$$\mathrm{dl}_L(FG^+) = \mathrm{dl}_L(FG^-) = \mathrm{dl}(U^+(FG)) = \mathrm{dl}_L(FG) = \mathrm{dl}(U(FG))$$

$$= \lceil \log_2(|G'|+1) \rceil.$$

To the best of our knowledge, at the moment the largest class of group for which $\mathrm{dl}(U(FG))$ has already been determined is the groups with cyclic derived subgroup of odd order (see [2, 3]). It is worth mentioning that for a non-nilpotent group G the integers $\mathrm{dl}_L(FG^+)$, $\mathrm{dl}_L(FG^-)$ and $\mathrm{dl}_L(FG)$ can be pairwise different. However, for odd prime p, we do not know if there exists a p-abelian nilpotent group G such that the above-mentioned three integers are not equal.

2. Preliminaries

Our first lemma, which can be found partly in the proof of Lemma 2.5 of [3], will allow us to extend the results from finite p-group to nilpotent group.

Lemma 1. *Let G be a nilpotent p-abelian group. Then there exists a finite p-group P which is isomorphic to a subgroup of a factor group of G and $P' \cong G'$.*

Proof. Now we will construct the group P. Since the derived subgroup of G is a finite p-group, we conclude that G is an FC-group. According to [16, Theorem 1.7], G is isomorphic to a subgroup of the direct product of the torsion FC-group G/A and the torsion-free abelian group $G/T(G)$, where A is a maximal torsion-free central subgroup of G and $T(G)$ is the torsion part of G. Therefore $G' \cong (G/A)'$. Let $\{g_1', g_2', \dots, g_n'\}$ be a generator set of $(G/A)'$. Then we can choose elements $g_1, h_1, g_2, h_2, \dots, g_n, h_n$ from G/A such that $(g_i, h_i) = g_i'$. As a finitely generated torsion nilpotent group, $N = \langle g_1, h_1, g_2, h_2, \dots, g_n, h_n \rangle$ is finite, and it is the direct product of its Sylow subgroups. Let us denote by P the Sylow p-subgroup of N. Since G' is a p-group, we have $P' = N' = (G/A)' \cong G'$ and the proof is complete.

Now, we list some well-known identities which will be used in the following:

$$(gh-1) = (g-1)(h-1) + (g-1) + (h-1); \tag{2.1}$$

$$[vw, z] = v[w, z] + [v, z]w, \quad [v, wz] = w[v, z] + [v, w]z; \tag{2.2}$$

$$[a, b] = ba((a, b) - 1), \quad (a, b) = 1 + a^{-1}b^{-1}[a, b] \quad (a, b \text{ are units}). \tag{2.3}$$

Denote by $\omega(FG)$ the augmentation ideal of FG. As it is well known, $\omega(FG)$ is nilpotent if and only if G is a finite p-group and $\operatorname{char}(F) = p$. Using the identity (2.1) we can easily observe that

$$g^m - 1 \equiv m(g-1) (\operatorname{mod} \omega(FG)^2) \tag{2.4}$$

for every $g \in G$ and integer m.

In this section assume that F is a field of characteristic p for some odd prime p, and that G is a finite p-group with derived subgroup $G' = \langle x \mid x^{p^k} = 1 \rangle$ and a, b are elements of G such that $(a, b) = x$. It is easy to check (see, e.g., [11, p. 252]) that

$$\begin{aligned}[b^l a^m, b^s a^t] &\equiv [b^l a^m, a^t b^s] \\ &\equiv (ms - lt) b^{l+s} a^{m+t} (x-1) (\operatorname{mod} FG\omega(FG')^2).\end{aligned} \tag{2.5}$$

Denote by I_n the ideal $\omega(FG)^3\omega(FG')^{2^n-1} + FG\omega(FG')^{2^n}$ of FG. We will also use freely the easy-verifiable fact that the value of the product $(g-1)(h-1)(x-1)$ is independent of the order of its factors modulo I_1 for all $g, h \in G$.

Note finally that the sets $\delta^{[n]}(FG^+)$ and $\delta^{[n]}(FG^-)$ form linear spaces over the field F for all $n \geq 0$.

Lemma 2. *There exist* $v_1, w_1, z_1 \in \delta^{[1]}(FG^+)$ *and* $v_1', w_1', z_1' \in \delta_1(U^+(FG))$ *such that*

$$v_1 \equiv v_1' - 1 \equiv (a-1)(b-1)(x-1) (\text{mod } I_1);$$

$$w_1 \equiv w_1' - 1 \equiv (a-1)^2(x-1) (\text{mod } I_1);$$

$$z_1 \equiv z_1' - 1 \equiv (b-1)^2(x-1) (\text{mod } I_1).$$

Proof. Let $u_0 = (a-1)(a^{-1}-1)$, $v_0 = (b-1)(b^{-1}-1)$ and $w_0 = (ab-1)((ab)^{-1}-1)$. Clearly, they are symmetric elements. Using the identity (2.2) let us calculate the Lie commutator $[u_0, v_0]$:

$$\begin{aligned}[u_0, v_0] &= (a-1)[a^{-1}, (b-1)(b^{-1}-1)] \\ &\quad + [a, (b-1)(b^{-1}-1)](a^{-1}-1) \\ &= (a-1)(b-1)[a^{-1}, b^{-1}] + (a-1)[a^{-1}, b](b^{-1}-1) \\ &\quad + (b-1)[a, b^{-1}](a^{-1}-1) + [a, b](b^{-1}-1)(a^{-1}-1).\end{aligned}$$

Using (2.5) and (2.1) we get that

$$[u_0, v_0] \equiv 4(a-1)(b-1)(x-1) (\text{mod } I_1).$$

Since 4 is an invertible element of F, so v_1 can be chosen as

$$v_1 = 4^{-1}[u_0, v_0] \equiv (a-1)(b-1)(x-1) (\text{mod } I_1)$$

and by (2.3)

$$v_1' = (1 + 4^{-1}u_0,\ 1 + v_0) = 1 + (1 + 4^{-1}u_0)^{-1}(1 + v_0)^{-1}[4^{-1}u_0,\ v_0]$$

$$= 1 + ((1 + 4^{-1}u_0)^{-1}(1 + v_0)^{-1} - 1)v_1 + v_1$$

$$\equiv 1 + v_1 \equiv 1 + (a - 1)(b - 1)(x - 1)(\text{mod } I_1).$$

Furthermore,

$$[u_0,\ w_0] = (a - 1)[a^{-1},\ (ab - 1)((ab)^{-1} - 1)]$$

$$+ [a,\ (ab - 1)((ab)^{-1} - 1)](a^{-1} - 1)$$

$$= (a - 1)(ab - 1)[a^{-1},\ (ab)^{-1}] + (a - 1)[a^{-1},\ ab]((ab)^{-1} - 1)$$

$$+ (ab - 1)[a,\ (ab)^{-1}](a^{-1} - 1) + [a,\ ab]((ab)^{-1} - 1)(a^{-1} - 1).$$

Applying first (2.5), then (2.1) and (2.4), we have

$$[u_0,\ w_0] \equiv 4(a - 1)(b - 1)(x - 1) + 4(a - 1)^2(x - 1)(\text{mod } I_1).$$

Let us choose w_1 as

$$w_1 = 4^{-1}[u_0,\ w_0] - v_1 \equiv (a - 1)^2(x - 1)(\text{mod } I_1),$$

and let $w_1' = (1 + 4^{-1}u_0,\ 1 + w_0 - v_0)$. Then by (2.3)

$$w_1' = 1 + (1 + 4^{-1}u_0)^{-1}(1 + w_0 - v_0)^{-1}[4^{-1}u_0,\ w_0 - v_0]$$

$$= 1 + ((1 + 4^{-1}u_0)^{-1}(1 + w_0 - v_0)^{-1} - 1)w_1 + w_1$$

$$\equiv 1 + (a - 1)^2(x - 1)(\text{mod } I_1).$$

Similar computations show that

$$[w_0,\ v_0] \equiv 4(a - 1)(b - 1)(x - 1) + 4(b - 1)^2(x - 1)(\text{mod } I_1),$$

and we can choose z_1 and z_1' as

$$z_1 = 4^{-1}[w_0,\ v_0] - v_1 \equiv (b - 1)^2(x - 1)(\text{mod } I_1),$$

and

$$z_1' = (1 + w_0 - u_0, 1 + 4^{-1}v_0) \equiv 1 + (b-1)^2(x-1) \pmod{I_1}.$$

In the next lemma we need the equivalence

$$(g^{-1} - 1) \equiv (g-1)^2 - (g-1) \pmod{\omega(FG)^3} \tag{2.6}$$

which holds for any $g \in G$. Indeed, if g is of order p^s, then

$$\begin{aligned} g^{-1} &= ((g-1)+1)^{p^s-1} \\ &\equiv \binom{p^s-1}{2}(g-1)^2 + \binom{p^s-1}{1}(g-1) + 1 \pmod{\omega(FG)^3}, \end{aligned}$$

by the binomial formula. Since $\binom{p^s-1}{2} \equiv 1 \pmod{p}$ and $\binom{p^s-1}{1} \equiv -1 \pmod{p}$, we get the required equivalence.

Lemma 3. *There exist elements* $v_1, w_1, z_1 \in \delta^{[1]}(FG^-)$ *such that*

$$v_1 \equiv (a-1)(b-1)(x-1) \pmod{I_1};$$

$$w_1 \equiv (a-1)^2(x-1) \pmod{I_1};$$

$$z_1 \equiv (b-1)^2(x-1) \pmod{I_1}.$$

Proof. Let $t_0 = a - a^{-1}$, $v_0 = b - b^{-1}$, $w_0 = ab - b^{-1}a^{-1}$ and $z_0 = a^{-1}b - b^{-1}a$. Evidently, they are skew symmetric elements. Let us calculate the Lie commutator of t_0 and v_0 with the help of (2.5):

$$\begin{aligned} [t_0, v_0] &= [a, b] - [a, b^{-1}] - [a^{-1}, b] + [a^{-1}, b^{-1}] \\ &\equiv ba(x-1) + b^{-1}a(x-1) + ba^{-1}(x-1) + b^{-1}a^{-1}(x-1) \\ &\equiv (ba-1)(x-1) + (b^{-1}a-1)(x-1) + (ba^{-1}-1)(x-1) \\ &\quad + (b^{-1}a^{-1}-1)(x-1) + 4(x-1) \pmod{FG\omega(FG')^2}. \end{aligned}$$

Using the identities (2.1) and (2.6) we have that

$$[t_0, v_0] \equiv 2(a-1)^2(x-1) + 2(b-1)^2(x-1) + 4(x-1)\,(\mathrm{mod}\ I_1).$$

Similar computations show that

$$[t_0, z_0] \equiv -4(a-1)(b-1)(x-1) + 4(a-1)^2(x-1)$$

$$+ 2(b-1)^2(x-1) + 4(x-1)\,(\mathrm{mod}\ I_1);$$

$$[t_0, w_0] \equiv 4(a-1)(b-1)(x-1) + 4(a-1)^2(x-1)$$

$$+ 2(b-1)^2(x-1) + 4(x-1)\,(\mathrm{mod}\ I_1);$$

$$[w_0, v_0] \equiv 4(a-1)(b-1)(x-1) + 2(a-1)^2(x-1)$$

$$+ 4(b-1)^2(x-1) + 4(x-1)\,(\mathrm{mod}\ I_1).$$

Since 2 and 8 are invertible elements in the field F, we get that

$$v_1 = 8^{-1}[t_0, w_0] - 8^{-1}[t_0, z_0] \equiv (a-1)(b-1)(x-1)\,(\mathrm{mod}\ I_1)$$

and

$$w_1 = 2^{-1}([t_0, w_0] - [t_0, v_0] - 4v_1) \equiv (a-1)^2(x-1)\,(\mathrm{mod}\ I_1).$$

Let finally $z_1 = 2^{-1}([w_0, v_0] - [t_0, v_0] - 4v_1)$. The calculations above show that

$$z_1 \equiv (b-1)^2(x-1)\,(\mathrm{mod}\ I_1),$$

and the proof is complete.

In the next lemma $\gamma_i(G)$ denotes the ith term of the lower central series of the group G. Evidently, the conditions of the lemma hold when G' is cyclic.

Lemma 4 (Lemma 3 in [2])**.** *Let G be a finite p-group such that for $i \geq 1$, $\gamma_{i+1}(G) \subseteq (\gamma_i(G))^p$. Then for all $k, l, m, n \geq 1$*

(i) $[\omega(FG')^m, \omega(FG)^l] \subseteq \omega(FG)^{l-1}\omega(FG')^{m+1}$;

(ii) $[\omega(FG)^k, \omega(FG)^l] \subseteq \omega(FG)^{k+l-2}\omega(FG')$;

(iii) $[\omega(FG)^k \omega(FG')^m,\ \omega(FG)^l (FG')^n] \subseteq \omega(FG)^{k+l-2} \omega(FG')^{n+m+1}$.

3. Proof of Theorem

Assume that G is a finite p-group with cyclic derived subgroup of order p^k, where p is an odd prime. Keeping the notations as before we are going to show by induction that there exist elements v_n, w_n, z_n in both $\delta^{[n]}(FG^+)$ and $\delta^{[n]}(FG^-)$, further v'_n, w'_n, z'_n in $\delta_n(U^+(FG))$ such that

$$v_n \equiv v'_n - 1 \equiv \alpha_n(a-1)(b-1)(x-1)^{2^n-1} \pmod{I_n};$$

$$w_n \equiv w'_n - 1 \equiv \beta_n(a-1)^2(x-1)^{2^n-1} \pmod{I_n};$$

$$z_n \equiv z'_n - 1 \equiv \gamma_n(b-1)^2(x-1)^{2^n-1} \pmod{I_n},$$

where α_n, β_n and γ_n are nonzero elements of the field F. In view of Lemma 2 and Lemma 3, this is clear for $n = 1$. Assume now the truth of the assertion for n and

$$v_n = \alpha_n(a-1)(b-1)(x-1)^{2^n-1} + \hat{v}_n;$$

$$w_n = \beta_n(a-1)^2(x-1)^{2^n-1} + \hat{w}_n;$$

$$z_n = \gamma_n(b-1)^2(x-1)^{2^n-1} + \hat{z}_n,$$

for some $\hat{v}_n$, $\hat{w}_n$ and $\hat{z}_n$ belong to I_n. Then by Lemma 4,

$$\begin{aligned}[w_n, z_n] &\equiv \beta_n\gamma_n[(a-1)^2(x-1)^{2^n-1}, (b-1)^2(x-1)^{2^n-1}] \\ &\equiv \beta_n\gamma_n[(a-1)^2, (b-1)^2](x-1)^{2^{n+1}-2} \pmod{I_{n+1}},\end{aligned}$$

furthermore, by (2.2) and (2.3)

$$\begin{aligned}[(a-1)^2, (b-1)^2] &= (a-1)(b-1)[a, b] + (a-1)[a, b](b-1) \\ &\quad + (b-1)[a, b](a-1) + [a, b](b-1)(a-1) \\ &\equiv 4(a-1)(b-1)(x-1) \pmod{I_1}\end{aligned}$$

and hence the element $v_{n+1} = [w_n, z_n]$ is contained in $\delta^{[n+1]}(FG^+)$, and it is congruent to $4\beta_n\gamma_n(a-1)(b-1)(x-1)^{2^{n+1}-1}$ modulo I_{n+1}. Similarly,

$$[w_n, v_n] \equiv \beta_n\alpha_n[(a-1)^2(x-1)^{2^n-1}, (a-1)(b-1)(x-1)^{2^n-1}]$$

$$\equiv \beta_n\alpha_n[(a-1)^2, (a-1)(b-1)](x-1)^{2^{n+1}-2} (\mathrm{mod}\ I_{n+1})$$

and since

$$[(a-1)^2, (a-1)(b-1)] = (a-1)^2[a, b] + (a-1)[a, b](a-1)$$

$$\equiv 2(a-1)^2(x-1) (\mathrm{mod}\ I_1),$$

so $w_{n+1} = [w_n, v_n] \equiv 2\beta_n\alpha_n(a-1)^2(x-1)^{2^{n+1}-1} (\mathrm{mod}\ I_{n+1})$. One can now get the suitableness of the element $z_{n+1} = [z_n, v_n]$ analogously.

Assume finally by induction the existence of the elements v'_n, w'_n and z'_n such that

$$v'_n \equiv 1 + v_n\ (\mathrm{mod}\ I_n);$$

$$w'_n \equiv 1 + w_n (\mathrm{mod}\ I_n);$$

$$z'_n \equiv 1 + z_n\ (\mathrm{mod}\ I_n)$$

and let

$$v'_{n+1} = (w'_n, z'_n), \quad w'_{n+1} = (w'_n, v'_n), \quad z'_{n+1} = (z'_n, v'_n).$$

Then v'_{n+1}, w'_{n+1} and z'_{n+1} belong to $\delta_{n+1}(U(FG^+))$ and by (2.3),

$$v'_{n+1} = (w'_n, z'_n) = 1 + (w'_n)^{-1}(z'_n)^{-1}[w'_n, z'_n] \equiv 1 + (w'_n)^{-1}(z'_n)^{-1}[w_n, z_n]$$

$$\equiv 1 + ((w'_n)^{-1}(z'_n)^{-1} - 1)v_{n+1} + v_{n+1} \equiv 1 + v_{n+1}\ (\mathrm{mod}\ I_{n+1}).$$

Repeating this calculation for w'_{n+1} and z'_{n+1} we get the truth of the assertion for arbitrary n.

Evidently, v_n and v_n' are nonzero elements while $2^n - 1 < p^k$. Indeed, to this it is enough to show that the nonzero element $y = (a-1)(b-1)(x-1)^{2^n - 1}$ does not belong to I_n. According to Jenning's theory [10], the element $x - 1$ is of weight $t \geq 2$, so y has weight $2 + t(2^n - 1)$, which means that $y \in \omega(FG)^{2+t(2^n-1)} \setminus \omega(FG)^{3+t(2^n-1)}$. Since $\omega(FG)^3$ has an F-basis consisting of regular elements of weight not less than 3, the inclusion $\omega(FG)^3 \omega(FG')^{2^n - 1} \subseteq \omega(FG)^{3+t(2^n-1)}$ holds. Therefore the element y cannot be in the ideal $\omega(FG)^3 \omega(FG')^{2^n - 1}$. Thus, if y was in the ideal I_n, then it should belong to $FG(x-1)^{2^n}$. But this is impossible, since then y would be expressed as $y'(x-1)^{2^n}$ for some $y' \in FG$, and the equalities

$$(a-1)(b-1)(x-1)^{p^k - 1} = (a-1)(b-1)(x-1)^{2^n - 1}(x-1)^{p^k - 2^n}$$

$$= y'(x-1)^{p^k} = 0$$

would hold.

Therefore, $\mathrm{dl}_L(FG^+)$, $\mathrm{dl}_L(FG^-)$, $\mathrm{dl}(U^+(FG))$ and so $\mathrm{dl}_L(FG)$ and $\mathrm{dl}(U(FG))$ are not less than $\lceil \log_2(|G'|) + 1 \rceil$. In view of Lemma 1 this remains true under the conditions of the theorem as well. The converse inequalities follow from [11].

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