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# A GENERALIZATION OF THE COMPOSITION OF THE DISTRIBUTIONS $x_+^\lambda \ln^m x_+$ AND $x_+^\mu$

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**Abstract**

Let $F$ be a distribution and $f$ be a locally summable function. The neutrix composition $F(f)$, of $F$ and $f$, is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. It is proved that the neutrix composition of the distributions $x_+^\lambda \ln^m x_+$ and $x_+^\mu$ exists and is equal to $\mu^m x_+^{\lambda\mu} \ln^m x_+$, for $\lambda < 0, \mu > 0$ and $\lambda, \lambda\mu \neq -1, -2, \ldots$.

## Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$, $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and $\mathcal{D}'[a, b]$ be the space of distributions defined on $\mathcal{D}[a, b]$.

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We define the locally summable function $x_+^\lambda \ln^m x_+$ for $\lambda > -1$ and $m = 0, 1, 2, \ldots$ by

$$x_+^\lambda \ln^m x_+ = \begin{cases} x^\lambda \ln^m x, & x > 0, \\ 0, & x < 0. \end{cases}$$

The distribution $x_+^\lambda \ln^m x_+$ is then defined inductively for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $m = 0, 1, 2, \ldots$, by the equation

$$(x_+^\lambda \ln^{m+1} x_+)' = \lambda x_+^{\lambda-1} \ln^{m+1} x_+ + (m+1) x_+^{\lambda-1} \ln^m x_+.$$

The distribution $x_-^\lambda \ln^m x_-$ is then defined for $\lambda \neq -1, -2, \ldots$ and $m = 0, 1, 2, \ldots$ by

$$x_-^\lambda \ln^m x_- = (-x)_+^\lambda \ln^m (-x)_+$$

and the distribution $|x|^\lambda \ln^m |x|$ is defined for $\lambda \neq -1, -2, \ldots$ and $m = 0, 1, 2, \ldots$ by

$$|x|^\lambda \ln^m |x| = x_+^\lambda \ln^m x_+ + x_-^\lambda \ln^m x_-.$$

It follows that if $r$ is a positive integer and $-r - 1 < \lambda < -r$, then

$$\langle x_+^\lambda \ln^m x_+, \varphi(x) \rangle = \int_0^1 x^\lambda \ln^m x \left[ \varphi(x) - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} x^k \right] dx + \sum_{k=0}^{r-1} \frac{(-1)^m m!\, \varphi^{(k)}(0)}{k!\, (\lambda + k + 1)^{m+1}}, \tag{1}$$

for an arbitrary function $\varphi$ in $\mathcal{D}[-1, 1]$.

We now let $N$ be the neutrix, see [1], having domain $N'$ the positive integers and range $N''$ the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n: \quad \lambda > 0, \ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x) dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots,$ it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now $f$ is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [2] and was originally called the *composition of distributions*.

**Definition 1.** Let $F$ be a distribution in $\mathcal{D}'$ and $f$ be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$, with $-\infty < a < b < \infty$, if

$$N - \lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle$$

for all $\varphi$ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$.

In particular, we say that the composition $F(f(x))$ exists and is equal to $h$ on the open interval $(a, b)$ if

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x)) \varphi(x) dx = \langle h(x), \varphi(x) \rangle$$

for all $\varphi$ in $\mathcal{D}[a, b]$.

Note that taking the neutrix limit of a sequence is equivalent to taking the normal limit of Hadamard's finite part of the sequence.

The following four theorems were proved in [4], [6], [3] and [5], respectively.

**Theorem 1.** *The neutrix composition* $(x_+^r)^{-1}$ *exists and*

$$(x_+^r)^{-1} = x_+^{-r} + (-1)^r \frac{2c(\rho) - r\phi(r-1)}{r!} \delta^{(r-1)}(x),$$

*for* $r = 1, 2, \ldots,$ *where*

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} 1/i, & i \geq 1, \\ 0, & i = 0. \end{cases}$$

**Theorem 2.** *If* $F_{\lambda,m}(x)$ *denotes the distribution* $x_+^{\lambda} \ln^m x_+$, *then the neutrix composition* $F_{\lambda,m}(x_+^{\mu})$ *exists and*

$$F_{\lambda,m}(x_+^{\mu}) = \mu^m x_+^{\lambda\mu} \ln^m x_+$$

*for* $-1 < \lambda < 0$, $\mu > 0$ *and* $\lambda\mu \neq -1, -2, \ldots$.

**Theorem 3.** *The distribution* $(x_+^{\mu})^{-1} \ln^m |x_+^{\mu}|$ *exists and*

$$(x_+^{\mu})^{-1} \ln^m |x_+^{\mu}| = \mu^m x_+^{-\mu} \ln^m x_+,$$

*for* $\mu > 0$, $\mu \neq 1, 2, \ldots$ *and* $m = 1, 2, \ldots$.

*In particular,* $(x_+^{\mu})^{-1}$ *exists and*

$$(x_+^{\mu})^{-1} = \mu^m x_+^{-\mu},$$

*for* $\mu > 0$ *and* $\mu \neq 1, 2, \ldots$.

**Theorem 4.** *If* $F_m(x)$ *denotes the distribution* $x^{-1} \ln^m |x|$, *then the distribution* $F_m(x^r)$ *exists and*

$$F_m(x^r) = r^m x^{-r} \ln|x|,$$

*for* $m, r = 1, 2, \ldots$.

To prove the next theorem, we need the following lemmas which can easily be proved by induction.

**Lemma 1.**

$$\int_{-1}^{1} u^i \rho^{(s)}(u) du = \begin{cases} 0, & 0 \le i < r, \\ (-1)^s s!, & i = r, \end{cases}$$

*for* $s = 0, 1, 2, \ldots$.

**Lemma 2.**

$$\int_1^n u^\alpha \ln^r v dv = \frac{(-1)^r r! (1 - n^{\alpha+1})}{(\alpha+1)^{r+1}} + \sum_{i=0}^{r-1} \frac{(-1)^i r! n^{\alpha+1} \ln^{r-i} n}{(r-i)! (\alpha+1)^{i+1}}$$

*for* $r = 1, 2, \ldots$.

We now prove the following generalization of Theorem 2.

**Theorem 5.** *If* $F_{\lambda, m}(x)$ *denotes the distribution* $x_+^\lambda \ln^m x_+$, *then the neutrix composition* $F_{\lambda, m}(x_+^\mu)$ *exists and*

$$F_{\lambda, m}(x_+^\mu) = \mu^m x_+^{\lambda\mu} \ln^m x_+ \tag{2}$$

*for* $\lambda < 0$, $\mu > 0$ *and* $\lambda, \lambda\mu \neq -1, -2, \ldots$.

*In particular, the composition* $F_{\lambda, m}(x_+^\mu)$ *exists if* $1 + \lambda\mu > 0$.

**Proof.** We suppose that $1 - s > \lambda > -s$ for some positive integer $s$. We then put $G_{\lambda, m}(x) = (x_+^{\lambda+s} \ln^m x_+)^{(s)}$ and note that $G_{\lambda, m}(x)$ is of the form

$$G_{\lambda, m}(x) = \sum_{i=0}^{m} a_{\lambda, m, i} x_+^\lambda \ln^i x_+, \tag{3}$$

where $a_{\lambda,m,i} = 0$ if $i < m - s$. Since $x_+^\lambda \ln^i x_+$ is an infinitely differentiable function on any closed interval not containing the origin, it follows that

$$F_{\lambda, i}(x_+^\mu) = \mu^i x_+^{\lambda\mu} \ln^i x_+$$

and thus

$$G_{\lambda,m}(x_+^{\mu}) = \sum_{i=0}^{m} a_{\lambda,m,i}\,\mu^i x_+^{\lambda\mu} \ln^i x_+ \tag{4}$$

on any closed interval not containing the origin.

Putting

$$\begin{aligned} G_{\lambda,m,n}(x) &= (x_+^{\lambda+s} \ln^m x_+)^{(s)} * \delta_n(x) \\ &= \int_{-1/n}^{1/n} ((x-t)_+^{\lambda+s} \ln^m (x-t)_+)\delta_n^{(s)}(t)\,dt \\ &= \begin{cases} \displaystyle\int_{-1/n}^{1/n} (x-t)^{\lambda+s} \ln^m (x-t)\delta_n^{(s)}(t)\,dt, & 1/n < x, \\ \displaystyle\int_{-1/n}^{x} (x-t)^{\lambda+s} \ln^m (x-t)\delta_n^{(s)}(t)\,dt, & -1/n \le x \le 1/n, \\ 0, & x < -1/n, \end{cases} \end{aligned}$$

we have

$$G_{\lambda,m,n}(x_+^{\mu}) = \begin{cases} \displaystyle\int_{-1/n}^{1/n} (x^{\mu}-t)^{\lambda+s} \ln^m (x^{\mu}-t)\delta_n^{(s)}(t)\,dt, & 1/n < x^{\mu}, \\ \displaystyle\int_{-1/n}^{x^{\mu}} (x^{\mu}-t)^{\lambda+s} \ln^m (x^{\mu}-t)\delta_n^{(s)}(t)\,dt, & 0 \le x^{\mu} \le 1/n, \\ \displaystyle\int_{-1/n}^{0} (-t)^{\lambda+s} \ln^m (-t)\delta_n^{(s)}(t)\,dt, & x < 0. \end{cases} \tag{5}$$

Our problem now is to evaluate

$$\begin{aligned} &\int_{-1}^{1} x^k G_{\lambda,m,n}(x_+^{\mu})\,dx \\ &= \int_0^{n^{-1/\mu}} x^k \int_{-1/n}^{x^{\mu}} (x^{\mu}-t)^{\lambda+s} \ln^m (x^{\mu}-t)\delta_n^{(s)}(t)\,dt\,dx \\ &\quad + \int_{n^{-1/\mu}}^{1} x^k \int_{-1/n}^{1/n} (x^{\mu}-t)^{\lambda+s} \ln^m (x^{\mu}-t)\delta_n^{(s)}(t)\,dt\,dx \end{aligned}$$

$$+\int_{-1}^{0} x^k \int_{-1/n}^{0} (-t)^{\lambda+s} \ln^m(-t)\delta_n^{(s)}(t)dtdx$$

$$= \frac{n^{-\lambda-(k+1)/\mu}}{\mu}\int_0^1 v^{(k+1)/\mu-1}\int_{-1}^{v}(v-u)^{\lambda+s}[\ln(v-u)-\ln n]^m \rho^{(s)}(u)dudv$$

$$+\frac{n^{-\lambda-(k+1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(u)\int_1^n v^{(k+1)/\mu-1}(v-u)^{\lambda+s}[\ln(v-u)-\ln n]^m dvdu$$

$$+n^{-\lambda}\int_{-1}^{0} x^k \int_{-1}^{0}(-u)^{\lambda}\ln^m(-u)\rho^{(s)}(u)dudx$$

$$= I_1 + I_2 + I_3, \tag{6}$$

where the substitutions $u = nt$ and $v = nx^{\mu}$ have been made.

It follows immediately that

$$N-\lim_{n\to\infty} I_1 = N-\lim_{n\to\infty} I_3 = 0 \tag{7}$$

for $k = 0, 1, 2, \ldots$.

Further,

$$I_2 = \frac{n^{-\lambda-(k+1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(u)\int_1^n v^{(k+1)/\mu-1}(v-u)^{\lambda+s}$$

$$\times[\ln(1-u/v)+\ln v-\ln n]^m dvdu$$

$$= \frac{n^{-\lambda-(k+1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(u)\int_1^n v^{(k+1)/\mu-1}(v-u)^{\lambda+s}$$

$$\times[\ln(1-u/v)+\ln v]^m dvdu + E(\ln n)$$

$$= \frac{n^{-\lambda-(k+1)/\mu}}{\mu}\sum_{i=0}^{m}\binom{m}{i}\int_{-1}^{1}\rho^{(s)}(u)\int_1^n v^{(k+1)/\mu+\lambda+s-1}$$

$$\times(1-u/v)^{\lambda+s}\ln^i(1-u/v)\ln^{m-i} vdvdu + E(\ln n)$$

$$= \sum_{i=0}^{m} J_i + E(\ln n), \tag{8}$$

where $E(\ln n)$ denotes the terms containing powers of $\ln n$ and so are negligible and the term containing $\ln^m v$ is zero, since $\int_{-1}^{1} \rho^{(s)}(u)du = 0$ for $s = 1, 2, \ldots$ by Lemma 1.

We note that $(1 - u/v)^{\lambda+s} \ln^i(1 - u/v)$ can be expanded in the form

$$(1 - u/v)^{\lambda+s} \ln^i(1 - u/v) = \sum_{p=0}^{\infty} \frac{c_{\lambda,i,p} u^p}{v^p},$$

where $c_{\lambda,i,p} = 0$ for $p = 0, 1, \ldots, i - 1$ and then

$$n^{-\lambda-(k+1)/\mu} \int_{-1}^{1} \rho^{(s)}(u) \int_{1}^{n} v^{(k+1)/\mu+\lambda+s-1}(1 - u/v)^{\lambda+s}$$

$$\times \ln^i(1 - u/v) \ln^{m-i} v dv du$$

$$= n^{-\lambda-(k+1)/\mu} \sum_{p=0}^{\infty} c_{\lambda,i,p} \int_{-1}^{1} u^p \rho^{(s)}(u) \int_{1}^{n} v^{(k+1)/\mu+\lambda+s-p-1} \ln^{m-i} v dv du$$

$$= \sum_{p=0}^{\infty} \frac{c_{\lambda,i,p}(-1)^{m-i}(m-i)!\,[n^{s-p} - n^{-\lambda-(k+1)/\mu}]}{[(k+1)/\mu + \lambda + s - p]^{m-i+1}} \int_{-1}^{1} u^p \rho^{(s)}(u)du + E(\ln n)$$

on using Lemma 2.

It follows that

$$\underset{n\to\infty}{N-\lim} J_i = 0, \tag{9}$$

for $i = s + 1, s + 2, \ldots$ and using Lemma 1, we have

$$\underset{n\to\infty}{N-\lim} J_i = \frac{c_{\lambda,i,s}(-1)^{m+s-i}(m-i)!\,s!}{\mu[(k+1)/\mu + \lambda]^{m-i+1}} \binom{m}{i} \tag{10}$$

for $i = 0, 1, 2, \ldots, s$. It then follows from equations (7) to (10) that

$$\underset{n\to\infty}{N-\lim} I_2 = \sum_{i=0}^{m} \frac{c_{\lambda,i,s}(-1)^{m+s-i}\mu^{m-i}(m-i)!\,s!}{(k+1+\lambda\mu)^{m-i+1}} \binom{m}{i} \tag{11}$$

for $k = 0, 1, 2, \ldots$.

It now follows from equations (6), (7) and (11) that

$$N-\lim_{n\to\infty}\int_{-1}^{1} x^k G_{\lambda,m,n}(x_+^{\mu})dx = \sum_{i=0}^{m}\frac{c_{\lambda,i,s}(-1)^{m+s-i}\mu^{m-i}(m-i)!\,s!}{(k+1+\lambda\mu)^{m-i+1}}\binom{m}{i}$$

$$= \sum_{i=0}^{m}\frac{c_{\lambda,m-i,s}(-1)^{s-i}\mu^{i}i!\,s!}{(k+1+\lambda\mu)^{i+1}}\binom{m}{i} \tag{12}$$

for $k = 0, 1, 2, \ldots$.

Note that in particular, if $1+\lambda\mu > 0$, then the usual limits exist in equations (9) to (12).

We now consider the case $k = r$, where $r$ is chosen so that $0 < \lambda\mu + r + 1 < 1$, and let $\psi$ be an arbitrary continuous function. Then

$$\int_0^{n^{-1/\mu}} x^r\psi(x)G_{\lambda,m,n}(x_+^{\mu})dx = \frac{n^{-\lambda-(r+1)/\mu}}{\mu}\int_0^1 v^{(r+1)/\mu-1}\int_{-1}^{v}\psi[(v/n)^{1/\mu}]$$

$$\times (v-u)^{\lambda+s}[\ln(v-u)-\ln n]^m\rho^{(s)}(u)dudv$$

and it follows that

$$\lim_{n\to\infty}\int_0^{n^{-1/\mu}} x^r\psi(x)G_{m,\lambda,n}(x_+^{\mu})_n dx = 0. \tag{13}$$

When $x \le 0$, we have

$$\int_{-1}^{0} x^r\psi(x)G_{m,\lambda,n}(x_+^{\mu})_n dx$$

$$= n^{-\lambda}\int_{-1}^{0} x^r\psi(x)\int_{-1}^{0}(-u)^{\lambda+s}[\ln(v-u)-\ln n]^m\rho^{(s)}(u)dudx$$

and it follows that

$$N-\lim_{n\to\infty}\int_{-1}^{0} x^r\psi(x)G_{m,\lambda,n}(x_+^{\mu})_n dx = 0. \tag{14}$$

When $x^{\mu} \geq 1/n$, we have

$$G_{\lambda, m, n}(x_+^{\mu}) = \int_{-1/n}^{1/n} (x^{\mu} - t)^{\lambda+s} \ln^m (x^{\mu} - t)\delta_n^{(s)}(t)dt$$

$$= n^s x^{\mu(\lambda+s)} \int_{-1}^{1} \left(1 - \frac{u}{nx^{\mu}}\right)^{\lambda+s} \left[\ln x^{\mu} + \ln\left(1 - \frac{u}{nx^{\mu}}\right)\right]^m \rho^{(s)}(u)du$$

$$= x^{\mu(\lambda+s)} \sum_{i=0}^{m} \int_{-1}^{1} \binom{m}{i} \ln^i x^{\mu} \sum_{p=0}^{\infty} \frac{c_{\lambda, m-i, p}\, u^p}{n^{p-s} x^{\mu p}} \rho^{(s)}(u)du$$

$$= (-1)^s s! \sum_{i=0}^{m} \binom{m}{i} c_{\lambda, m-i, s}\, \mu^i x^{\lambda\mu} \ln^i x + O(n^{-1}) \tag{15}$$

on using Lemma 1.

Letting $n$ tend to infinity, it follows that

$$G_{\lambda, m}(x^{\mu}) = (-1)^s s! \sum_{i=0}^{m} \binom{m}{i} c_{\lambda, m-i, s}\, \mu^i x^{\lambda\mu} \ln^i x \tag{16}$$

for $x > 0$.

Comparing equations (4) and (16), we see that

$$a_{\lambda, m, i} = (-1)^s s! \binom{m}{i} c_{\lambda, m-i, s}, \tag{17}$$

for $i = 0, 1, 2, \ldots, m$.

We also see from equation (15) that

$$\left| \int_{-1/n}^{1/n} (x^{\mu} - t)^{\lambda+s} \ln^m (x^{\mu} - t)\delta_n^{(s)}(t)dt \right|$$

$$\leq s! \sum_{i=0}^{m} \binom{m}{i} |c_{\lambda, m-i, s}\, \mu^i x^{\lambda\mu} \ln^i x| + O(n^{-1}),$$

for $x^{\mu} \geq 1/n$.

If now $n^{-1/\mu} < \eta < 1$, then

$$\int_{n^{-1/\mu}}^{\eta} x^r \left| \int_{-1/n}^{1/n} (x^\mu - t)^{\lambda+s} \ln^m (x^\mu - t) \delta_n^{(s)}(t) dt \right| dx$$

$$\leq s! \sum_{i=0}^{m} \binom{m}{i} \mu^i |c_{\lambda, m-i, s}| \int_{n^{-1/\mu}}^{\eta} [x^{\lambda\mu+r} |\ln^i x| + O(n^{-1})] dx$$

$$= O(\eta^{\lambda\mu+r+1} |\ln^m \eta|) + \eta O(n^{-1}).$$

It follows that

$$\lim_{n \to \infty} \int_{n^{-1/\mu}}^{\eta} x^r \psi(x) \left| \int_{-1/n}^{1/n} (x^\mu - t)^{\lambda+s} \ln^m (x^\mu - t) \delta_n^{(s)}(t) dt \right| dx$$

$$= O(\eta^{\lambda\mu+r+1} |\ln^m \eta|). \tag{18}$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^r}{r!} \varphi^{(r)}(\xi x),$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\langle G_{\lambda, m, n}(x_+^\mu), \varphi(x) \rangle &= \int_{-1}^{1} G_{\lambda, m, n}(x_+^\mu) \varphi(x) dx \\
&= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k G_{\lambda, m, n}(x_+^\mu) dx \\
&\quad + \int_{\eta}^{1} \frac{x^r}{r!} G_{\lambda, m, n}(x_+^\mu) \varphi^{(r)}(\xi x) dx \\
&\quad + \int_{n^{-1/\mu}}^{\eta} \frac{x^r}{r!} G_{\lambda, m, n}(x_+^\mu) \varphi^{(r)}(\xi x) dx \\
&\quad + \int_{0}^{n^{-1/\mu}} \frac{x^r}{r!} G_{\lambda, m, n}(x_+^\mu) \varphi^{(r)}(\xi x) dx \\
&\quad + \int_{-1}^{0} \frac{x^r}{r!} G_{\lambda, m, n}(x_+^\mu) \varphi^{(r)}(\xi x) dx.
\end{aligned}$$

Using equations (1), (11) to (14), (16) and (17), it follows that

$$N-\lim_{n\to\infty}\langle G_{\lambda,m,n}(x_+^{\mu}),\ \varphi(x)\rangle$$

$$=\sum_{k=0}^{r-1}\sum_{i=0}^{m}\frac{(-1)^{s-i}c_{\lambda,m-i,s}\mu^i i!\,s!\,\varphi^{(k)}(0)}{(k+1+\lambda\mu)^{i+1}k!}\binom{m}{i}$$

$$+(-1)^s s!\sum_{i=0}^{m}\binom{m}{i}c_{\lambda,m-i,s}\mu^i\int_\eta^1\frac{x^{\lambda\mu+r}\ln^i x}{r!}\varphi^{(r)}(\xi x)\,dx+O(\eta^{\lambda\mu+r+1}|\ln^m\eta|)$$

$$=\sum_{k=0}^{r-1}\sum_{i=0}^{m}\frac{(-1)^i a_{\lambda,m,i}\mu^i i!\,\varphi^{(k)}(0)}{(k+1+\lambda\mu)^{i+1}k!}$$

$$+\sum_{i=0}^{m}a_{\lambda,m,i}\mu^i\int_0^1\frac{x^{\lambda\mu+r}\ln^i x}{r!}\varphi^{(r)}(\xi x)\,dx$$

$$=\sum_{k=0}^{r-1}\sum_{i=0}^{m}\frac{(-1)^i a_{\lambda,m,i}\mu^i i!\,\varphi^{(k)}(0)}{(k+1+\lambda\mu)^{i+1}k!}$$

$$+\sum_{i=0}^{m}a_{\lambda,m,i}\mu^i\int_0^1 x^{\lambda\mu}\ln^i x\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx$$

$$=\sum_{i=0}^{m}a_{\lambda,m,i}\mu^i\left\{\int_0^1 x^{\lambda\mu}\ln^i x\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx+\frac{(-1)^i i!\,\varphi^{(k)}(0)}{(k+1+\lambda\mu)^{i+1}k!}\right\}$$

$$=\sum_{i=0}^{m}a_{\lambda,m,i}\mu^i\langle x_+^{\lambda\mu}\ln^i x_+,\ \varphi(x)\rangle,$$

since $\eta$ can be made arbitrarily small. This proves the existence of $G_{\lambda,m}(x_+^{\mu})$ and

$$G_{\lambda,m}(x_+^{\mu})=\sum_{i=0}^{m}a_{\lambda,m,i}\mu^i x_+^{\lambda\mu}\ln^i x_+ \tag{19}$$

on the interval $[-1,1]$ for $m=0,1,2,\ldots$. However, equation (3) (or (19)) clearly holds on any interval not containing the origin for $\lambda<0$, $\mu>0$ and $\lambda,\mu\neq-1,-2,\ldots$.

In particular, when $m = 0$, we have

$$G_{\lambda,0}(x_+^{\mu}) = x_+^{\lambda\mu} = F_{\lambda,0}(x_+^{\mu}) \tag{20}$$

for $\lambda < 0$, $\mu > 0$ and $\lambda, \mu \neq -1, -2, \ldots$.

Now suppose that

$$F_{\lambda,i}(x_+^{\mu}) = \mu^i x_+^{\lambda\mu} \ln^i x_+ \tag{21}$$

for $i = 0, 1, 2, \ldots, m-1$ for some $m$, $\lambda < 0$, $\mu > 0$ and $\lambda, \mu \neq -1, -2, \ldots$. This is true by equation (20) when $m = 1$.

Note that equation (3) can be written in the form

$$G_{\lambda,m}(x) = \sum_{i=0}^{m} a_{\lambda,m,i} F_{\lambda,i}(x). \tag{22}$$

Since $G_{\lambda,m}(x_+^{\mu})$ exists and $F_{\lambda,i}(x_+^{\mu})$ exists by our assumption for $i = 0, 1, 2, \ldots, m-1$, it follows from equation (22) that $F_{\lambda,i}(x_+^{\mu})$ exists and

$$\begin{aligned} G_{\lambda,m}(x_+^{\mu}) &= \sum_{i=0}^{m-1} a_{\lambda,m,i} F_{\lambda,i}(x_+^{\mu}) + a_{\lambda,m,m} F_{\lambda,m}(x_+^{\mu}) \\ &= \sum_{i=0}^{m-1} a_{\lambda,m,i} \mu^i x_+^{\lambda\mu} \ln^i x_+ + a_{\lambda,m,m} F_{\lambda,m}(x_+^{\mu}) \\ &= \sum_{i=0}^{m} a_{\lambda,m,i} \mu^i x_+^{\lambda\mu} \ln^i x_+ \end{aligned}$$

on using equations (19) and (21). It follows that

$$F_{\lambda,m}(x_+^{\mu}) = \mu^i x_+^{\lambda\mu} \ln^m x_+$$

and equation (2) follows by induction.

Since the usual limit exists in equation (12), it follows that the composition $F_{\lambda,m}(x_+^{\mu})$ exists, if $1 + \lambda\mu > 0$. This completes the proof of the theorem.

Replacing $x$ by $-x$ in Theorem 5, we get

**Theorem 6.** *If* $F_{\lambda, m}(x)$ *denotes the distribution* $x_-^{\lambda} \ln^m x_-$, *then the neutrix composition* $F_{\lambda, m}(x_-^{\mu})$ *exists and*

$$F_{\lambda, m}(x_-^{\mu}) = \mu^m x_-^{\lambda\mu} \ln^m x_- \tag{23}$$

*for* $\lambda < 0$, $\mu > 0$ *and* $\lambda, \lambda\mu \neq -1, -2, \ldots$.

*In particular, the composition* $F_{\lambda, m}(x_-^{\mu})$ *exists if* $1 + \lambda\mu > 0$.

The proof of the next theorem is similar to the proof of Theorem 5 and is left as an exercise.

**Theorem 7.** *If* $F_{\lambda, m}(x)$ *denotes the distribution* $|x|^{\lambda} \ln^m |x|$, *then the neutrix composition* $F_{\lambda, m}(|x|^{\mu})$ *exists and*

$$F_{\lambda, m}(|x|^{\mu}) = \mu^m |x|^{\lambda\mu} \ln^m |x| \tag{24}$$

*for* $\lambda < 0$, $\mu > 0$ *and* $\lambda, \lambda\mu \neq -1, -2, \ldots$.

*In particular, the composition* $F_{\lambda, m}(|x|^{\mu})$ *exists if* $1 + \lambda\mu > 0$.

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