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ON LOCAL CONTROLLABILITY OF A QUASILINEAR HYPERBOLIC EQUATION

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Abstract

In this article, we prove controllability of a one-dimensional wave equation with control functions on the right-hand side of the equation, and give a priori estimates for the controls.

Introduction

We prove controllability of the one-dimensional wave equation

$$u_{tt}(x, t) - u_{xx(x, t)} = \left(\frac{x}{l} - 1\right) f(t) + \frac{x}{l} g(t), \quad (0 < t < T, \ 0 < x < l)$$

where f and g are control functions to be found so that the system can be driven from the initial state

$$u(x, 0^+) = u_t(x, 0^+) = 0 \quad (0 \le x \le l)$$

to a final state

$$u(x, t)_{|\, t=T} = \psi_1(x), \quad u_t(x, t)_{|\, t=T} = \psi_2(x) \quad 0 \le x \le l.$$

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We show that this problem is equivalent to a homogeneous case studied by Il'in and Tikhomirov [1], with control functions on the boundary.

A priori estimates for the control functions are also given.

0. Preliminary Results

Throughout the present article

$$Q_T = (0,\, l) \times (0,\, T).$$

We consider the one-dimensional wave equation

$$v_{tt} - v_{xx} = 0, \quad (x,\, t) \in Q_T \tag{1}$$

$$v(x,\, t)|_{t=0} = \varphi(x), \quad v_t(x,\, t)|_{t=0} = \psi(x), \quad 0 \le x \le l \tag{2}$$

$$v(x,\, t)|_{x=0} = f(t), \quad v_t(x,\, t)|_{x=l} = g(t), \quad 0 \le t \le T \tag{3}$$

where

$$\varphi \in W_2^2(0,\, l), \;\; \psi \in W_2^1(0,\, l), \;\; f \in W_2^2(0,\, T), \;\; g \in W_2^2(0,\, T).$$

Moreover, we assume that the following compatibility conditions are satisfied

$$f(0) = \varphi(0), \;\; g(0) = \varphi(l), \;\; f'(0) = \psi(0), \;\; g'(0) = \psi(l). \tag{4}$$

The problem can be stated in the following way:

In time $T > 0$, we propose to drive the system from its initial state (2) to a given final state

$$v(x,\, t)|_{t=T} = \varphi_1(x), \quad v_t(x,\, t)|_{t=T} = \psi_1(x), \tag{5}$$

with $\varphi_1 \in W_2^2(0,\, l)$ and $\psi_1 \in W_2^1(0,\, l)$ by means of control functions f and g to be found.

In other words, for fixed $T > 0$, find the Dirichlet boundary control functions f and g such that the solutions v of the problem (1-4) satisfies (5).

Let us recall some results in [1].

Definition 1 [1, p.693]**.** We say that a function $(x, t) \mapsto v(x, t)$ belongs to the class $\hat{W}_2^2(Q_T)$, if v and its first partial derivatives are continuous in $\overline{Q}_T$, and if this function admits all second generalised partial derivatives belonging to $L_2(0, l)$ for all $x \in [0, l]$.

Definition 2 [1, p.694]**.** We say that a function $v \in \hat{W}_2^2(Q_T)$ is a solution of a mixed problem (1)-(3), if v satisfies the wave equation (1.1) for all $t \in (0, T)$ and almost everywhere for $x \in (0, l)$, and therefore for all $x \in (0, l)$, and almost everywhere for $t \in (0, T)$. Moreover v satisfies the initial conditions (2) and the boundary conditions (3).

Theorem 1 [1, p.698]**.** *Let* $\varphi \in W_2^2(0, l)$, $\psi \in W_2^1(0, l)$, $\varphi_1 \in W_2^2(0, l)$ *and* $\psi_1 \in W_2^1(0, l)$. *There exist control functions* f, $g \in W_2^2(0, l)$ *satisfying the compatibility conditions* $($*for* $T = l)$:

$$f(l) = \varphi_1(0), \quad g(l) = \varphi_1(l), \quad f'(l) = \psi_1(0), \quad g'(l) = \psi_1(l) \tag{6}$$

and such that the solution $v \in \hat{W}_2^2(Q_l)$ *of the mixed problem*

$$v_{tt} - v_{xx} = 0 \quad \text{in } Q_l$$

$$v\,|_{t=0} = \varphi(x), \quad v_t\,|_{t=0} = \psi(x), \quad 0 \le x \le l$$

$$v\,\big|_{x=0} = f(t), \quad v\,\big|_{x=l} = g(t), \quad 0 \le t \le l$$

satisfies the conditions

$$v\,|_{t=l} = \varphi_1(x), \quad v_t\,\big|_{x=l} = \psi_1(x), \quad 0 \le x \le l$$

if and only, *the following three constraints are satisfied*:

$$\psi(0) + \varphi'(0) - \psi_1(l) + \varphi_1'(l) = 0 \tag{7}$$

$$\psi(l) + \varphi'(l) - \psi_1(0) + \varphi_1'(0) = 0 \tag{8}$$

$$\int_0^l \psi(x)dx + \varphi(0) + \varphi(l) + \int_0^l \psi_1(x)dx - \varphi_1(0) - \varphi_1(l) = 0 \tag{9}$$

from which we obtain boundary control functions f and g given by:

$$f(t) = \frac{1}{2}\int_0^t \psi(x)dx + \frac{1}{2}\varphi(t) + \frac{1}{2}\varphi(0)$$

$$+ \frac{1}{2}\int_{l-t}^{l} \psi_1(x)dx - \frac{1}{2}\varphi_1(l) + \frac{1}{2}\varphi_1(l-t) \tag{10}$$

$$g(t) = \frac{1}{2}\int_0^t \psi_1(x)dx + \frac{1}{2}\varphi_1(t) - \frac{1}{2}\varphi_1(0)$$

$$+ \frac{1}{2}\int_{l-t}^{l} \psi(x)dx + \frac{1}{2}\varphi(l) + \frac{1}{2}\varphi(l-t). \tag{11}$$

Theorem 2. *There exists only one solution* $v \in \hat{W}_2^2(Q_l)$ *for the mixed problem* (1)-(3).

1. A Priori Estimates of Control Functions

Proposition 1. *We can show that the following a priori estimates hold*

$$\| f \|_{\hat{W}_2^2(0,\, l)} \le c(l)(\| \psi \|_{W_2^1(0,\, l)}$$

$$+ \| \psi_1 \|_{W_2^1(0,\, l)} + \| \varphi \|_{W_2^2(0,\, l)} + \| \varphi_1 \|_{W_2^2(0,\, l)}) \tag{12}$$

$$\| g \|_{W_2^2(0,\, l)} \le c(l)(\| \psi \|_{W_2^1(0,\, l)}$$

$$+ \| \psi_1 \|_{W_2^1(0,\, l)} + \| \varphi \|_{W_2^2(0,\, l)} + \| \varphi_1 \|_{W_2^2(0,\, l)}). \tag{13}$$

Let us prove (12).

From (10), we get

$$f'(t) = \frac{1}{2}\psi(t) + \frac{1}{2}\varphi'(t) - \frac{1}{2}\psi_1(l-t) + \frac{1}{2}\varphi_1'(l-t)$$

and

$$f''(t) = \frac{1}{2}\psi'(t) + \frac{1}{2}\varphi''(t) - \frac{1}{2}\psi_1'(l-t) + \frac{1}{2}\varphi_1''(l-t).$$

So we have

$$|f(t)|^2 = \frac{1}{4}\left[\left|\int_0^t \psi(x)dx + \varphi(t) + \varphi(0) + \int_{l-t}^{l} \psi_1(x)dx - \varphi_1(l) + \varphi_1(l-t)\right|\right]^2$$

$$\leq \frac{1}{4}\left[\left|\int_0^t \psi(x)dx\right| + |\varphi(t)| + |\varphi(0)|\right.$$

$$\left. + \left|\int_{l-t}^{l} \psi_1(x)dx\right| + |\varphi_1(l)| + |\varphi_1(l-t)|\right]^2.$$

The inequality $(a_1 + \cdots + a_n)^2 \leq n(|a_1|^2 + \cdots + |a_n|^2)$ implies

$$|f(t)|^2 \leq \frac{3}{2}[l\|\psi(x)\|^2_{L_2(0,l)} + |\varphi(t)|^2 + |\varphi(0)|^2$$

$$+ l\|\psi_1(x)\|^2_{L_2(0,l)} + |\varphi_1(t)|^2 + |\varphi_1(l-t)|^2]$$

$$\leq \frac{3}{2}[C_1^2 l\|\psi(x)\|_{W_2^1(0,l)} + |\varphi(t)|^2 + C_2^2\|\varphi(x)\|^2_{W_2^2(0,l)}$$

$$+ C_3^2 l\|\psi_1(x)\|^2_{W_2^1(0,l)} + C_4^2\|\varphi_1(x)\|^2_{W_2^2(0,l)} + |\varphi_1(l-t)|^2].$$

Moreover,

$$|f'(t)|^2 \leq \frac{1}{4}[|\psi(t) + \varphi'(t) + \psi_1(l-t) - \varphi_1'(l-t)|]^2$$

$$\leq \frac{1}{4}[|\psi(t)| + |\varphi'(t)| + |\psi_1(l-t)| + |\varphi_1'(l-t)|]^2$$

$$\leq |\psi(t)|^2 + |\varphi'(t)|^2 + |\psi_1(l-t)|^2 + |\varphi_1'(l-t)|^2$$

and

$$|f''(t)|^2 \leq \frac{1}{4}[|\psi'(t) + \varphi''(t) - \psi_1'(l-t) - \varphi_1''(l-t)|]^2$$

$$\leq \frac{1}{4}[|\psi'(t)| + |\varphi''(t)| + |\psi_1'(l-t)| + |\varphi_1''(l-t)|]^2$$

$$\leq |\psi'(t)|^2 + |\varphi''(t)|^2 + |\psi_1'(l-t)|^2 + |\varphi_1''(l-t)|^2.$$

Hence,

$$
\begin{aligned}
\| f \|^2_{W_2^2(0,\, l)} &= \int_0^l [| f(t) |^2 + | f'(t) |^2 + | f''(t) |^2] dt \\
&\leq \frac{3}{2} \int_0^l [C_1^2 l \| \psi(x) \|^2_{W_2^1(0, l)} + C_2^2 \| \varphi(x) \|^2_{W_2^2(0, l)} \\
&\quad + C_3^2 l \| \psi_1(x) \|^2_{W_2^1(0, l)} + C_4^2 \| \varphi_1(x) \|^2_{W_2^2(0, l)}] dt \\
&\quad + \int_0^l (| \psi(t) |^2 + | \psi'(t) |^2) dt + \int_0^l (| \psi_1(l-t) |^2 + | \psi_1'(l-t) |^2) dt \\
&\quad + \int_0^l \left[\frac{3}{2} | \varphi_1(t-l) |^2 + | \varphi_1'(t-l) |^2 + | \varphi_1''(t-l) |^2 \right] dt \\
&= \left(\frac{3}{2} C_1^2 l^2 + 1 \right) \| \psi(x) \|^2_{W_2^1(0, l)} + \left(\frac{3}{2} C_3^2 l^2 + 1 \right) \| \psi_1(x) \|^2_{W_2^1(0, l)} \\
&\quad + \int_0^l \left[\frac{3}{2} | \varphi(t) |^2 + | \varphi'(t) |^2 + | \varphi''(t) |^2 \right] dt \\
&\quad + \int_0^l \left[\frac{3}{2} | \varphi_1(t-l) |^2 + | \varphi_1'(t-l) |^2 + | \varphi_1''(t-l) |^2 \right] dt \\
&\quad + \frac{3}{2} C_2^2 l \| \varphi(x) \|^2_{W_2^2(0, l)} + \frac{3}{2} C_4^2 l \| \varphi_1(x) \|^2_{W_2^2(0, l)} \\
&= \left(\frac{3}{2} C_1^2 l^2 + 1 \right) \| \psi(x) \|^2_{W_2^1(0, l)} + \left(\frac{3}{2} C_3^2 l^2 + 1 \right) \| \psi_1(x) \|^2_{W_2^1(0, l)} \\
&\quad + \left(\frac{3}{2} C_2^2 l + 1 \right) \| \varphi(x) \|^2_{W_2^2(0, l)} + \frac{1}{2} \| \varphi(x) \|^2_{L_2(0, l)} \\
&\quad + \left(\frac{3}{2} C_4^2 l + 1 \right) \| \varphi_1(x) \|^2_{W_2^2(0, l)} + \frac{1}{2} \| \varphi_1(x) \|^2_{L_2(0, l)} \\
&\leq \left(\frac{3}{2} C_1^2 l^2 + 1 \right) \| \psi(x) \|^2_{W_2^1(0, l)} + \left(\frac{3}{2} C_3^2 l^2 + 1 \right) \| \psi_1(x) \|^2_{W_2^1(0, l)} \\
&\quad + \left(\frac{3}{2} C_2^2 l + \frac{1}{2} C_5^2 + 1 \right) \| \varphi(x) \|^2_{W_2^2(0, l)} \\
&\quad + \left(\frac{3}{2} C_4^2 l + \frac{1}{2} C_6^2 + 1 \right) \| \varphi_1(x) \|^2_{W_2^2(0, l)}
\end{aligned}
$$

and, setting

$$C(l) = \max\left(\frac{3}{2}C_1^2 l^2 + 1,\ \frac{3}{2}C_3^2 l^2 + 1,\ \frac{3}{2}C_2^2 l + \frac{1}{2}C_5^2 + 1,\ \frac{3}{2}C_4^2 l + \frac{1}{2}C_6^2 + 1\right)$$

we get

$$\| f \|^2_{W_2^2(0,l)} \le C(l)(\| \psi \|^2_{W_2^1(0,l)} + \| \psi_1 \|^2_{W_2^1(0,l)} + \| \varphi \|^2_{W_2^2(0,l)} + \| \varphi_1 \|^2_{W_2^2(0,l)}).$$

Thus,

$$\| f \|_{W_2^2(0,l)} \le C(l)(\| \psi \|_{W_2^1(0,l)} + \| \psi_1 \|_{W_2^1(0,l)} + \| \varphi \|_{W_2^2(0,l)} + \| \varphi_1 \|_{W_2^2(0,l)}).$$

The proof of (13) is similar.

Proposition 2. *For* $\varphi \in W_2^4(0, l)$, $\psi \in W_2^3(0, l)$, $f \in W_2^5(0, T)$, $g \in W_2^5(0, T)$, $\varphi_1 \in W_2^5(0, l)$, $\psi_1 \in W_2^5(0, l)$, *Theorems* 3 *and* 4 *hold and the solution* v *of the mixed problem* (1)-(3) *belongs to the space* $W_2^4(Q_T)$; *and the following estimates hold*:

$$\| f \|_{W_2^5(0,l)} \le C(l)(\| \psi \|_{W_2^3(0,l)} + \| \psi_1 \|_{W_2^4(0,l)} + \| \varphi \|_{W_2^4(0,l)} + \| \varphi_1 \|_{W_2^5(0,l)}) \tag{14}$$

$$\| g \|_{W_2^5(0,l)} \le C(l)(\| \psi \|_{W_2^3(0,l)} + \| \psi_1 \|_{W_2^4(0,l)} + \| \varphi \|_{W_2^4(0,l)} + \| \varphi_1 \|_{W_2^5(0,l)}). \tag{15}$$

The inequalities (14) and (15) are established like the inequality (12).

Example 1. Consider the problem

$$\tilde{v}_{tt} - \tilde{v}_{xx} = 0 \tag{16}$$

$$\tilde{v}\,|_{t=0} = C_7\left(1 - \frac{x}{l}\right) + C_8\,\frac{x}{l}, \quad \tilde{v}_t\,|_{t=0} = C_9\left(1 - \frac{x}{l}\right) \tag{17}$$

$$\tilde{v}\,|_{x=0} = f(t), \quad \tilde{v}\,|_{x=l} = g(t) \tag{18}$$

$$\tilde{v}\,|_{t=l} = \psi_1(x), \quad v_t\,|_{t=l} = \psi_2(x) \tag{19}$$

where

$$\psi_1 \in W_2^5(0, l) \cap \overset{0}{W_2^1}(0, l), \quad \psi_2 \in W_2^4(0, l) \cap \overset{0}{W_2^1}(0, l).$$

Let

$$\varphi(x) = C_7\left(1 - \frac{x}{l}\right) + C_8 \frac{x}{l}, \quad \psi(x) = C_9\left(1 - \frac{x}{l}\right)$$

$$\varphi_1(x) = \chi_1(x), \quad \psi_1(x) = \chi_2(x),$$

and assume that conditions (7)-(8) are satisfied, i.e.,

$$\chi_1'(l) + \frac{C_7}{l} - \frac{C_8}{l} + C_9 = 0 \tag{20}$$

$$-\chi_1'(0) - \frac{C_7}{l} + \frac{C_8}{l} = 0 \tag{21}$$

$$\int_0^l \chi_2(x)dx + C_7 + C_8 + \frac{l}{2}C_9 = 0. \tag{22}$$

Then, according to Theorem 3, there exits a unique solution (f, g) of problem (16)-(19) and the controls f and g are given by (10), (11).

From (10), we have

$$\begin{aligned} f(t) &= \frac{1}{2}\int_0^t \psi(x)dx + \frac{1}{2}\varphi(t) + \frac{1}{2}\varphi(0) \\ &\quad + \frac{1}{2}\int_{l-t}^l \psi_1(x)dx - \frac{1}{2}\varphi_1(t) + \frac{1}{2}\varphi_1(0) \\ &= \frac{1}{2}\int_0^t C_9\left(1 - \frac{x}{l}\right)dx + \frac{1}{2}\left[C_7\left(1 - \frac{t}{l}\right) + C_8 \frac{t}{l}\right] \\ &\quad + \frac{1}{2}C_7 + \frac{1}{2}\int_{l-t}^l \psi_2(x)dx + \frac{1}{2}\psi_1(l-t) \end{aligned}$$

so that

$$\begin{aligned} f(t) &= \frac{1}{2}\int_0^t \chi_2(x)dx + \frac{1}{2}C_9\left(t - \frac{t^2}{2l}\right) + \frac{1}{2}C_7 \\ &\quad + \frac{1}{2}\left[C_7\left(1 - \frac{t}{l}\right) + C_8 \frac{t}{l}\right] + \frac{1}{2}\chi_1(l-t). \end{aligned} \tag{23}$$

With the same process, using (11), we get

$$g(t) = \frac{1}{2}\int_0^t \psi_2(x)dx - \frac{1}{2}C_9\left(l - t - \frac{(l-t)^2}{2l}\right)$$

$$+ \frac{1}{2}C_8 + \frac{1}{2}\left[C_8\left(1 - \frac{t}{l}\right) + C_7\frac{t}{l}\right] + C_9\frac{l}{4} + \frac{1}{2}\chi_1(t). \tag{24}$$

Taking the derivatives of f and g from (23) and (24), we get

$$f'(t) = \frac{1}{2}\chi_2(l-t) + \frac{1}{2}C_9\left(1 - \frac{t^2}{2l}\right) + \frac{1}{2}\left(\frac{-C_7}{l} + \frac{C_8}{l}\right) - \frac{1}{2}\chi_1'(l-t) \tag{25}$$

and

$$g'(t) = \frac{1}{2}\chi_2(t) + \frac{1}{2}\left(\frac{C_7}{l} - \frac{C_8}{l}\right) + C_9\frac{t}{2l} + \frac{1}{2}\chi_1(t). \tag{26}$$

From (20)-(22), and (23)-(26), we get

$$f(0) = C_7 = \varphi(0), \ \ g(0) = C_8 = \varphi(l), \ \ f'(0) = C_9 = \psi(0), \ \ g'(0) = 0 = \psi(l)$$

and

$$f(l) = 0 = \varphi_1(0), \ \ g(l) = 0 = \varphi_1(l), \ \ f'(l) = 0 = \psi_1(0), \ \ g'(l) = 0 = \psi_1(l)$$

so that the compatibility conditions with functions φ and ψ for $t = 0$, and with functions φ_1 and ψ_1 for $T = l$ hold.

2. The Main Result

Remark 1. If $\tilde{f}(t) = f''(t)$, $\tilde{g}(t) = g''(t)$ and $w(x, t) = \left(1 - \frac{x}{l}\right)f(t) + \frac{x}{l}g(t)$, then $w \in W_2^5(Q_T)$ and satisfies the following problem

$$w_{tt} - w_{xx} = \left(1 - \frac{x}{l}\right)\tilde{f}(t) + \frac{x}{l}\tilde{g}(t) \tag{27}$$

$$w\,|_{t=0} = C_7\left(1 - \frac{x}{l}\right) + C_8\frac{x}{l}, \quad w_t\,|_{t=0} = C_9\left(1 - \frac{x}{l}\right) \tag{28}$$

$$w\,|_{x=0} = f(t), \quad w_t\,|_{x=t} = g(t) \tag{29}$$

$$w\,|_{t=l} = 0, \quad w_t\,|_{t=l} = 0. \tag{30}$$

Remark 2. Let $u = \tilde{v} - w$. In this case, the function u is a solution of the following problem

$$u_{tt} - u_{xx} = F(x, t) \tag{31}$$

$$u\,|_{t=0} = 0, \quad w_t\,|_{t=0} = 0 \tag{32}$$

$$u\,|_{x=0} = 0, \quad v\,|_{x=l} = 0 \tag{33}$$

$$u\,|_{t=l} = \chi_1(x), \quad u_t\,|_{t=l} = \chi_2(x) \tag{34}$$

where $F(x, t) = \left(\frac{x}{l} - 1\right)\tilde{f}(t) - \frac{x}{l}\,\tilde{g}(t).$

Remark 3. Although we shall hereforth deal with the study of problem (31)-(34), the controllability problem in this case is formulated in the following way:

in time $T > 0$ we propose to drive the system from an initial state (32) to a given final state (34) by means control functions $\tilde{f}$ and $\tilde{g}$.

Theorem 3. *There exists a unique solution of the problem* (31)-(34).

Proof. The existence of control functions $\tilde{f}$ and $\tilde{g}$ follows from the foregoing results. We can proceed to prove their uniqueness.

Let $(\tilde{f}, \tilde{g})$ be any solution of the problem (31)-(34) with $\chi_1(x) = 0$ and $\chi_2(x) = 0$. Set

$$f(t) = \int_0^t \int_0^s \tilde{f}(\sigma) d\sigma ds + C_{10} t + C_{11}$$

$$g(t) = \int_0^t \int_0^s \tilde{g}(\sigma) d\sigma ds + C_{12} t + C_{13}$$

and

$$w(x, t) = \left(1 - \frac{x}{l}\right) f(t) + \frac{x}{l}\, g(t).$$

In this case, w satisfies the following system

$$w_{tt} - w_{xx} = \left(1 - \frac{x}{l}\right)\tilde{f}(t) + \frac{x}{l}\tilde{g}(t) \tag{35}$$

$$w|_{t=0} = C_{11}\left(1 - \frac{x}{l}\right) + C_{13}\frac{x}{l}, \quad w_t|_{t=0} = C_{10}\left(1 - \frac{x}{l}\right) + C_{12}\frac{x}{l} \tag{36}$$

$$w|_{x=0} = f(t), \quad w|_{x=t} = g(t) \tag{37}$$

$$w|_{t=l} = \left(1 - \frac{x}{l}\right)f(T) + \frac{x}{l}g(T), \quad w_t|_{t=l} = \left(1 - \frac{x}{l}\right)f'(T) + \frac{x}{l}g'(T). \tag{38}$$

According to (31)-(34), (35)-(38), the change of variables $v = w + u$ gives the following new problem

$$v_{tt} - v_{xx} = 0, \quad (x, t) \in Q_T = (0, l) \times (0, T),$$

$$v|_{t=0} = \varphi(x), \quad v_t|_{t=0} = \psi(x), \quad 0 \le x \le l$$

$$v|_{x=0} = f(t), \quad v|_{x=l} = g(t), \quad 0 \le t \le T$$

$$v(x, t)|_{t=T} = \varphi_1(x), \quad v_t(x, t)|_{t=T} = \psi_1(x),$$

where

$$\varphi(x) = C_{11}\left(1 - \frac{x}{l}\right) + C_{13}\frac{x}{l}, \quad \psi(x) = C_{10}\left(1 - \frac{x}{l}\right) + C_{12}\frac{x}{l},$$

$$\varphi_1(x) = \left(1 - \frac{x}{l}\right)f(T) + \frac{x}{l}g(T), \quad \psi_1(x) = \left(1 - \frac{x}{l}\right)f'(T) + g'(T)\frac{x}{l};$$

which means that we have obtained problem (1)-(3), (5); and using conditions (7)-(9), we get $\tilde{f}(t) = 0$ and $\tilde{g}(t) = 0$.

Therefore, there exists a unique solution of problem (31)-(34) and the following estimates hold.

$$\|\tilde{f}(\ )\|_{W_2^3(0,l)} \le C(l)(\|\chi_2(x)\|_{W_2^4(0,l)} + \|\chi_1(x)\|_{W_2^5(0,l)}) \tag{39}$$

$$\|\tilde{g}(\ )\|_{W_2^3(0,l)} \le C(l)(\|\chi_2(x)\|_{W_2^4(0,l)} + \|\chi_1(x)\|_{W_2^5(0,l)}). \tag{40}$$

Let us prove inequality (39). Consider the system of three equations (20)-(21) in three unknown variables C_7, C_8, C_9

$$\frac{C_7}{l} - \frac{C_8}{l} + C_9 = -\chi_1'(l)$$

$$-\frac{C_7}{l} + \frac{C_8}{l} = \chi_1'(0)$$

$$C_7 + C_8 + \frac{l}{2} C_9 = -\int_0^l \chi_2(x)dx.$$

According to Cramer rule,

$$D = \begin{vmatrix} \frac{1}{l} & -\frac{1}{l} & 1 \\ -\frac{1}{l} & \frac{1}{l} & 0 \\ 1 & 1 & \frac{l}{2} \end{vmatrix} = -\frac{2}{l} \neq 0;$$

$$D_1 = \begin{vmatrix} -\chi_1'(l) & -\frac{1}{l} & 1 \\ \chi_1'(0) & \frac{1}{l} & 0 \\ -\int_0^l \chi_2(x)dx & 1 & \frac{l}{2} \end{vmatrix} = \frac{1}{l}\int_0^l \chi_2(x)dx - \frac{1}{2}\chi_1'(l) + \frac{3}{2}\chi_1'(0),$$

$$D_2 = \begin{vmatrix} \frac{1}{l} & -\chi_1'(l) & 1 \\ -\frac{1}{l} & \chi_1'(0) & 0 \\ 1 & -\int_0^l \chi_2(x)dx & \frac{l}{2} \end{vmatrix} = \frac{1}{l}\int_0^l \chi_2(x)dx - \frac{1}{2}\chi_1'(l) - \frac{1}{2}\chi_1'(0),$$

$$D_3 = \begin{vmatrix} \frac{1}{l} & -\frac{1}{l} & -\chi_1'(l) \\ -\frac{1}{l} & \frac{1}{l} & \chi_1'(0) \\ 1 & 1 & -\int_0^l \chi_2(x)dx \end{vmatrix} = \frac{2}{l}\chi_1'(l) - \frac{2}{l}\chi_1'(0).$$

Thus

$$C_7 = \frac{D_1}{D} = -\frac{1}{2}\int_0^l \chi_2(x)dx + \frac{l}{4}\chi_1'(l) - \frac{3l}{4}\chi_1'(0),$$

$$C_8 = \frac{D_2}{D} = -\frac{1}{2}\int_0^l \chi_2(x)dx + \frac{l}{4}\chi_1'(l) + \frac{l}{4}\chi_1'(0),$$

$$C_9 = \frac{D_3}{D} = \chi_1'(0) - \chi_1'(l).$$

This implies, using (23) and (24),

$$f(t) = \frac{1}{2}\int_{l-t}^l \chi_2(x)dx - \frac{1}{2}\int_0^l \chi_2(x)dx + \frac{1}{2}\chi_1(l-t)$$

$$+ \frac{1}{2}\left(\frac{l}{2} - t + \frac{t^2}{2l}\right)\chi_1'(l) + \frac{1}{2}\left(-\frac{3l}{2} + 2t - \frac{t^2}{2l}\right)\chi_1'(0) \tag{41}$$

and

$$g(t) = \frac{1}{2}\int_0^t \chi_2(x)dx - \frac{1}{2}\int_0^t \chi_2(x)dx + \frac{1}{2}\chi_1(t)$$

$$- \frac{1}{2}\left(l - t - \frac{(t-l)^2}{2l}\right)(\chi_1'(0) - \chi_1'(l)) + \frac{l}{2}\chi_1'(0) - \frac{t}{2}\chi_1'(0). \tag{42}$$

By (41),

$$f'(t) = \frac{1}{2}\chi_2(l-t) - \frac{1}{2}\chi_1'(l-t) + \frac{1}{2}\left(-1 + \frac{t}{l}\right)\chi_1'(l) + \frac{1}{2}\left(2 - \frac{t}{l}\right)\chi_1'(0),$$

$$f''(t) = -\frac{1}{2}\chi_2'(l-t) + \frac{1}{2}\chi_1''(l-t) + \frac{1}{2l}\chi_1'(l) + \frac{1}{2l}\chi_1'(0),$$

$$f'''(t) = \frac{1}{2}\chi_2''(l-t) - \frac{1}{2}\chi_1'''(l-t),$$

$$f^{(4)}(t) = -\frac{1}{2}\chi_2'''(l-t) + \frac{1}{2}\chi_1^{(4)}(l-t),$$

and

$$f^{(5)}(t) = \frac{1}{2}\chi_2^{(4)}(l-t) - \frac{1}{2}\chi_1^{(5)}(l-t).$$

Thus

$$
\begin{aligned}
|f''(t)|^2 &= \frac{1}{4}\left[\left| -\chi_2'(l-t) + \chi_1''(l-t) + \frac{1}{l}\chi_1'(l) - \frac{1}{l}\chi_1'(0)\right|\right]^2 \\
&\leq \frac{1}{4}\left[|\chi_2'(l-t)| + |\chi_1''(l-t)| + \left|\frac{1}{l}\chi_1'(l)\right| + \left|\frac{1}{l}\chi_1'(0)\right|\right]^2 \\
&\leq |\chi_2'(l-t)|^2 + |\chi_1''(l-t)|^2 + \frac{1}{l^2}|\chi_1'(l)|^2 + \frac{1}{l^2}|\chi_1'(0)|^2 \\
&\leq |\chi_2'(l-t)|^2 + |\chi_1''(l-t)|^2 + \frac{C_4^2 + C_5^2}{l^2}\|\chi_1'(x)\|^2_{W_2^3(0,l)} \\
&\leq |\chi_2'(l-t)|^2 + |\chi_1''(l-t)|^2 + \frac{C_4^2 + C_5^2}{l^2}\|\chi_1'(x)\|^2_{W_2^4(0,l)},
\end{aligned}
$$

$$
\begin{aligned}
|f'''(t)|^2 &= \frac{1}{4}[|\chi_2''(l-t) - \chi_1'''(l-t)|]^2 \\
&\leq \frac{1}{4}[|\chi_2''(l-t)|^2 + |\chi_1'''(l-t)|^2] \\
&\leq |\chi_2''(l-t)|^2 + |\chi_1'''(l-t)|^2,
\end{aligned}
$$

$$
\begin{aligned}
|f^{(4)}(t)|^2 &= \frac{1}{4}[|\chi_2'''(l-t) + \chi_1^{(4)}(l-t)|]^2 \\
&\leq |\chi_2'''(l-t)|^2 + |\chi_1^{(4)}(l-t)|^2,
\end{aligned}
$$

and

$$|f^{(5)}(t)|^2 \leq |\chi_2^{(4)}(l-t)|^2 + |\chi_1^{(5)}(l-t)|^2.$$

Therefore

$$
\begin{aligned}
\|\tilde{f}\|^2_{W_2^3(0,l)} &= \int_0^l [|\tilde{f}(t)|^2 + |\tilde{f}'(t)|^2 + |\tilde{f}''(t)|^2 + |\tilde{f}'''(t)|^2]dt \\
&= \int_0^l [|f''(t)|^2 + |f'''(t)|^2 + |f^{(4)}(t)|^2 + |f^{(5)}(t)|^2]dt
\end{aligned}
$$

$$\leq \| \chi_2'(l-t) \|^2_{W_2^3(0,l)} + \| \chi_1''(l-t) \|^2_{W_2^3(0,l)}$$

$$+ \frac{C_{14}^2 + C_{15}^2}{l^2} \| \chi_1'(x) \|^2_{W_2^4(0,l)}$$

$$\leq \| \chi_2(x) \|^2_{W_2^4(0,l)} + \left(1 + \frac{C_{14}^2 + C_{15}^2}{l^2} \right) \| \chi_1(x) \|^2_{W_2^5(0,l)}.$$

Hence

$$\| \widetilde{f} \|^2_{W_2^3(0,l)} \leq C(l) (\| \chi_2(x) \|_{W_2^4(0,l)} + \| \chi_1(x) \|^2_{W_2^5(0,l)}),$$

where

$$f''(t) = -\frac{1}{2} \chi_2'(l-t) + \frac{1}{2} \chi_1''(l-t) + \frac{1}{2l} \chi_1'(l) - \frac{1}{2l} \chi_1'(0),$$

$$f'''(t) = \frac{1}{2} \chi_2''(l-t) - \frac{1}{2} \chi_1'''(l-t),$$

$$C(l) = \max \left(1,\, 1 + \frac{C_{14}^2 + C_{15}^2}{l^2} \right).$$

The proof of (40) is similar to that of (39).

Reference

[1] V. A. Il'in and V. V. Tikhomirov, Volnovje ouravnjenje…" The wave equation with the Dirichlet boundary conditions (Soviet Maths), Differentialnie ouravnienie 35(5) (1999), 692-704.