Advances and Applications in Statistics
Volume 10, Number 2, 2008, Pages 295-301
Published Online: January 27, 2009
This paper is available online at http://www.pphmj.com

UP-CROSSINGS RATE BY THE PRODUCT OF TWO GAUSSIAN PROCESSES

Q. D. KATATBEH*, M. T. ALODAT and K. M. ALUDAAT

*Department of Mathematics and Statistics
University of Science and Technology
Irbid, Jordan

Department of Statistics
Yarmouk University
Irbid, Jordan
e-amil: malodat@yu.edu.jo

Abstract

We derived the up-crossings rate of the product of two stationary, differentiable and independent Gaussian random processes. We compared our results with an example.

1. Introduction

Let $X(t)$, $t \in [0, T]$, be a stationary and differentiable random process. Let $\dot{X}(t)$ denote the derivative of $X(t)$. The process $X(t)$ is said to have an *up-crossing* of the level x at $t_0 \in [0, T]$ if $X(t_0) = x$ and $\dot{X}(t_0) > 0$. Let $N_x(X, T)$ denote the number of up-crossings of x by $X(T)$ in $[0, T]$. The mean value of $N_x(X, T)$ is given by the following well-known theorem:

Theorem 1 (Rice's formula)**.** *Let* $X(t)$, $t \in [0, T]$, *be a differentiable stationary process with derivative* $\dot{X}(t)$. *If* $f_{X(0)}(x)$ *denotes the density of*

2000 Mathematics Subject Classification: 60G10.

Keywords and phrases: Gaussian process, up-crossing, Rice's formula.

Received August 28, 2008

$X(0)$, *then*

$$E\{N_x(X, T)\} = Tf_X(x)E\{\dot{X}(0)^+ \mid X(0) = x\},$$

where $X^+ = \max\{0, X\}$.

The problem of finding the probability, $P\{\sup_{[0, T]} X(t) \geq x\}$, for large x, is of central interest in many applications of processes to the analysis and design of systems in engineering. Since it is not possible in general to find the exact value of this probability for many processes, one can find an approximation to it. Adler [1] gives the following upper bound:

$$\begin{aligned} P\{\sup_{t \in [0, T]} X(t) \geq x\} &= P\{X(0) \geq x \text{ or } N_x(X, T) \geq 1\} \\ &\leq P\{X(0) \geq x\} + \mathbb{E}N_x(X, T) \\ &= P\{X(0) \geq x\} + T\mathbb{E}N_x(X, 1). \end{aligned} \tag{1}$$

In (1), for wide class of processes, high x and large T, it can be shown that the second term is the dominating one, so

$$P\{\sup_{t \in [0, T]} X(t) \geq x\} \approx T\mathbb{E}N_x(X, 1). \tag{2}$$

From equation (2), we can approximate $P\{\sup_{t \in [0, T]} X(t) \geq x\}$ if we approximate $\mathbb{E}N_x(X, 1)$.

The length of the interval between an up-crossing and the subsequent down-crossing of a process $Y(t)$ is called the *duration of* $Y(t)$. The mean of this random quantity is called in engineering the *average duration of fades* and has special interest in electrical engineering. The average duration of fades can be approximated as follows (Aldous [2]):

$$E\{\text{average of fades}\} = \frac{TP\{Y(0) > y\}}{\mathbb{E}N_y(Y, T)}.$$

Leadbetter and Spaniolo [5] give the rate of up-crossings by a Gaussian process. Hasofer [4] gives the rate of up-crossings by a Rayleigh process. Barnett and Kedem [3] derived the zero up-crossings rate for the product of two stationary Gaussian processes. In general, the up-crossings rate by the product of two independent and stationary Gaussian processes is missing in the literature. In this paper, we will find the up-crossing rate of the product of two stationary, differentiable, independent Gaussian random processes. The following notation will be used in the next sections: $N(0, \sigma^2)$ is a normal distribution with mean 0 and variance σ^2. The symbol ~ means 'distributed as'.

2. Up-crossing Rate of $Y(t)$

Let $X_1(t)$ and $X_2(t)$, $t \in [0, T]$, be two independent, stationary and differentiable Gaussian random processes. Let $\lambda = \operatorname{Var}\{\dot{X}_i(0)\}$ for $i = 1, 2$. Define the processes $Y(t)$, $t \in [0, T]$ as $Y(t) = X_1(t)X_2(t)$. In this section we will be interested in finding $\mathbb{E}N_y(Y, 1)$ for a given level y. If $\dot{Y}(t)$ denotes the derivative of $Y(t)$, then $\dot{Y}(t) = X_1(t)\dot{X}_2(t) + \dot{X}_1(t)X_2(t)$. To find $\mathbb{E}N_y(Y, 1)$, we need to find the joint density function of $Y = Y(0)$ and $\dot{Y} = \dot{Y}(0)$ as well as the density of Y. To do this, let $X_i = X_i(0)$, $\dot{X}_i = \dot{X}_i(0)$, and consider the following transformation:

$$y_1 = x_1 x_2, \quad y_2 = x_1 \dot{x}_2, \quad y_3 = \dot{x}_1 x_2, \quad y_4 = \dot{x}_2.$$

The inverse of this transformation is

$$x_2 = \frac{y_1 y_4}{y_2}, \quad x_1 = \frac{y_2}{y_4}, \quad \dot{x}_1 = \frac{y_2 y_3}{y_1 y_4}, \quad \dot{x}_2 = y_4,$$

and the Jacobian is

$$\begin{vmatrix} 0 & \frac{1}{y_4} & 0 & \frac{-y_2}{y_4^2} \\ \frac{-y_2 y_3}{y_4 y_1^2} & \frac{y_3}{y_1 y_4} & \frac{y_2}{y_1 y_4} & \frac{-y_2 y_3}{y_1 y_4^2} \\ \frac{y_4}{y_2} & \frac{-y_1 y_4}{y_2^2} & 0\,\frac{y_1}{y_2} & \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{|\, y_1 y_4 \,|}.$$

From the theory of random processes X_1 and X_2 are independent $N(0, 1)$, and $\dot{X}_1$ and $\dot{X}_2$ are independent $N(0, \lambda)$, then we have that

$$f(x_1, x_2, \dot{x}_1, \dot{x}_2) = (4\pi^2\lambda)^{-1} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2\lambda}(\dot{x}_1^2 + \dot{x}_2^2)\right\},$$

where $-\infty < x_i, \dot{x}_i < \infty$ for $i = 1, 2$. From the above transformation, the density function of Y_1, Y_2, Y_3 and Y_4 is given by

$$h(y_1, y_2, y_3, y_4)$$

$$= f\left(\frac{y_2}{y_4}, \frac{y_1 y_4}{y_2}, \frac{y_2 y_3}{y_1 y_4}, y_4\right)\frac{1}{|y_1 y_4|}$$

$$= \frac{1}{4\pi^2\lambda|y_1 y_4|}\exp\left\{-\frac{1}{2}\left(\frac{y_2^2}{y_4^2} + \frac{y_1^2 y_4^2}{y_2^2}\right) - \frac{1}{2\lambda}\left(\frac{y_2^2 y_3^2}{y_1^2 y_4^2} + y_4^2\right)\right\}$$

$$= \frac{1}{4\pi^2\lambda|y_1|}\exp\left\{-\frac{1}{2}\left(y_2^2 + \frac{y_2^2 y_3^2}{\lambda y_1^2}\right)\frac{1}{y_4^2} - \frac{1}{2}\left(\frac{y_1^2}{y_2^2} + \frac{1}{\lambda}\right)y_4^2\right\}\frac{1}{|y_4|},$$

where $-\infty < y_i < \infty$ for $i = 1, 2$. So the joint density function of Y_1, Y_2 and Y_3 is given by

$$h(y_1, y_2, y_3)$$

$$= \frac{1}{2\pi^2\lambda|y_1|}\int_0^\infty \frac{1}{y_4}\exp\left\{-\frac{1}{2y_4^2}\left(y_2^2 + \frac{y_2^2 y_3^2}{\lambda y_1^2}\right) - \frac{1}{2}\left(\frac{y_1^2}{y_2^2} + \frac{1}{\lambda}\right)y_4^2\right\}dy_4$$

$$= \frac{1}{2\pi^2\lambda|y_1|}K_0\left(\sqrt{y_2^2 + \frac{y_2^2 y_3^2}{\lambda y_1^2}}\sqrt{\frac{y_1^2}{y_2^2} + \frac{1}{\lambda}}\right)$$

$$= \frac{1}{2\pi^2\lambda|y_1|}K_0\left(\frac{1}{|y_1|}\sqrt{y_1^2 + \frac{y_3^2}{\lambda}}\sqrt{y_1^2 + \frac{y_2^2}{\lambda}}\right),$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero. Consider again the following transformation:

$$v_1 = y_1, \quad v_2 = y_2 \text{ and } v_3 = y_2 + y_3.$$

The inverse of this transformation is

$$y_1 = v_1, \quad y_2 = v_2 \text{ and } y_3 = v_3 - v_2,$$

and therefore the Jacobian is 1. So the joint density of $V_1 = Y_1$, $V_2 = Y_2$ and $V_3 = Y_2 + Y_3$ is

$$h(v_1, v_2, v_3) = \frac{1}{2\pi^2\lambda|v_1|} K_0\left(\frac{1}{|v_1|}\sqrt{v_1^2 + \frac{(v_3 - v_2)^2}{\lambda}}\sqrt{v_1^2 + \frac{v_2^2}{\lambda}}\right).$$

By integrating v_2 out we get that, for $-\infty < v_1, v_3 < \infty$,

$$h(v_1, v_3) = \frac{1}{2\pi^2\lambda|v_1|}\int_{-\infty}^{\infty} K_0\left(\frac{1}{|v_1|}\sqrt{v_1^2 + \frac{(v_3 - v_2)^2}{\lambda}}\sqrt{v_1^2 + \frac{v_2^2}{\lambda}}\right)dv_2,$$

the joint density function of $Y(0)$ and $\dot{Y}(0)$. The density function of $Y = X_1X_2$ is given in Springer [6] as follows:

$$h(v_1) = \frac{K_0(|v_1|)}{\pi}, \quad \text{for } -\infty < v_1 < \infty.$$

We are ready now to find the conditional density of $\dot{Y}$ given $Y = y$. So

$$\begin{aligned} h(v_3 \mid y) &= \frac{h(y, v_3)}{h(y)} \\ &= \frac{1}{2\pi\lambda|y|K_0(|y|)}\int_{-\infty}^{\infty} K_0\left(\frac{1}{|y|}\sqrt{y^2 + \frac{(v_3 - v_2)^2}{\lambda}}\sqrt{y^2 + \frac{v_2^2}{\lambda}}\right)dv_2. \end{aligned}$$

By Rice's formula,

$$\begin{aligned} \mathbb{E}\{N_y(Y, 1)\} &= h(y)\mathbb{E}\{V_3^+ \mid V_1 = y\} \\ &= \frac{1}{2\pi\lambda|y|}\int_0^{\infty} v_3 h(v_3 \mid y)dv_3. \end{aligned}$$

The last equation is not a closed form and has to be obtained using numerical calculations.

3. Example

Consider the following two random processes:

$$X_1(t) = Z_1 \sin(t) + Z_2 \cos(t) \quad \text{and} \quad X_2(t) = Z_3 \sin(t) + Z_4 \cos(t),$$

where $t \in [0, \pi]$ and Z_1, Z_2, Z_3 and Z_4 are i.i.d. $N(0, 1)$. Note that $\lambda = 1$ since $\dot{X}_i \sim N(0, 1)$. Re-scale the two processes as follows:

$$X_1(t) = R_1 \cos(t - \theta_1) \quad \text{and} \quad X_2(t) = R_2 \cos(t - \theta_2),$$

where θ_1 and θ_2 are i.i.d. $U(0, 2\pi)$, and R_1 and R_2 are i.i.d. according to

$$f(r) = \begin{cases} r\exp(-r^2/2), & r \geq 0; \\ 0, & r < 0. \end{cases}$$

Let $Y(t) = X_1(t)X_2(t)$. It follows that

$$\begin{aligned} \sup_{[0,\pi]} Y(t) &= R_1R_2 \sup_{[0,\pi]} \cos(t - \theta_1)\cos(t - \theta_2) \\ &= \frac{1}{2} R_1R_2(1 + \cos(\theta_2 - \theta_1)). \end{aligned}$$

Therefore,

$$\begin{aligned} P\{\sup_{[0,\pi]} Y(t) > y\} &= P\{\sup_{[0,\pi]} X_1(t)X_2(t) > y\} \\ &= P\{R_1R_2(1 + \cos(\theta_2 - \theta_1)) > 2y\}. \qquad (3) \end{aligned}$$

The probability on the right hand side of equation (3) is has to be computed by simulation. Table 1 contains the values of $\mathbb{E}\{N_y(Y, T)\}$ and $P\{\sup_{[0,\pi]} Y(t) > y\}$ for different values of y are very closed. From Table 1, we note that $\mathbb{E}\{N_y(Y, 1)\}$ approximates $P\{\sup_{[0,2\pi]} Y(t) > u\}$ well.

4. Conclusion

In this paper we derive the up-crossing rate of the product of two differentiable, stationary and Gaussian random processes. This rate was not in a closed form but written in terms of integrals which can be computed numerically. The example in Section 3 shows that the rate of up-crossings approximates the tail distribution of $\sup Y(t)$ well for large levels.

Table 1. Values of $\mathbb{E}\{N_y(Y, 1)\}$ and $P\{\sup_{[0, \pi]} Y(t) > y\}$ for different levels

y	$\mathbb{E}\{N_y(Y, 1)\}$	$P\{\sup_{[0, \pi]} Y(t) > y\}$
1.5	0.16651	0.1665
2.0	0.09996	0.1002
2.5	0.06019	0.0602
3.0	0.03629	0.0362
3.5	0.21905	0.0220
4.0	0.01322	0.0133

References

[1] R. Adler, The Geometry of Random Fields, John Wiley & Sons, New York, 1981.

[2] D. Aldous, Probability Approximation via the Poisson Clumping Heuristic, Springer-Verlag, New York, 1989.

[3] J. T. Barnett and B. Kedem, Zero-crossing rates of mixtures and products of Gaussian processes, IEEE Trans. Inform. Theo. 44(4) (1998), 1672-1677.

[4] A. M. Hasofer, On the derivative and the upcrossings of the Rayleigh process, Austral. J. Statist. 12 (1970), 150-151.

[5] M. R. Leadbetter and G. V. Spaniolo, On statistics at level crossings by a stationary process, Statist. Neerlandica 56(2) (2002), 152-164.

[6] M. D. Springer, The Algebra of Random Variables, Wiley, New York, 1979.

■