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USING FEASIBLE DIRECTIONS TO PROVE THE EXISTENCE OF STRONG EFFICIENT UNIT IN DATA ENVELOPMENT ANALYSIS

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Abstract

The existence of at least one strong efficient decision making unit among the set of the supposed decision making units is studied by using the concept of the feasible directions.

1. Introduction

Data envelopment analysis is a simple, yet powerful approach in measuring efficiency of decision making units with multiple inputs and outputs. Data envelopment analysis (DEA) was originated by Charnes, Cooper and Rhodes in 1978, and the first model in DEA was called CCR ([3]). The objective of DEA models is to evaluate the relative efficiency of a set of decision making units (DMUs) involved in a production process. DEA models provide efficiency scores that assess the performance of the different DMUs in terms of either the use of several inputs or the production of certain outputs (or even simultaneously). In this paper, we suppose that there are n decision making units (DMU) which DMU_j $(j = 1, \ldots, n)$ use m inputs x_{ij} $(i = 1, \ldots, m)$ to produce s outputs

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y_{rj} $(r = 1, \ldots, s)$. The following model, which evaluates DMU_o, is known as "CCR fractional model"

$$\text{maximize} \quad \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

$$\text{subject to} \quad \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, \quad j = 1, \ldots, n$$

$$\frac{u_r}{\sum_{i=1}^{m} v_i x_{io}} \geq \varepsilon, \quad r = 1, \ldots, s$$

$$\frac{v_i}{\sum_{i=1}^{m} v_i x_{io}} \geq \varepsilon, \quad i = 1, \ldots, m. \tag{1}$$

$\varepsilon > 0$ is a non-Archimedean element defined to be smaller than any positive real number. This means that is not a real number. In fact, infinitesimal non-Archimedean element ε in this model makes u_r and v_i positive, see Charnes et al. [3] for further discussions. With following Charnes-Cooper's transformations:

$$\mu_r = t u_r, \quad r = 1, \ldots, s,$$

$$\nu_i = t v_i, \quad i = 1, \ldots, m, \quad t = \frac{1}{\sum_{i=1}^{m} v_i x_{io}}$$

problem (1) is converted to the following problem which is easier to work:

$$\text{maximize} \quad \sum_{r=1}^{s} \mu_r y_{ro}$$

$$\text{subject to} \quad -\sum_{i=1}^{m} \nu_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} \leq 0, \quad j = 1, \ldots, n$$

$$\sum_{i=1}^{m} \nu_i x_{io} = 1$$

$$-\nu_i \leq -\varepsilon, \quad i = 1, \dots, m$$

$$-\mu_r \leq -\varepsilon, \quad r = 1, \dots, s. \tag{2}$$

Let $*$ show the optimal value. According to Charnes et al. [3], the optimal value of the objective function of model (2) is defined as efficiency score of DMU_o, see also Cooper et al. [2]. Therefore, the condition of efficiency for DMU_o by problem (2) is stated as follows:

$$\sum_{r=1}^{s} \mu_r^* y_{ro} = 1. \tag{3}$$

The dual of the above problem will be as follows:

$$\begin{aligned} \text{minimize} \quad & \theta - \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\ \text{subject to} \quad & 0 = \theta x_{io} - \sum_{j=1}^{n} x_{ij}\lambda_j - s_i^-, \quad i = 1, \dots, m \\ & y_{ro} = \sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+, \quad r = 1, \dots, s \end{aligned} \tag{4}$$

in which s_i^-, s_r^+ and λ_j are non-negative.

Suppose that $X = \{x : Gx \leq g, \, x \geq 0\}$ is feasible region of a linear program in which $G \in R^{m*n}$ and $g \in R^m$. Therefore, we will have the following definition.

Definition. Let $x_o \in X$. $d \neq \mathbf{0}$ is a *feasible direction* for X at point x_o if there exists $\delta > 0$, so that for any $0 < \varepsilon < \delta$, $x_o + \varepsilon d \in X$.

2. The Existence of Strong Efficient Unit

In what follows, we prove directly that at least one of the decision making units is efficient.

Theorem 1. *At least one of the constraints in the problem* (1) *will be as equality for optimal solution.*

Proof. We consider DMU_o and evaluate it with fractional CCR model

$$\begin{aligned}
&\text{maximize} && \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\
&\text{subject to} && \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, \ldots, n \\
& && u_r, v_i \ge 0. \qquad (5)
\end{aligned}$$

Using Charnes-Cooper's transformations the above problem is changed to the following linear programming problem:

$$\begin{aligned}
&\text{maximize} && \sum_{r=1}^{s} \mu_r y_{ro} \\
&\text{subject to} && -\sum_{i=1}^{m} \nu_i x_{ij} + \sum_{r=1}^{s} \mu_r y_{rj} \le 0, \quad j = 1, \ldots, n \\
& && \sum_{i=1}^{m} \nu_i x_{io} = 1 \\
& && \mu_r, \nu_i \ge 0, \quad r = 1, \ldots, s, \quad i = 1, \ldots, m. \qquad (6)
\end{aligned}$$

Now, if in optimal solution none of the n constraints of the fractional problem obtains its high bound (means one), therefore, for $x^* = (\nu_1^*, \ldots, \nu_m^*, \mu_1^*, \ldots, \mu_s^*)$ the corresponding optimal solution in its equivalent linear programming problem, the first n inequalities will be satisfied strictly. If we call the matrix of corresponding coefficients for these n constraints A_1 and the matrix of the coefficients for the last constraint A_2, means $A_2 = [x_{io}, \ldots, x_{mo}, o, \ldots, 0]$, therefore, it is clear that $A_2 d = \mathbf{0}$ will have non-zero solution. Because it is enough that the first m components for d are selected zero and the last s components are selected non-zero (and desired). We make a feasible direction in point x^* and we proceed in its length until obtaining a new point for which the amount of the objective function is better than the optimal amount and then we will reach to contradiction.

Since $F(x) = A_1 x < 0$ is continuous, so there exists $\delta > 0$ so that $F(x^* + \delta \overline{d}) = A_1(x^* + \delta \overline{d}) < 0$ in which $\overline{d}$ is a non-zero solution of $A_2 d = \mathbf{0}$. If $\mathbf{0} = \overline{d} = (0, ..., 0, y_{1o}, ..., y_{so})^t$, then it is obvious that $x^* + \delta \overline{d}$ satisfies in the last constraint of the linear programming problem because $A_2(x^* + \delta \overline{d}) = A_2 x^* + A_2 \overline{d} = A_2 x^* + \mathbf{0} = 1$, so $\overline{d}$ is a feasible direction for the solution space of the above linear programming problem. But the amount of the objective function for $(x^* + \delta \overline{d})$ is better than the optimal amount of the objective function. In fact,

$$C(x^* + \delta \overline{d}) = (0, ..., 0, y_{1o}, ..., y_{so})(\nu_1^*, ..., \nu_m^*, \mu_1^* + \delta y_{1o}, ..., \mu_s + \delta y_{so})^t$$

$$= \sum_{r=1}^{s} \mu_r^* y_{ro} + \delta \sum_{r=1}^{s} y_{ro}^2 > \sum_{r=1}^{s} \mu_r^* y_{ro}$$

in which $C = (0, ..., 0, y_{1o}, ..., y_{so})$ is coefficients vector of the objective function of (6). So it is in contrast with $(\nu_1^*, ..., \nu_m^*, \mu_1^*, ..., \mu_s^*)$ which is optimal solution of (6). This contradiction was created because we supposed that none of the n constraints of CCR fractional model obtains its own upper bound (the amount of unity). Therefore, the proof is complete.

Theorem 2. *At least one of the decision making units is strong efficient.*

Proof. It is clear from the previous theorem that in evaluating the arbitrary DMU $(j = o)$ at least one of the fractional constraints will be as equality for optimal solution. It means that if $(v_1^*, ..., v_m^*, u_1^*, ..., u_s^*)$ is the optimal solution for that k-th constraint occurs as equality, since in evaluating all of DMUs with CCR fractional model the space of solution are always the same, so in evaluating k-th DMU the amount of the objective function will be unity for $(v_1^*, ..., v_m^*, u_1^*, ..., u_s^*)$. If $(\nu_1^*, ..., \nu_m^*, \mu_1^*, ..., \mu_s^*)$ is the corresponding solution in its equivalent linear programming problem, then $\sum_{r=1}^{s} \mu_r^* y_{ro} = 1$. Therefore, according to (3) DMU_k is efficient, so the proof is complete.

3. Conclusion

Although, the discussion of the existence of at least an efficient DMU among a set of supposed DMUs is obvious, intuitively. However, we provided an alternative proof based on feasible directions. We believe that the discussions such as those of the current paper are fruitful to improve the mathematics of data envelopment analysis.

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Appendix: The continuity of $F(x) = A_1 x$

Using a norm which is consistent with matrix norm we must prove that:

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\text{ such that } \| x - \bar{x} \| < \delta \Rightarrow \| F(x) - F(\bar{x}) \| < \varepsilon$$

for given ε, we will define a suitable δ hereunder,

$$\| F(x) - F(\bar{x}) \| = \| A_1 x - A_1 \bar{x} \| = \| A_1 (x - \bar{x}) \| \leq \| A_1 \| \| x - \bar{x} \| < \varepsilon$$

means it must:

$$\| x - \bar{x} \| < \frac{\varepsilon}{\| A_1 \|}, \quad \| A_1 \| \neq 0$$

so it is enough to define $\delta = \dfrac{\varepsilon}{\| A_1 \|}$. Therefore, $F(x) = A_1 x$ is continuous.

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