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ON THE INTERSECTION OF MAXIMAL (MINIMAL PRIME) IDEALS CONTAINING AN IDEMPOTENT

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Abstract

Elementwise characterization of the intersection of maximal (minimal prime) ideals containing an idempotent (regular element) is given. Using these facts, we generalize several important results such as Nakayama's Lemma and Krull's intersection Theorem.

1. Introduction

Throughout this paper, R is a commutative ring with identity. We let $e \in R$ be an idempotent element and as the notations in (1) and (2) we suppose that M_e is the intersection of all maximal ideals containing e and P_e is the intersection of all minimal prime ideals containing e. It is clear that $M_0 = J(R)$, the Jacobson radical of R and $P_0 = rad(R)$, the prime radical of R. Furthermore, $M_e = J\left(\frac{R}{(e)}\right)$ and $P_e = rad\left(\frac{R}{(e)}\right)$. Whenever $a \in R$ is a regular element, i.e., if there exists some $c \in R$ such that $a = a^2c$, then $e = ac$ is an idempotent element. In this case clearly $M_e = M_a$ and $P_e = P_a$. Thus any argument concerning the

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intersection of maximal (minimal prime) ideals containing a regular element may be restricted to those of containing an idempotent element.

We denote by $C(X)$, the ring of all continuous real valued functions on a completely regular Hausdorff space X. In $C(X)$, M_f and P_f for any arbitrary $f \in C(X)$ are characterized in [1] and [2].

2. Elementwise Characterizations

In this section we will give elementwise characterizations of M_e and P_e. Some applications are also given in this section.

Proposition 2.1. *Let e be an idempotent element and $M_e(P_e)$ be the intersection of all maximal (minimal prime) ideals containing e in R. Then*

(a) $M_e = \{b \in R : 1 - r(1-e)b$ *is unit for every* $r \in R\}$.

(b) $P_e = \{b \in R : (1-e)b^n = 0$ *for some* $n \in \mathbb{N}\}$.

Proof. (a) Let $1 - r(1-e)b$ be unit for every $r \in R$, but $b \notin M_e$. Then there exists a maximal ideal M such that $e \in M$ and $b \notin M$. Hence $M + (b) = R$ and so there is $r_0 \in R$ for which $1 = m + r_0 b$ for some $m \in M$. But $1 - r_0 b + r_0 eb = m + r_0 eb$ implies that $m + r_0 eb \in M$ and hence $1 - r_0(1-e)b \in M$, which is a contradiction. Conversely suppose that $b \in M_e$ and there exists $r_0 \in R$ such that $1 - r_0(1-e)b$ is a nonunit. Hence, there exists a maximal ideal M such that $1 - r_0(1-e)b \in M$. Now $e^2 = e$ implies that $e(1-e) = 0 \in M$. Thus either $e \in M$ or $1 - e \in M$. If $1 - e \in M$, then $1 \in M$, a contradiction and if $e \in M$, then $b \in M$ and therefore $r_0(1-e)b \in M$. This implies that $1 \in M$, a contradiction.

(b) Let P be a minimal prime ideal containing e, $b \in R$ and there exists $n \in \mathbb{N}$ such that $(1-e)b^n = 0$. But $1 - e \notin P$, then $b^n \in P$. Now $b \in P$ implies that $b \in P_e$. On the other hand, suppose that $b \in P_e$ and

$(1-e)b^n \neq 0$ for all $n \in \mathbb{N}$, then $S = \{(1-e)b^n : n \in \mathbb{N}\}$ is a multiplicatively closed set in R and $0 \notin S$. Hence there exists a minimal prime ideal P such that $P \cap S = \varnothing$. Now if $e \in P$, then $b \in P$ and hence $(1-e)b \in P$. But $(1-e)b \in S$, a contradiction. If $e \notin P$, then $1-e \in P$ implies that $(1-e)b \in P$, a contradiction, for $(1-e)b \in S$.

Corollary 2.2. (a) $J(R) = \{b \in R : 1 - rb$ *is unit for all* $r \in R\}$.

(b) $rad(R) = \{b \in R : b^n = 0$ *for some* $n \in \mathbb{N}\}$.

Proof. Take $e = 0$ in Proposition 2.1.

In any Artinian ring R, it is well known that $M_0 = J(R)$ is a nilpotent ideal, see Theorem 41.8 in [4]. We generalize this fact for M_e.

Corollary 2.3. *In any Artinian ring* R, $M_e^n = (e)$ *for some* $n \in \mathbb{N}$.

Proof. Evident.

Proposition 2.4. *Let* I *be an ideal of the ring* R. *Then* $I \subseteq M_e$ *if and only if each element of the coset* $1 + (1-e)I$ *is unit*.

Proof. We begin by assuming that $I \subseteq M_e$ and that there is some element $x \in I$ for which $1 + (1-e)x$ is nonunit and get a contradiction. Therefore, the element $1 + (1-e)x$ must belong to some maximal ideal M. By Proposition 2.1, $1 - r(1-e)x$ is unit for any $r \in R$. Letting $r = -1$, the element $1 + (1-e)x$ is unit, which is impossible by our assumption. Conversely, suppose that $I \not\subseteq M_e$, then there exists a maximal ideal M such that $e \in M$ but $I \not\subseteq M$. Taking $x \in I - M$, we have $M + (x) = R$ for M is maximal. Thus, $1 = m + rx$ for some $m \in M$ and $r \in R$. But $1 - rx + rex = m + rex \in M$, i.e., $1 + (1-e)(-rx) \in M$ and we have $1 + (1-e)(-rx) \in 1 + (1-e)I$ for $x \in I$. Now $1 + (1-e)(-rx)$ is unit, by our hypothesis a contradiction.

Corollary 2.5. *If* $x \in M_e$ *is an idempotent element of the ring* R, *then* $x = ex$.

Proof. Since $x \in M_e$, by Proposition 2.1, the element $1-(1-e)x$ is unit. Thus there exists $s \in R$ such that $(1-x+ex)s = 1$ and multiplying both sides by e we have $es - esx + e^2 sx = e$. This implies that $es = e$. Now we have $x = x.1 = x(1-x+ex)s = ex^2 s = esx = ex$.

Corollary 2.6. *If I is a nilideal of R, then* $(1-e)I \subseteq M_e$.

Proof. Suppose that $x \in (1-e)I$, then there is $y \in I$ for which $x = (1-e)y$. Hence $1 - r(1-e)x = 1 - r(1-e)y$ for every $r \in R$. But $r(1-e)y \in I$ implies that $(r(1-e)y)^n = 0$ for some $n \in \mathbb{N}$. Therefore, $1 - r(1-e)y$ is unit for every $r \in R$. Now by Proposition 2.1, we have $y \in M_e$ and hence $x \in M_e$.

Proposition 2.7. *If x and y are two idempotent elements of the ring R such that $x - y \in P_e$, then $x - y = e(x - y)$.*

Proof. By the formula $((x-y)-e(x-y))(1-(x+y)+e(x+y)) = 0$, it is enough to show that $1-(x+y)+e(x+y)$ is a unit element of R. Now one may write $1-(x+y)+e(x+y)$ in the form $(1-2x+2ex)+(x-y)(1-e)$. Since $x-y \in P_e$, by Proposition 2.1, $(x-y)(1-e)$ is a nilpotent element. On the other hand, $1-2x+2ex$ is unit, for

$$(1-2x+2ex)^2 = 1-2x+2ex-2x+4x^2-4ex^2+2ex-4ex^2+4e^2x^2 = 1.$$

Now $1-(x+y)+e(x+y)$ is the sum of a nilpotent element and a unit element, which will be necessarily a unit element in R.

3. Generalizations

In this section using Proposition 2.1, we generalize several important results, such as Nakayama's Lemma and Krull's intersection Theorem. First we need the following lemma.

Lemma 3.1. *Let I and J be two ideals of the ring R and I be finitely generated. If $IJ = I$, then there exists an element $r \in J$ such that* $(1-r)I = (0)$.

Proof. See Lemma in [3, p. 242].

The proof of the following corollary is similar to that of Lemma in [3, p. 242].

Corollary 3.2. *Let K be a finitely generated R-module and I be an ideal of R. If $K = KI$, then there exists an element $r \in I$ such that* $(1 - r)K = (0)$.

Proposition 3.3 (Nakayama's Lemma, generalized). *Let K be a finitely generated R-module, I be an ideal of R and $I \subseteq M_e$. If $K = KI$, then there exists an element $r \in M_e$ such that $K = reK$.*

Proof. By Corollary 3.2, there exists some $r \in I$ such that $(1 - r)K = (0)$. Thus $(1 - r)K + reK = reK$ and hence $(1 - r + re)K = reK$. But $I \subseteq M_e$ implies that $r \in M_e$. By Proposition 2.1, the element $1 - r(1 - e)t$ is unit for every $t \in R$. Let $t = 1$. Therefore, there exists $s \in R$ such that $(1 - r + re)s = 1$. Thus $(1 - r + re)sK = resK$ and consequently $K = resK$. But $es = e$ implies that $K = reK$.

Lemma 3.4 (Generalized Krull's intersection Theorem). *Let I be an ideal of the Noetherian ring R. If $I \subseteq M_e$, then there exists $r \in M_e$ such that $\bigcap_{n=1}^{\infty} I^n = re(\bigcap_{n=1}^{\infty} I^n)$.*

Proof. Let $A = \bigcap_{n=1}^{\infty} I^n$. Then we have $A = IA$. By Proposition 3.3, there exists $r \in M_e$ for which, $A = reA$, i.e., $\bigcap_{n=1}^{\infty} I^n = re(\bigcap_{n=1}^{\infty} I^n)$.

Corollary 3.5. *Let K be a finitely generated R-module, N be an R-submodule of K and $I \subseteq M_e$. If $N + IK = K$, then there exists $r \in M_e$ such that $K = N + reK$.*

Proof. We have $I\left(\dfrac{K}{N}\right) = \left(\dfrac{N + IK}{N}\right) = \left(\dfrac{K}{N}\right)$. By Proposition 3.3, there exists $r \in M_e$ such that $\left(\dfrac{K}{N}\right) = re\left(\dfrac{K}{N}\right)$. This implies that $K = N + reK$.

Let I be an ideal in R and $e \in R$ be an idempotent element. Then clearly $e + I$ is an idempotent element of the residue class ring $\frac{R}{I}$. We assume that $\mathcal{M}_{e+I}(\mathcal{P}_{e+I})$ is the intersection of all maximal (minimal prime) ideals containing $e + I$ in $\frac{R}{I}$. Then $\mathcal{M}_{e+I}$ and $(\mathcal{P}_{e+I})$ may be represented in terms of M_e and P_e.

Corollary 3.6. *Let I be an ideal and e be an idempotent element in R.*

(a) $\mathcal{M}_{e+I} \supseteq \frac{M_e + I}{I}$.

(b) *If* $I \subseteq M_e$, *then* $\mathcal{M}_{e+I} = \frac{M_e + I}{I}$.

(c) $\mathcal{P}_{e+I} \supseteq \frac{P_e + I}{I}$.

(d) *If* $I \subseteq P_e$, *then* $\mathcal{P}_{e+I} = \frac{P_e + I}{I}$.

Proof. Evident.

Proposition 3.7. $\bigcup P = \{r \in R : rb \in P_e$ *for some* $b \notin P_e\}$, *where* P *runs through the set of minimal prime ideals containing e.*

Proof. Suppose that $r \in \bigcup P$. Hence $r \in P$ for some minimal prime ideal P containing e. Hence $r^n b = 0$ for some $b \notin P$ and $n \in \mathbb{N}$, consequently $b \notin P_e$ but $rb \in P_e$. On the other hand, let $r \in R$, $rb \in P_e$ and $b \notin P_e$. Then there exists a minimal prime ideal P containing e such that $b \notin P$. Also by Proposition 2.1, $(1 - e)(rb)^n = 0$ for some $n \in \mathbb{N}$. Thus $(1 - e)(rb)^n \in P$. But neither $1 - e \notin P$ nor $b^n \notin P$. This implies that $r^n \in P$ so $r \in P$ and hence $r \in \bigcup P$.

We conclude the article by a result concerning the ring of power series. For details of the ring, see [3] and [4].

Proposition 3.8. *Let e be an idempotent element in R and* $\mathcal{M}_e$ *be the intersection of all maximal ideals containing e in* $R[[x]]$. *Then* $\mathcal{M}_e = (M_e, x)$.

Proof. Suppose that $f \in (M_e, x)$ and $\mathcal{M}$ is a maximal ideal in $R[[x]]$ containing e. Hence there exists a maximal ideal M in R containing e such that $\mathcal{M} = (M, x)$. But $f(x) = b + xg(x)$ for some $b \in M_e$ and $g \in R[[x]]$. Since $b \in M$ and $xg(x) \in (x)$, $f(x) \in (M, x)$. Thus $f \in \mathcal{M}$ and so $f \in \mathcal{M}_e$. On the other hand, let $f \in \mathcal{M}_e$ and M be a maximal ideal in R containing e. Then (M, x) is a maximal ideal in $R[[x]]$ containing e. Therefore, $f \in (M, x)$ and hence $f(x) = b + xg(x)$ for some $b \in M$ and $g \in R[[x]]$. Take $x = 0$, then $f(0) = b \in M$ implies that $f(0) \in M_e$ and finally we have $f(x) \in (M_e, x)$.

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