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NONLINEAR IMPULSIVE PERIODIC EVOLUTION EQUATIONS

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Abstract

In this paper, we consider the question of the existence of periodic solutions of nonlinear impulsive differential equations monitored by the strongly nonlinear evolution equations $\dot{x}(t) + A(t, x(t)) = g(t, x(t))$, $0 < t < T$. Here, $V \hookrightarrow H \hookrightarrow V^*$ is an evolution triple, $A : I \times V \to V^*$ is a uniformly monotone operator and $g : I \times H \to V^*$ is a Caratheodary mapping.

1. Introduction

In recent years, impulsive periodic systems have attracted much attention since many evolution processes are subject to short term impulsive perturbations. In this paper, we consider the following periodic boundary value problem of an impulsive differential equation

$$\dot{x}(t) + A(t, x(t)) = g(t, x(t)),\ t \neq t_i,\ 0 < t < T, \tag{1a}$$

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$$x(t_i^+) - x(t_i) = F_i(x(t_i)),\ i = 1, 2, \ldots, n, \tag{1b}$$

$$x(0) = x(T), \tag{1c}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n < T$, A is a nonlinear monotone operator, g is a nonlinear nonmonotone perturbation in a Banach space, $\Delta x(t_i) = x(t_i^+) - x(t_i^-) \equiv x(t_i^+) - x(t_i)$, $i = 1, 2, \ldots, n$ and F_i's are some operators. The impulsive condition (1b) represents the jump in the state x at time t_i; with F_i determining the size of the jump at time t_i (for the definitions of the operators A, g and F_i will be given in Section 2). Some interesting examples of impulsive periodic systems can be found in the modelling of nanoelectronic devices (see for instance [6, p. 307]).

Impulsive evolution equations with an unbounded linear operator A of the form

$$\dot{x}(t) + A(t) = g(t, x(t)),\ t > 0,\ t \neq t_i,$$

$$\Delta x(t_i) = F_i(x(t_i)),\ i = 1, 2, \ldots, n,$$

$$x(0) = x(T),$$

have been considered by Hinpang [3]. The questions of the existence and regularity of solutions have been discussed. However, these questions still open when the operator A is nonlinear.

The purpose of this paper is to study the existence of periodic solutions of the strongly nonlinear impulsive evolution equations on $(0, T)$ and we will apply these results to impulsive control of periodic systems.

2. System Descriptions

The mathematical setting of our problem is the following. Let H be a real separable Hilbert space, V be a dense subspace of H having the structure of a reflexive Banach space, with the continuous embedding $V \hookrightarrow H \hookrightarrow V^*$, where V^* is the topological dual space of V. The system model considered here is based on this evolution triple. Let the embedding $V \hookrightarrow H$ be compact.

Let $\langle x, y\rangle$ denote the pairing of an element $x \in V^*$ and an element $y \in V$. If $x, y \in H$, then $\langle x, y\rangle = (x, y)$, where (x, y) is the scalar product on H. The norm in any Banach space X will be denoted by $\|\cdot\|_X$.

Let $I \equiv (0, T)$ be a finite subinterval of the real line and $\bar{I} \equiv [0, T]$. Let $p, q \geq 1$ be such that $1/p + 1/q = 1$ where $2 \leq p < +\infty$. For p and q satisfying the preceding conditions, it follows from reflexivity of V that both $L_p(I, V)$ and $L_q(I, V^*)$ are reflexive Banach spaces and the pairing between $L_p(I, V)$ and $L_q(I, V^*)$ is denoted by $\langle\langle\ ,\ \rangle\rangle$.

Define

$$W_{pq}(I) = W_{pq}(0, T) = \{x : x \in L_p(I, V),\ \dot{x} \in L_q(I, V^*)\}$$

and

$$\| x \|_{W_{pq}(I)} = \| x \|_{L_p(L,\, V)} + \| \dot{x} \|_{L_q(L,\, V^*)},$$

where $\dot{x}$ denotes the derivative of x in the generalized sense. Furnished with the norm $\|\cdot\|_{W_{pq}(I)}$, the space $(W_{pq}(I), \|\cdot\|_{W_{pq}(I)})$ becomes a Banach space which is clearly reflexive and separable. Moreover, the embedding $W_{pq}(I) \hookrightarrow C(\bar{I}, H)$ is continuous. Let us assume further that there is an embedding constant $0 < \eta < 1$ such that $\| x \|_{C[\bar{I},\, H]} \leq \eta \| x \|_{W_{pq}(I)}$. If the embedding $V \hookrightarrow H$ is compact, the embedding $W_{pq}(I) \hookrightarrow L_p(I, H)$ is also compact (see [7], Problem 23.13 (b)).

For a partition $0 < t_1 < t_2 < \cdots < t_n < T$ on $(0, T)$, we define the *set* $PW_{pq}(0, T) = \{x \in W_{pq}(t_i, t_{i+1}),\ i = 1, 2, \ldots, n,$ where $t_0 = 0,\ t_{n+1} = T\}$. Moreover, for each $x \in PW_{pq}(0, T)$, we define $\| x \|_{PW_{pq}(0,\, T)} = \sum_{i=1}^{n} \| x \|_{W_{pq}(t_i, t_{i+1})}$. As a result, the space $(PW_{pq}(0, T), \|\cdot\|_{PW_{pq}(0,\, T)})$ becomes a Banach space. Let $PC([0, T], H) = \{x : x$ is a map from $[0, T]$ into H such that x is continuous at every point $t \neq t_i$, left continuous at $t = t_i$, and possesses the right-hand limit $x(t_i^+)$ for $i = 1, 2, \ldots, n\}$. Equipped with the supremum norm topology, it is a Banach space.

Definition 1. By a (classical) solution x of problem (1), we mean a function $x \in PW_{pq}(0, T) \cap PC([0, T], H)$ such that $x(0) = x(T)$ and $\Delta x(t_i) = F_i(x(t_i))$ for $i = 1, 2, \ldots, n$ which satisfies

$$\langle \dot{x}(t), v \rangle + \langle A(t, x), v \rangle = \langle g(t, x), v \rangle$$

for all $v \in V$ and μ-a.e. on I, where μ is the Lebesgue measure on I.

We need the following hypothesis on the data of problem (1).

(A) $A : I \times V \to V^*$ is an operator such that

(1) $t \mapsto A(t, x)$ is weakly measurable, i.e., the function $t \mapsto \langle A(t, x), v \rangle$ is μ-measurable on I for all $x, v \in V$.

(2) For each $t \in I$, the operator $A(t) : V \to V^*$ is uniformly monotone and hemicontinuous, that is, there is a constant $c_1 \geq 0$ such that

$$\langle A(t, x_1) - A(t, x_2), x_1 - x_2 \rangle \geq c_1 \| x_1 - x_2 \|_V^p$$

for all $x_1, x_2 \in V$, and the map $s \mapsto \langle A(t, x + sz), y \rangle$ is continuous on [0, 1] for all $x, y, z \in V$.

(3) Growth condition: There exists a constant $c_2 > 0$ and a nonnegative function $a_1(\cdot) \in L_q(I)$ such that

$$\| A(t, x) \|_{V^*} \leq c_2 \| x \|_V^{p-1}$$

for all $x \in V$, $t \in I$.

(4) Coerciveness: There exists a constant $c_3 > 0$ such that

$$\langle A(t, x), x \rangle \geq c_3 \| x \|_V^p$$

for all $x \in V$, $t \in I$.

Without loss of generality, we can assume that $A(t, 0) = 0$ for all $t \in \bar{I}$.

(G) $g : I \times H \to V^*$ is an operator such that

(1) $t \mapsto g(t, x)$ is weakly measurable.

(2) $g(t, x)$ is Hölder continuous with respect to x with exponent $0 < \alpha \leq 1$ in H and uniformly in t. That is, there is a constant L such that

$$\| g(t, x_1) - g(t, x_2) \|_{V^*} \leq L \| x_1 - x_2 \|_H^{\alpha}$$

for all $x_1, x_2 \in H$ and $t \in I$. This assumption implies that the map $x \mapsto g(t, x)$ is continuous.

(3) There exists a constant $c_4 > 0$ such that

$$\| g(t, x) \|_{V^*} < c_4 \| x \|_H^{k-1}$$

for all $x \in V$, $t \in I$, where $1 \leq k < p$ is constant.

(F) $F_i : H \to H$ is locally Lipschitz continuous on H, i.e., for any $\rho > 0$, there exists a constant $L_i(\rho)$ such that

$$\| F_i(x_1) - F_i(x_2) \|_H \leq L_i(\rho) \| x_1 - x_2 \|_H$$

for all $\| x_1 \|_H, \| x_2 \|_H < \rho$ $(i = 1, 2, \ldots, n)$.

It is sometimes convenient to rewrite system (1) into an operator equation. To do this, we set $X = L_p(I, V)$ and hence $X^* = L_q(I, V^*)$. Moreover, we set

$$A(x)(t) = A(t, x(t)),$$

$$G(x)(t) = g(t, x(t)), \tag{2}$$

for all $x \in X$ and for all $t \in (0, T)$. Then the original problem (1) is equivalent to the following operator equation (see [7, Theorem 30.A]):

$$\dot{x} + Ax = G(x),$$
$$x(0) = x(T),$$
$$\Delta x(t_i) = F_i(x(t_i)), \tag{3}$$

for $i = 1, 2, ..., n$ and $0 < t_1 < t_2 < \cdots < t_n < T$.

Remark. It follows from Theorem 30.A of Zeidler [7] that equation (3) defines an *operator* $A : X \to X^*$ such that A is uniformly monotone, hemicontinuous, coercive, bounded and satisfied

$$\| Ax \|_{X^*} \leq \gamma \| x \|_X^{p-1}$$

for some constants $\gamma > 0$ and for all $x \in X$. Moreover, by using hypothesis (G)(3) and using the same technique as in Theorem 30.A, one can show that the operator $G : L_p(I, H) \to X^*$ is also bounded and satisfies

$$\| G(u) \|_{X^*} \leq \delta \| u \|_{L_p(I,H)}^{k-1}$$

for some constants $\delta > 0$ and for all $u \in L_p(I, H)$.

3. Existence of a Periodic Solution

In order to get a periodic solution of equation (1) in the space $PW_{pq}(I)$, we firstly consider the following Cauchy problem:

$$\dot{x}(t) + A(t, x(t)) = g(t, x(t)),$$
$$x(0) = x_0 \in H,$$
$$\Delta x(t_i) = F_i(x(t_i)), \tag{4}$$

where $i = 1, 2, ..., n$ and $0 < t_1 < t_2 < \cdots < t_n < T$. By a solution of system (4), we mean a *function* $x(t)$ as defined in Definition 1 except that $x(t)$ must satisfy the initial condition $x(0) = x_0$.

Lemma 1. *Under assumptions* (A), (F) *and* (G), *system* (4) *has a unique solution* $x \in PW_{pq}(0, T) \cap PC([0, T], H)$ *and the solution depends continuously on the initial condition.*

Proof. The existence of the solution of system (4) follows from [5] Theorem B. We can use hypothesis G(2) in proving the solution's uniqueness. To see this, suppose that system (4) has two solutions $x_1, x_2 \in PW_{pq}(0, T) \cap PC([0, T], H)$. Then it follows from the integration by parts formula and monotonicity of $A(t, x)$ that

$$\| x_1(t) - x_2(t) \|_H^2 - \| x_1(0) - x_2(0) \|_H^2$$

$$= 2\int_0^t \langle \dot{x}_1(s) - \dot{x}_2(s),\ x_1(s) - x_2(s) \rangle_{V^*-V} ds$$

$$= 2\int_0^t \langle \dot{x}_1(s) - \dot{x}_2(s),\ x_1(s) - x_2(s) \rangle_{V^*-V} ds$$

$$+ 2\int_0^t \langle g(s,\ x_1(s)) - g(s,\ x_2(s)),\ x_1(s) - x_2(s) \rangle_{V^*-V} ds$$

$$\leq 2\int_0^t \langle g(s,\ x_1(s)) - g(s,\ x_2(s)),\ x_1(s) - x_2(s) \rangle_{V^*-V} ds$$

$$\leq 2\int_0^t \| g(s,\ x_1(s)) - g(s,\ x_2(s)) \|_{V^*} \| x_1(s) - x_2(s) \|_V ds$$

$$\leq 2L\int_0^t \| x_1(s) - x_2(s) \|_H \| x_1(s) - x_2(s) \|_V ds$$

$$\leq 2Lc\int_0^t \| x_1(s) - x_2(s) \|_H^2 ds,$$

for some positive constant c. By Gronwall's lemma, we get

$$\| x_1(t) - x_2(t) \|_H^2 \leq \| x_1(0) - x_2(0) \|_H^2 e^{2Lc(t-0)}. \tag{5}$$

Note that we can derive the uniqueness result for system (4) by simply setting $x_1(0) = x_2(0)$ and using equation (5). Furthermore, equation (5) also implies that the solution of system (4) depends continuously on the initial condition. This proves Lemma 1.

Definition 2. For each $y \in H$, we write $x(t;\, 0,\, y)$ to mean the solution of equation (4) corresponding to the initial condition $x(0) = y$.

Corollary to Lemma 1. *The map* $y \to x(t;\, 0,\, y)$ *is well defined and is continuous from H into H.*

Proof. The proof follows immediately from Theorem 1.

Lemma 2. *Assume that* $p = 2$ *and* $k = 1$. *Let* $r > 0$ *and* $x_0 \in H$. *If* $\| x_0 \| \le r$, *then* $\| x \|_{C([\bar{I}, H])} \le r$, *where x is the solution of* (4) *corresponding to the initial condition* x_0.

Proof. Let x be the solution of (4) corresponding to the initial condition $x(0) = x_0$. Then $x \in W_{pq}(I)$. Let $X = L_p(I, V)$ and $X^* = L_q(I, V^*)$, it follows from equation (3) that

$$\langle\langle \dot{x},\, x \rangle\rangle + \langle\langle A(x),\, x \rangle\rangle = \langle\langle G(x),\, x \rangle\rangle.$$

Since A is coercive (Hypothesis (A)),

$$c_3 \| x \|_X^p \le \langle\langle G(x),\, x \rangle\rangle - \langle\langle \dot{x},\, x \rangle\rangle.$$

Using integration by part, Hölder's inequality and Hypothesis (G); we get

$$c_3 \| x \|_X^p$$

$$\le \langle\langle G(x),\, x \rangle\rangle - \frac{1}{2} [\| x(T) \|_H^2 - \| x(0) \|_H^2]$$

$$\le \left(\int_0^T \| g(t,\, x) \|_{V^*}^q \, dt \right)^{1/q} \left(\int_0^T \| x(t) \|_V^p \right)^{1/p}$$

$$- \frac{1}{2} [\| x(T) \|_H^2 - \| x(0) \|_H^2]$$

$$\le c_4 \left(\int_0^T (\| x(t) \|_H)^q \, dt \right)^{1/q} (\| x \|_X) + \frac{1}{2} \| x(0) \|_H^2$$

$$\leq c_4 \| x \|_{L_q(I, H)} \| x \|_X + \frac{1}{2} \| x(0) \|_H^2$$

$$\leq c_5 \| x \|_{L_p(I, V)} \| x \|_X + \frac{1}{2} \| x(0) \|_H^2 \ (\because L_p(I, V) \hookrightarrow L_q(I, H) \ [7, \text{ p. } 407])$$

$$\leq c_5 \| x \|_X^2 + \frac{1}{2} \| x(0) \|_H^2,$$

for some constants $c_5 > 0$. We finally get $c_3 \| x \|_X^p \leq c_5 \| x \|_X^2 + \frac{1}{2} \| x(0) \|$. Substituting $p = 2$, we get

$$(c_3 - c_5) \| x \|_X^2 \leq \frac{1}{2} \| x(0) \|_H^2.$$

By choosing some suitable constants c_3 and c_5, we can assume that $c_3 - c_5 \geq 1$.

Hence

$$\| x \|_X^2 \leq \frac{1}{(c_3 - c_5)} \| x(0) \|_H^2 \leq \frac{1}{2} \| x(0) \|_H^2. \tag{6}$$

It follows from equation (6) that if $\| x(0) \| \leq r$, then $\| x \|_X \leq (1/\sqrt{2}) r$ for each fixed $r > 0$. Next, we shall estimate $\| \dot{x} \|$. Let $\phi \in X$. Then it follows from equation (3) that

$$\langle\langle \dot{x}, \phi \rangle\rangle + \langle\langle A(x), \phi \rangle\rangle = \langle\langle G(x), \phi \rangle\rangle.$$

Applying Hölder's inequality, we get

$$| \dot{x}(\phi) | \leq \| A(x) \|_{X^*} \| \phi \|_X + \| G(x) \|_{X^*} \| \phi \|_X.$$

By using Remark at the end of Section 2, we have

$$| \dot{x}(\phi) | \leq (\gamma \| x \|_X^{p-1} + \delta \| x \|_{L_p(I, H)}^{k-1}) \| \phi \|_X. \tag{7}$$

Then by substituting $p = 2$, $k = 1$ into (7) and by choosing sufficiently small γ and δ, we get

$$\| \dot{x} \| \leq \gamma \| x \|_X + \delta \leq \gamma \left(\frac{1}{\sqrt{2}} r \right) + \delta \leq \frac{1}{4} r. \tag{8}$$

Hence for a given $r > 0$, we get from equations (6) and (8) that

$$\| x \|_{W_{pq}(I)} = \| x \|_X + \| \dot{x} \|_{X^*} \leq \frac{1}{\sqrt{2}} r + \frac{1}{4} r < r.$$

Finally, we note that the embedding $W_{pq}(I) \hookrightarrow C[\bar{I}, H]$ is continuous; then

$$\| x \|_{C[\bar{I},\, H]} \leq \eta \| x \|_{W_{pq}(I)}.$$

Since we assume that the embedding constant $0 < \eta < 1$,

$$\| x \|_{C[\bar{I},\, H]} \leq r.$$

The assertion follows:

Theorem 1. *Let conditions* (A), (F) *and* (G) *hold. Then equation* (1) *has a T-periodic solution if and only if there exists* $x_0 \in H$ *such that*

$$x(T) = x_0 = x(0). \tag{9}$$

Proof. The necessary condition is obvious.

Sufficiency: Consider the following Cauchy problem:

$$\begin{aligned} &\dot{x}(t) + A(t, x(t)) = g(t, x(t)), \\ &x(0) = x_0, \\ &\Delta x(t_i) = F_i(x(t_i)),\ i = 1, 2, \ldots, n. \end{aligned} \tag{10}$$

It follows from Lemma 1 that system (10) has a solution on $[0, T]$. Since condition (9) is satisfied, this solution must be a T-periodic impulsive solution.

Lemma 3 (Bohl-Brower fixed point theorem)**.** *Let B be a non-empty compact convex subset of* R^m *and let the operator* $U : B \to B$ *be continuous. Then U has a fixed point* $x \in B$.

We are now ready to prove the existence of a periodic solution of system (1) in the special case that $H = R^m$.

Theorem 2. *Let the following conditions hold*:

1. *Conditions* (A), (F) *and* (G) *are met*.

2. *Assumptions of Lemma* 2 *are met*.

Then equation (2) *has a T-periodic solution* $x \in PW_{pq}(0, T) \cap PC([0, T], H)$.

Proof. Let $t, s \in (0, T]$ be such that $s \leq t \leq T$ and $y \in H$, and $x(t; s, y)$ be the solution equation (3) for which $x(s^+; s, y) = y$ (i.e., the initial condition is $x(s^+) = y$). We define the *operator* $U(t, s) : H \to H$ by the formula $U(t, s)y = x(t; s, y)$. It follows from Lemma 1 that the *operator* $U(t, s)$ is defined uniquely for each $y \in H$. Let $x(t) \equiv x(t; 0, y)$ be the solution of (3) corresponding to the initial condition $x(0) = y \in R^m$. Let $t > 0$ be a positive real number and let $D = clB(0, r)$ ($clB(0, r) \equiv$ closure of the ball in R^m centered at the origin and of radius r) which is a compact subset of R^m. Define an *operator* $U = U(T, 0) : D \to R^m$ as follows:

$$U(y) = x(T; 0, y).$$

By Lemma 1, the solution of (2) is unique and hence the operator U is well defined and continuous. It follows from Lemma 2 that the operator $U : D \to D$. Hence, Lemma 3 implies that there is a point $x_0 \in D$ such that

$$U(T, 0)(x_0) = x_0 \text{ or } x(T; 0, x_0) = x_0.$$

Hence $x(0) = x_0 = x(T)$ and by Theorem 1, system (2) has a periodic solution.

4. Control of Impulsive Periodic Systems

In this section, we study the existence of admissible control pairs. We model the control space by a separable reflexive Banach space E. By $P_f(E)(P_{fc}(E))$, we denote a class of nonempty closed (closed and convex) subsets of E. Recall that (see for example [4]) a multifunction $\Gamma : I \to P_f(E)$ is said to be *graph measurable* if

$$Gr(U) = \{(t, v) \in I \times E : v \in U(t)\} \in B(I) \times B(E),$$

where $B(I)$ and $B(E)$ are the Borel σ-fields of I and E respectively. For $2 \leq q < +\infty$, we define the *admissible space* U_{ad} to be the set of all $L_q(I, E)$-selections of $\Gamma(\cdot)$, i.e.,

$$U_{ad} = \{u \in L_q(I, E) : u(t) \in \Gamma(t)\mu \text{ - a.e. on } I\},$$

where μ is the Lebesgue on I. Note that the admissible space $U_{ad} \neq \phi$ if $\Gamma : I \to P_f(E)$ is graph measurable and the map

$$t \to |\Gamma(t)| := \sup\{\| v \|_E : v \in \Gamma(t)\} \in L_q(I).$$

The control problem (P) under consideration is the following:

$$\dot{x}(t) + A(t, x(t)) = g(t, x(t)) + B(t)u(t),\ t \neq t_i,\ 0 \leq t \leq T, \tag{11a}$$

$$x(t_i^+) - x(t_i) = F_i(x(t_i)),\ i = 1, 2, \ldots, n, \tag{11b}$$

$$x(0) = x(T). \tag{11c}$$

Here, we require the operators A, g and F_i's of equation (11) to satisfy hypotheses (A), (G) and (F), respectively as in Section 2. We now give new hypotheses for the remaining data.

(U) $U : I \to P_{fc}(E)$ is a measurable multifunction such that the map

$$t \to |U(t)| = \sup\{\| u \|_E : u \in U(t)\}$$

belongs to $L_q(I)$.

(B) $B \in L_\infty(I, L(E, H))$, where by $L(E, H)$, we denote the space of all bounded, linear operators from V into H.

By using the same notation as in equation (3), we can rewrite the control systems (11a)-(11c) into an equivalent operator equation as follows:

$$\dot{x} + A(x) = G(x) + B(u),\ 0 < t < T, \tag{12a}$$

$$x(0) = x_0 \in H, \tag{12b}$$

$$\Delta x(t_i) = F_i(x(t_i)), \tag{12c}$$

where $i = 1, 2, \ldots, n$ $(0 < t_1 < t_2 < \cdots < t_n < T)$ and the operators A, G and F_i $(i = 1, 2, \ldots, n)$ are the same as in equation (4). We set $B(u)(t) = B(t)u(t)$.

This relation defines an *operator* $B : L_q(I, E) \to L_q(I, H)$ which is linear and continuous.

It follows immediately from hypothesis (U) that the admissible space $U_{ad} \neq \phi$ and U_{ad} is a bounded closed convex subset of $L_q(I, E)$. Any solution x of equations (12a)-(12c) is referred to as a state trajectory of the evolution system corresponding to $u \in U_{ad}$ and the pair (x, u) is called an *admissible pair*. Let

$$A_{ad} = \{(x, u) \in PW_{pq}(I) \times U_{ad} : (x, u) \text{ is an admissible pair}\},$$

$$X_{ad} = \{x \in PW_{pq}(I) : \exists u \in U_{ad} \text{ such that } (x, u) \in A_{ad}\}.$$

Theorem 3. *Assume that the hypotheses* (A), (G), (B) *and* (U) *hold. Then the admissible set* $A_{ad} \neq \phi$ *and* X_{ad} *is bounded in* $PW_{pq}(I) \cap PC(I, H)$.

Proof. Let $u \in U_{ad}$, define

$$g_u(t, x) = g(t, x) + B(t)u(t).$$

Since $B \in L_\infty(I, L(E, H))$, one can see that $g_u : I \times H \to V^*$ satisfies hypothesis (G). Hence, by virtue of Theorem B, equation (12) has a solution.

Next, we shall show that X_{ad} is bounded in $PW_{pq}(I)$ by considering each case separately. Let $x \in X_{ad}$.

Case 1. $t \in (0, t_1)$. By Lemma 2, $\| x \|$ is bounded in $W_{pq}(0, t_1)$. Hence,

$$\| x \|_{W_{pq}(0, t_1)} \leq M_1 \text{ and } \| x \|_{C([0, t_1], H)} \leq M_1.$$

Case 2. $t \in (t_1, t_2)$. Since $\| x(0) \|_H$ and $\| x(t_1) \|_H \leq M_1$, by hypothesis (F), we have

$$\| x(t_1^+) \|_H \leq \| x(t_1) \|_H + \| F(x(t_1)) \|_H$$

$$\leq M_1[1 + 2L_1(M_1)] + \| F(x(0)) \|_H,$$

where $L(M_1)$ is real constant depending on M_1. Hence, $\| x(t_1^+) \|_H$ is bounded. Using Lemma 2 again, we have

$$\| x \|_{W_{pq}(0, t_2)} \leq M_2 \text{ and } \| x \|_{C([t_1, t_2], H)} \leq M_2.$$

After a finite step, there exists $M > 0$ such that

$$\| x \|_{W_{pq}(0, T)} \leq M \text{ and } \| x \|_{C([\bar{I}, H])} \leq M.$$

Hence, X_{ad} is bounded in $PW_{pq}(0, T) \cap PC(\bar{I}, H)$.

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