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THE CHARACTERIZATION OF COMPACT SUPPORT OF FOURIER TRANSFORM FOR SCALING FUNCTION AND ORTHONORMAL WAVELETS OF $L^2(R^s)$

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Abstract

In this paper, under mild condition, we give the sufficient and necessary condition that $\varphi(x)$ is a scaling function of $L^2(R^s)$, in view of support of Fourier transform for $\varphi(x)$. Furthermore, suppose $\{\psi_\mu\}$ is an orthonormal wavelet of $L^2(R^s)$ and the whole support of its Fourier

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transform is

$$\bigcup_{\mu} \mathrm{supp}\{\hat{\psi}_\mu\} = \prod_{i=1}^{s}[A_i, D_i] - \prod_{i=1}^{s}(B_i, C_i),$$

$$A_i \le B_i \le C_i \le D_i, \quad i = 1, 2, \ldots, s.$$

Under the weakest condition that each $|\hat{\psi}_\mu|$ is continuous for $\omega \in \partial\left(\prod_{i=1}^{s}[A_i, D_i]\right)$, we obtain results on the above whole support of $\{\psi_\mu\}$, which are characterized by some equalities and inequalities. We have improved completely Long's results and generalized Zhang's results.

1. Introduction and Preliminaries

The wavelet transform is a simple and practical mathematical tool that cuts up data or functions into different frequency components, and then studies each component with a resolution matched to its scale. Many mathematicians and physicians have noticed the wavelet transform. Indeed, engineers have discovered that it can be applied in all environments where the signal analysis is used. The main feature of the wavelets transform is to hierarchically decompose general function, as a signal or a process, into a set of approximations functions with different scales. In order to implement the transform, we need to construct various wavelets. The main purpose of this paper is to study compact support of Fourier transform for scaling function and orthonormal wavelets of $L^2(R^s)$. Wavelets are a fairly simple mathematical tool with a variety of possible application. Already they have led to exciting application in fractals [7], signal analysis [11], image processing [1, 4, 5] and design of orthonormal wavelet [2, 3, 6, 8, 9, 10, 12, 13] the last two decades.

Throughout this paper, Z is the set of integers, R is the set of real numbers, E_s denotes all the integer lattice points of cube $[0, M-1]^s$ $(M \ge 2, M \in Z)$ and Γ denotes all the vertices of cube $[0, 1]^s$. For $G \subset R^s$, ∂G denotes the boundary of G. $\hat{f}(\omega)$ denotes the Fourier transform of $f(x) \in L^2(R^s)$, defined by

$$\hat{f}(\omega) = \int_{R^s} f(t)e^{i\omega \cdot t}dt, \quad \omega \in R^s.$$

Let f be a measurable function, if $E = \text{clos}\{x \in R^s \mid f(x) \neq 0\}$, then we say the set E is the support of f, and write $\text{supp}f = E$.

Let $\{\psi_\mu(x)\} \in L^2(R^s)$ and $\{\psi_{\mu, k, l}\}$ be an orthonormal basis of $L^2(R^s)$. Then $\{\psi_\mu(x)\}$ is called an *orthonormal wavelet*, where $\psi_{\mu, k, l}(x) = M^{\frac{ks}{2}}\psi_\mu(M^k x - l)$, $\mu \in E_s - \{0\}$, $k \in Z$, $l \in Z^s$.

Definition 1. Let $\{V_k\}_{k\in Z}$ be a sequence of the closed subspaces of $L^2(R^s)$ satisfying:

(1) $V_k \subset V_{k+1}$, $k \in Z$;

(2) $\text{Clos}_{L^2(R^s)}\left(\bigcup_{k\in Z} V_k\right) = L^2(R^s)$, $\bigcap_{k\in Z} V_k = \{0\}$;

(3) $f(x) \in V_j$ if and only if $f(Mx) \in V_{j+1}$, $j \in Z$;

(4) There exists a function $\varphi(x) \in V_0$ such that $\{\varphi(x - n)\}_{n\in Z^s}$ is an orthonormal basis of V_0. Then $\{V_k\}_{k\in Z}$ is said to be a *multiresolution analysis* with dilation factor M, and $\varphi(x)$ is the corresponding scaling function. For $\varphi(x) \in V_0 \subset V_1$, then there exists a sequence $\{p_k\}_{k\in Z^s}$ such that $\varphi(x)$ satisfies the two-scale equation

$$\varphi(x) = M^s \sum_{k\in Z^s} p_k \varphi(Mx - k). \tag{1}$$

By taking Fourier transform on the both sides of (1), we have

$$\hat{\varphi}(M\omega) = m_0(\omega)\hat{\varphi}(\omega), \quad m_0(\omega) = \sum_{k\in Z^s} p_k e^{-ik\cdot\omega},$$

where $m_0(\omega)$ is said to be the *mask function*, and $m_0(\omega) \in L^2([0, 2\pi]^s)$.

Suppose the vector $\left(m_0\left(\omega + \frac{2\pi\nu}{M}\right)\right)_\nu$, $\nu \in E_s$ can be extended to a unitary matrix

$$M(\omega) = \left(m_\mu\left(\omega + \frac{2\pi\nu}{M}\right)\right)_{\nu, \mu}, \quad \mu, \nu \in E_s. \tag{2}$$

Then the function $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ is an orthonormal wavelet of $L^2(R^s)$, which is said to be an *M-MRA wavelet* generated by $\varphi(t)$, where $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ satisfying $\hat{\psi}_\mu(M\omega) = m_\mu(\omega)\hat{\varphi}(\omega)$, $\mu \in E_s - \{0\}$.

However, we know that, not all orthonormal wavelets are MRA wavelets. For example, the Journe's wavelet

$$\hat{\psi}(\xi) = \chi_{\left[\frac{4\pi}{7} \leq |\xi| \leq \pi\right]}(\xi) + \chi_{\left[4\pi \leq |\xi| \leq \frac{32\pi}{7}\right]}(\xi)$$

is a non-MRA wavelet (see [2, 4]), where χ_I is the characteristic function on I.

Using several theorems [10, Theorem 2.2.1, Theorem 3.6.7, Theorem 3.6.10, Equation 3.5.65] and [2, 3, 8, 9, 12, 13], we can derive easily the following results:

Proposition 1. *Let* $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ *be an orthonormal wavelet, and for some* $\delta_1, \delta_2 > 0$ *such that*

$$|\hat{\psi}_\mu(\omega)| = O(1 + |\omega|)^{-\delta_1}, \quad \hat{\psi}_\mu \in L^{2-\delta_2}, \quad \mu \in E_s - \{0\} \tag{3}$$

and $\bigcup_{\alpha \in Z^s} (\Omega + 2\pi\alpha) = R^s$, *where* $\Omega = \bigcup_{\mu \in E_s - \{0\}} \bigcup_{j<0} (M^j \mathrm{supp}\hat{\psi}_\mu)$. *Then* $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ *is an M-MRA wavelet.*

Proposition 2. *Let* $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ *be an orthonormal wavelet and* (3) *be valid. Then*

$$\sum_{\mu \in E_s - \{0\}} \sum_{j<0} \sum_{\alpha \in Z^s} |\hat{\psi}_\mu(M^{-j}(\omega + 2\pi\alpha))|^2 = n(\omega), \quad \textit{for a.e. } \omega \in R^s,$$

where $n(\omega)$ *is an integer-valued function.*

Proposition 3. *Let* $\{\psi_\mu(x)\}_{\mu\in E_s-\{0\}}$ *be an orthonormal wavelet. Then*

$$\sum_{\mu\in E_s-\{0\}}\sum_{j\in Z}|\hat{\psi}_\mu(M^j\omega)|^2 = 1, \quad \textit{for a.e. } \omega \in R^s$$

and

$$\sum_{\alpha\in Z^s}\overline{\hat{\psi}_\mu(\omega+2\pi\alpha)}\hat{\psi}_\nu(M^j(\omega+2\pi\alpha)) = \delta_{0,j}\delta_{\mu,\nu}, \quad \textit{for a.e. } \omega\in R^s,\ j\geq 0.$$

Proposition 4. *Let* $\{\psi_\mu(x)\}_{\mu\in E_s-\{0\}}$ *be an M-MRA wavelet, and the corresponding scaling function* φ *satisfies* $\lim_{\omega\to\infty}\hat{\varphi}(\omega) = 0$. *Then*

$$|\hat{\varphi}(\omega)|^2 = \sum_{\mu\in E_s-\{0\}}\sum_{j>0}|\hat{\psi}_\mu(M^j\omega)|^2, \quad \textit{for a.e. } \omega\in R^s. \tag{4}$$

Proposition 5 [3]**.** *Suppose* $\{V_j\}_{j\in Z}$ *is an orthonormal M-MRA with scaling function* $\varphi(x)$, *and* $|\hat{\varphi}(\omega)|$ *is a continuous at* 0. *Then* $\hat{\varphi}(0)\neq 0$.

2. Main Results

In this section, we borrow ideas from reference [12], to give characterization of compact support of Fourier transform for scaling function and orthonormal wavelets of $L^2(R^s)$. We have improved completely Long's results [10, Proposition 2.2.2, Theorem 3.5.11] and generalize Zhang's results [12, Theorem 1, Theorem 2]. We give the following results:

Theorem 1. *Let the function* $\varphi(x)\in L^2(R^s)$ *satisfy*

$$\operatorname{supp}\hat{\varphi} = \prod_{i=1}^{s}[a_i, b_i]. \tag{5}$$

Then $\varphi(x)$ *generates an orthonormal M-MRA if and only if the following conditions hold*:

$$|\hat{\varphi}(\omega)| = 1 \quad \textit{for a.e. } \omega\in\prod_{i=1}^{s}[a_i+\delta_i, b_i-\delta_i], \textit{ where } \delta_i = b_i - a_i - 2\pi, \tag{6}$$

$$2\pi \leq b_i - a_i \leq \frac{4M}{M+1}\pi, \frac{a_i}{M} + 2\pi \geq b_i, \frac{b_i}{M} - 2\pi \leq a_i, -2\pi \leq a_i < 0 < b_i \leq 2\pi, \quad (7)$$

$$\sum_{\nu \in E_s} |\hat{\varphi}(\omega - 2\pi\nu)|^2 = 1 \quad \textit{for a.e. } \omega \in [0, 2(M-1)\pi]^s. \quad (8)$$

Theorem 2. *Let* $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ *be an orthonormal wavelet with dilation M of* $L^2(R^s)$ *and the whole support of its Fourier transform be*

$$\bigcup_{\mu \in E_s - \{0\}} supp\{\hat{\psi}_\mu\} = \prod_{i=1}^{s} [A_i, D_i] - \prod_{i=1}^{s} (B_i, C_i), \quad A_i \leq B_i \leq C_i \leq D_i. \quad (9)$$

If each $|\hat{\psi}_\mu|$ *is continuous for* $\omega \in \partial\left(\prod_{i=1}^{s}[A_i, D_i]\right)$, *then for* $i = 1, 2, \ldots, s$, *we have*

$$B_i < 0 < C_i, MB_i \leq \frac{A_i}{M} < B_i, C_i < \frac{D_i}{M} \leq MC_i, C_i - 2\pi = \frac{A_i}{M}, B_i + 2\pi = \frac{D_i}{M}. \quad (10)$$

Theorem 3. *Suppose* $\hat{\varphi}(\omega) \in L^2(R^s) \cap C(R^s)$, $supp\,\hat{\varphi} = \prod_{j=1}^{s}[a_j, d_j] - \prod_{j=1}^{s}(b_j, c_j)$ $(b_j < c_j, c_j > 0, j = 1, 2, \ldots, s)$, *and* φ *generates an M-MRA. Then*

$$a_j < 0, \quad d_j > 0, \quad 2\pi < d_j - a_j \leq 4M\pi, \quad a_j < \frac{d_j}{M} \leq b_j. \quad (11)$$

Remark 1 [12]**.** Comparing Theorem 1 with Long's results [10, Proposition 2.2.2], it is clear that we remove the redundant condition $\hat{\varphi}(\omega) \in C(R^s)$.

Remark 2 [12]**.** Comparing Theorem 2 with Long's results [10, Theorem 3.5.11], it is clear that we remove many redundant conditions in Long's result as follows:

(i) $\{\psi_\mu\}$ is an MRA wavelet.

(ii) $\hat{\varphi}(\omega) \in C(R^s)$ and $\lim_{\omega \to \infty} \hat{\varphi}(\omega) = 0$.

(iii) $B_i \leq 0 \leq C_i$, $i = 1, 2, \ldots, s$.

Remark 3. When $M = 2$, we can prove that the conclusion of Theorem 1 and Theorem 2 implies that of Long's result [10, Proposition 2.2.2, Theorem 3.5.11] and Zhang's results [12, Theorem 1, Theorem 2].

Remark 4 [12]**.** In general, the condition that $|\hat{\psi}_\mu|$ is continuous for $\omega \in \partial\left(\prod_{i=1}^{s}[A_i, D_i]\right)$ cannot be removed. The well-known Shannon wavelet [7] $\psi(x)$ is a counterexample. In fact, $\hat{\psi}(\omega) = \chi_{[-2\pi, -\pi]}(\omega) + \chi_{[\pi, 2\pi]}(\omega)$. $\hat{\psi}(\omega)$ is discontinuous at the two endpoint $\omega = -2\pi, 2\pi$ but here $\frac{A}{2} = B$ $(A = 2\pi, B = \pi)$. This is contrary to $\frac{A}{2} < B$.

Remark 5. We generalized result of reference [12, Theorem 1, Theorem 2] as follows:

(i) dilation factor of scaling function is generalized to arbitrary positive integer $(M \geq 2)$ from 2;

(ii) we gave the characterization of compactly support of Fourier transform of scaling function in two different case, that is, $\operatorname{supp}\hat{\varphi} = \prod_{i=1}^{s}[a_i, b_i]$ and $\operatorname{supp}\hat{\varphi} = \prod_{j=1}^{s}[a_j, d_j] - \prod_{j=1}^{s}(b_j, c_j)$. But, only one case was discussed in reference [12], that is $\operatorname{supp}\hat{\varphi} = \prod_{i=1}^{s}[a_i, b_i]$.

3. Proof of Main Results

Proof of Theorem 1. We prove sufficiency. Suppose the conditions (6)-(8) are valid, we only need to prove that φ generated an M-MRA. In other words, we need to prove that $\{\varphi(\cdot - k) : k \in Z^s\}$ is an orthonormal basis for V_0 and $V_0 \subset V_1$. This is equivalent to proving $\Phi(\xi) = \sum_{k \in Z^s}|\hat{\varphi}(\xi + 2k\pi)|^2 \equiv 1$ and finding a $2\pi Z^s$-periodic function $m_0(\xi)$ satisfies

$$\hat{\varphi}(M\xi) = m_0(\xi)\hat{\varphi}(\xi), \quad \text{a.e. } \xi \in R^s. \tag{12}$$

From (5) and (8), we obtain

$$\Phi(\xi) = \sum_{k \in Z^s} |\hat{\varphi}(\xi + 2k\pi)|^2 = \sum_{\nu \in E_s} |\hat{\varphi}(\xi - 2\pi\nu)|^2 = 1 \text{ for a.e. } \xi \in [0, 2(M-1)\pi]^s$$

and $\Phi(\xi)$ is $2\pi Z^s$-periodic function. Hence, $\Phi(\xi) \equiv 1$.

Defining

$$m_0(\xi) = \begin{cases} \hat{\varphi}(2\xi)/\hat{\varphi}(\xi), & \xi \in \Delta_1 = \prod_{i=1}^{s} \left[\frac{a_i}{M}, \frac{b_i}{M}\right], \\ 0, & \xi \in \Delta_2 - \Delta_1 = \prod_{i=1}^{s} \left[-\pi + \frac{a_i + b_i}{M^2}, \pi + \frac{a_i + b_i}{M^2}\right] - \Delta_1, \end{cases} \tag{13}$$

which is then extended periodically by $2\pi Z^s$ from Δ_2 to R^s. It is clear that (12) is valid on Δ_2. We need to prove that (12) is valid on $\Delta_2 + 2\pi\alpha$, $\alpha \neq 0$, $\alpha \in Z^s$. This is equivalent to proving

$$\hat{\varphi}(M(\xi + 2\pi\alpha)) = m_0(\xi)\hat{\varphi}(\xi + 2\pi\alpha) \tag{14}$$

for $\xi \in \Delta_2$, $\alpha \neq 0$, $\alpha \in Z^s$.

Let α_k be the k-th coordinate of α. Suppose $\alpha_k \geq 1$, when $\xi \in \Delta_1$, we have

$$\xi_j + 2\pi\alpha_j \leq \frac{\alpha_j}{M} + 2\pi \geq b_j, \quad M(\xi_j + 2\pi\alpha_j) \leq Mb_j \geq b_j,$$

thus, the both sides of (14) are equal to 0, (14) is valid. When $\xi \in \Delta_2 - \Delta_1$, we have $m_0(\xi) = 0$,

$$M(\xi_j + 2\pi\alpha_j) \leq M\left(-\pi + \frac{a_j + b_j}{M^2} + 2\pi\right) = M\pi + \frac{a_j + b_j}{M} \geq b_j,$$

thus, $\hat{\varphi}(M(\xi + 2\pi\alpha)) = 0$. So (14) is valid. Analogously, when $\alpha_k \leq -1$, (14) is valid. Combining above results, we have

$$\hat{\varphi}(M\xi) = m_0(\xi)\hat{\varphi}(\xi), \text{ a.e. } \xi \in R^s.$$

The proof of sufficiency is completed.

We prove necessity. First, we prove that (7) is valid. The proof is processed in three steps:

(a) We prove that $b_j - a_j \geq 2\pi,\ j = 1, 2, ..., s.$

Since $\varphi(x)$ is a scaling function ($\varphi(x)$ generates an orthonormal M-MRA), we know that

$$\sum_{k \in Z^s} |\hat{\varphi}(\omega + 2\pi k)|^2 = 1, \quad \text{a.e. } \omega \in R^s. \tag{15}$$

Suppose that there exists some j such that $b_j - a_j < 2\pi.$ Without loss of generality, we set $j = 1,$ i.e., $b_1 - a_1 < 2\pi.$ Then $\forall \omega_1 \in (b_1, a_1 + 2\pi),$ we have

$$\omega_1 + 2k\pi \notin [a_1, b_1], \quad k \in Z.$$

Therefore

$$\sum_{k \in \mathbf{Z^s}} |\hat{\varphi}(\omega + 2\pi k)|^2 = 0, \quad \omega \in (b_1, a_1 + 2\pi) \times \prod_{i=2}^{s} [a_i, b_i].$$

This is contrary to (15). So $b_j - a_j \geq 2\pi,\ j = 1, 2, ..., s.$

(b) We prove that $a_j \leq 0,\ b_j \geq 0,\ j = 1, 2, ..., s.$

Suppose that there exists some j such that $a_j > 0.$ Without loss of generality, we set $j = 1,$ i.e., $a_1 > 0.$ Let ε_1 be a positive number such that $\varepsilon_1 < \min\left\{\frac{a_1}{M}, \min_j\left\{\frac{b_j - a_j}{M}\right\}\right\}.$ Then

$$a_i < a_i + M\varepsilon_1 < b_i, \quad i = 1, 2, ..., s, \quad \frac{a_1}{M} + \varepsilon_1 < \frac{2a_1}{M} \leq a_1.$$

Let $a = (a_1, a_2, ..., a_s),\ \varepsilon = (\varepsilon_1, \varepsilon_1, ..., \varepsilon_1).$ Then we have

$$a + M\varepsilon \in \prod_{i=1}^{s} [a_i, b_i], \quad \frac{a}{M} + \varepsilon \notin \prod_{i=1}^{s} [a_i, b_i].$$

So from (5), we have

$$\hat{\varphi}(a + M\varepsilon) \neq 0, \quad \hat{\varphi}\left(\frac{a}{M} + \varepsilon\right) = 0.$$

Since $\varphi(x)$ generates an orthonormal M-MRA, we know that

$$\hat{\varphi}(M\omega) = m_0(\omega)\hat{\varphi}(\omega),$$

where $m_0(\omega)$ is the mask function which is stated as in Section 1. A contradiction is achieved, so we obtain $a_j \leq 0,\ j = 1, 2, \ldots, s$. Similarly, we have $b_j \geq 0,\ j = 1, 2, \ldots, s$.

(c) We prove that $a_j \neq 0,\ b_j \neq 0,\ j = 1, 2, \ldots, s$.

From (b), we know $a_j \leq 0$. If there exists some $a_j = 0$, without loss of generality, we set $j = 1$, i.e., $a_1 = 0$. Then by (5), $\operatorname{supp}\hat{\varphi} \subset E_1$, where $E_1 = [0, \infty) \times R^{s-1}$. Now take a function $f \in L^2(R^s)$ such that $\operatorname{supp}\hat{f} \not\subset E_1$. Because $\left\{\varphi_{k,l}(x) = M^{\frac{ks}{2}}\varphi(M^k x - l),\ k \in Z,\ l \in Z^s\right\}$ is an orthonormal basis of $L^2(R^s)$, define $V_k = \{\varphi_{k,l}(x),\ l \in Z^s\}$. Then $f(x)$ can be represented as follows:

$$f(x) = \sum_{k \in Z,\ l \in Z^s} d_{k,l}\varphi_{k,l}(x).$$

By taking the Fourier transform on the both sides of above equation, we get

$$\hat{f}(\omega) = \sum_{k \in Z,\ l \in Z^s} d_{k,l}\hat{\varphi}_{\mu,k,l}(\omega).$$

By $\operatorname{supp}\hat{\varphi} \subset E_1$, we have $\operatorname{supp}\hat{\varphi}_{k,l}(\omega) = \operatorname{supp}\hat{\varphi}\left(\frac{\omega}{M^k}\right) \subset E_1$. Again by $\overline{\bigcup_{m \in Z} V_m} = L^2(R^s)$, we have $\operatorname{supp}\hat{f} \in E_1$. This is contrary to $\operatorname{supp}\hat{f} \not\subset E_1$. Therefore, $a_1 \neq 0$. Furthermore, we show that $a_j \neq 0$, $j = 1, 2, \ldots, s$. Similarly, we have $b_j \neq 0,\ j = 1, 2, \ldots, s$.

Using $\hat{\varphi}(M\omega) = m_0(\omega)\hat{\varphi}(\omega)$ and (5), we have that

$$\begin{cases} m_0(\omega) = 0,\ \omega \in \prod_{j=1}^{s}[a_j, b_j] - \prod_{j=1}^{s}\left[\frac{a_j}{M}, \frac{b_j}{M}\right], \\ m_0(\omega) \neq 0,\ \omega \in \prod_{j=1}^{s}\left(\frac{a_j}{M}, \frac{b_j}{M}\right). \end{cases}$$

By the periodicity of $m_0(\omega)$, we obtain that

$$\frac{b_j}{M} - a_j \le 2\pi, \quad b_j - \frac{a_j}{M} \le 2\pi.$$

Therefore

$$2\pi \le b_j - a_j \le \frac{4M}{M+1}\pi, \quad -2\pi \le a_j < 0 < b_j \le 2\pi, \quad j = 1, 2, ..., s.$$

So, (7) is valid.

Next, we prove that (6) is valid.

Let $\omega = (\omega_1, \omega_2, ..., \omega_s) \in \prod_{j=1}^{s} [a_j + \delta_j, b_j - \delta_j]$, where $\delta_j = b_j - a_j - 2\pi$. Then, we have that

$$\omega + 2\pi k \notin \prod_{j=1}^{s} [a_j, b_j], \quad k = (k_1, k_2, ..., k_s) \in Z^s, \quad k \ne 0.$$

By (5) and (15), we have

$$1 = \sum_{k \in Z^s} |\hat{\varphi}(\omega + 2\pi k)|^2 = |\hat{\varphi}(\omega)|^2, \quad \omega \in \prod_{j=1}^{s} [a_j + \delta_j, b_j - \delta_j].$$

So, (6) is valid.

Finally, we prove that (8) is valid.

Let $\xi \in [0, 2(M-1)\pi]^s$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_s) \in Z^s - E_s$. Then $\xi - 2\pi\alpha \notin (-2\pi, 2\pi)^s$. Furthermore, we have $\xi - 2\pi\alpha \notin \prod_{j=1}^{s} [a_j, b_j]$. By (5), (7) and (15), we have

$$\sum_{k \in Z^s} |\hat{\varphi}(\xi - 2\pi\nu)|^2 = \sum_{\nu \in E_s} |\hat{\varphi}(\xi - 2\pi\nu)|^2 = 1, \quad \xi \in [0, 2(M-1)\pi]^s.$$

So, (8) is valid. The proof of Theorem 2 is completed. □

Before proof of Theorem 1, we give three lemmas:

Lemma 1. *Under the assumptions of Theorem* 2, *we have* $A_i < 0$, $D_i > 0$, $i = 1, 2, ..., s$.

Proof. Suppose there exists some i such that $A_i \geq 0$. Without loss of generality, we set $i = 1$, i.e., $A_1 \geq 0$. Then by (9), $\operatorname{supp}\hat{\psi}_\mu \subset E_1 (\mu \in E_s - \{0\})$, where

$$E_1 = [0, \infty) \times R^{s-1}. \tag{16}$$

Now take a function $f \in L^2(R^s)$ such that $\hat{f}$ is continuous and $\operatorname{supp}\hat{f} \not\subset E_1$. Because $\{\psi_{\mu, k, l}(x)\}_{\mu \in E_s - \{0\}}$ is an orthonormal basis of $L^2(R^s)$. Then $f(x)$ can be represented as follows:

$$f(x) = \sum_{\mu \in E_s/\{0\}} \sum_{k \in Z, l \in Z^s} C_{\mu, k, l} \psi_{\mu, k, l}(x).$$

Furthermore, we get

$$\hat{f}(\omega) = \sum_{\mu \in E_s/\{0\}} \sum_{k \in Z, l \in Z^s} C_{\mu, k, l} \hat{\psi}_{\mu, k, l}(\omega). \tag{17}$$

By $\operatorname{supp}\hat{\psi}_\mu \subset E_1$, we have $\operatorname{supp}\hat{\psi}_{\mu, k, l}(\omega) = \operatorname{supp}\hat{\psi}_\mu\left(\frac{\omega}{M^k}\right) \subset E_1$. Again by (17), noticing that $\hat{f}(\omega)$ is continuous in R^s, we know that $\hat{f}(\omega) = 0$ for $\omega \in R^s - E_1$. Furthermore, we have $\operatorname{supp}\hat{f} \subset E_1$. This is contrary to $\operatorname{supp}\hat{f} \not\subset E_1$. So $A_i < 0, i = 1, 2, \ldots, s$. Similarly, we have $D_i > 0$, $i = 1, 2, \ldots, s$. □

Lemma 2. *Under the assumptions of Theorem* 2, *we have* $B_i < 0 < C_i$, $i = 1, 2, \ldots, s$.

Proof. First, we prove that, for any $\mu, \nu \in E_s - \{0\}$ and a sufficiently large $j > 0$, we have

$$\overline{\hat{\psi}_\mu(M^{-j}\omega)}\hat{\psi}_\nu(\omega) = 0, \quad \text{for a.e. } \omega \in R^s. \tag{18}$$

Since $\{\psi_\mu(x)\}_{\mu \in E_s - \{0\}}$ is an orthonormal wavelet in $L^2(R^s)$, by Proposition 3, we know that, for any μ, ν and $j > 0$,

$$\sum_{\alpha \in Z^s} \overline{\hat{\psi}_\mu(M^{-j}\omega + 2\pi\alpha)}\hat{\psi}_\nu(\omega + M^j 2\pi\alpha) = \delta_{0,j}\delta_{\mu,\nu}, \quad \text{for a.e. } \omega \in R^s. \tag{19}$$

By (9), we know that each $\hat{\psi}_\mu(\omega)$ has a compact support. So we can take a sufficiently large $j > 0$ such that for any $\alpha \neq 0$, $\omega \in \operatorname{supp}\hat{\psi}_\nu$, we always have $\omega + M^j 2\pi\alpha \notin \operatorname{supp}\hat{\psi}_\nu$. Namely, $\hat{\psi}_\nu(\omega + M^j 2\pi\alpha) = 0$. Again by (19), we know, for a sufficiently large $j > 0$,

$$\overline{\hat{\psi}_\mu(M^{-j}\omega)}\hat{\psi}_\nu(\omega) = 0, \quad \text{for a.e. } \omega \in \operatorname{supp}\hat{\psi}_\nu. \tag{20}$$

From this we obtain (18) immediately.

Next, by (19) it follows that, for a sufficiently large $j > 0$,

$$\left(\sum_{\mu \in E_s - \{0\}} |\hat{\psi}_\mu(M^{-j}\omega)|\right)\left(\sum_{\nu \in E_s - \{0\}} |\hat{\psi}_\nu(\omega)|\right) = 0, \quad \text{for a.e. } \omega \in R^s. \tag{21}$$

Let

$$Q = \operatorname{supp}\left(\sum_{\mu \in E_s - \{0\}} |\hat{\psi}_\mu(M^{-j}\omega)|\right) \cap \operatorname{supp}\left(\sum_{\nu \in E_s - \{0\}} |\hat{\psi}_\nu(\omega)|\right). \tag{22}$$

Combining (21) and (22), we know $\operatorname{mes} Q = 0$.

Finally, by (9), we know that

$$\operatorname{supp}\left(\sum_{\mu \in E_s - \{0\}} |\hat{\psi}_\mu(\omega)|\right) = \bigcup_{\mu \in E_s - \{0\}} \operatorname{supp}\hat{\psi}_\mu = \prod_{i=1}^{s} [A_i, D_i] - \prod_{i=1}^{s} (B_i, C_i).$$

Consequently,

$$\operatorname{supp}\left(\sum_{\mu \in E_s - \{0\}} |\hat{\psi}_\mu(M^{-j}\omega)|\right) = \prod_{i=1}^{s} [M^j A_i, M^j D_i] - \prod_{i=1}^{s} (M^j B_i, M^j C_i).$$

Putting this equation, (21), (22) and Lemma 1 together, we obtain that, for a sufficiently large $j > 0$,

$$\prod_{i=1}^{s} [A_i, D_i] \subset \prod_{i=1}^{s} (M^j B_i, M^j C_i).$$

Hence we have $M^j B_i \leq A_i \leq D_i \leq M^j C_i$ $(i = 1, 2, ..., s)$. Since $A_i < 0$, $D_i > 0$, we have $B_i < 0 < C_i$, $i = 1, 2, ..., s$. Proof of Lemma 2 is completed. □

Lemma 3. *Under the assumptions of Theorem* 2, *then*

(i) $\frac{A_i}{M} \leq B_i,\ C_i \leq \frac{D_i}{M}$, $i = 1, 2, ..., s$.

(ii) $D_i - A_i > 2M\pi$, $i = 1, 2, ..., s$.

(iii) $\{\psi_\mu\}_{\mu \in E_s - \{0\}}$ *is an M-MRA wavelet.*

Proof. First, we prove (i).

Suppose $\frac{A_1}{M} > B_1$. By Lemma 1 and Lemma 2, we know that $A_1 \leq B_1 < 0$. Let $F = \prod_{i=2}^{s}(B_i, C_i)$ and $\omega \in \left(B_1, \frac{A_1}{M}\right) \times F$. When $k \geq 1$, $M^k\omega \in (M^k B_1, M^{k-1}A_1) \times M^k F \subset (-\infty, A_1) \times M^k F$. When $k \leq 0$, $M^k \omega \in (B_1, 0) \times M^k F$, where $M^k F = \prod_{i=2}^{s}(M^k B_i, M^k C_i)$. Noticing that $B_i < 0 < C_i$, we have

$$M^k \omega \notin \prod_{i=1}^{s}[A_i, D_i] - \prod_{i=1}^{s}(B_i, C_i), \quad k \in Z.$$

Hence from (9), we obtain that

$$\hat{\psi}_\mu(M^k\omega) = 0, \quad \forall \omega \in \left(B_1, \frac{A_1}{M}\right) \times F, \quad \mu \in E_s - \{0\},\ k \in Z.$$

Take $f(x) \neq 0$, a.e. $x \in R^s$ such that $\operatorname{supp}\hat{f} \subset \left(B_1, \frac{A_1}{M}\right) \times F$. Then

$$\begin{aligned} f(x) &= \sum_{\mu \in E_s - \{0\}} \sum_{k \in Z} \sum_{l \in Z^s} \langle f, \psi_{\mu,k,l}\rangle \psi_{\mu,k,l}(x) \\ &= \sum_{\mu \in E_s - \{0\}} \sum_{k \in Z} \sum_{l \in Z^s} \frac{1}{(2\pi)^s} \langle \hat{f}, \hat{\psi}_{\mu,k,l}\rangle \psi_{\mu,k,l}(x) = 0. \end{aligned}$$

This is contrary to $f(x) \neq 0$, a.e. $x \in R^s$. Therefore, $\frac{A_1}{M} \leq B_1$.

Similarly, we can obtain $\frac{A_i}{M} \le B_i$, $C_i \le \frac{D_i}{M}$, $i = 1, 2, ..., s$.

Next, we prove (ii).

By (9), we know that $\operatorname{supp}\hat{\psi}_\mu \subset \prod_{i=1}^{s}[A_i, D_i]$, then

$$\hat{\psi}_\mu(\omega) = 0, \quad \omega \notin \prod_{i=1}^{s}[A_i, D_i], \quad \mu \in E_s - \{0\}. \tag{23}$$

Since $|\hat{\psi}_\mu|$ is continuous for $\omega \in \partial\left(\prod_{i=1}^{s}[A_i, D_i]\right)$, we have $\hat{\psi}_\mu(\omega) = 0$, $\omega \in \partial\left(\prod_{i=1}^{s}[A_i, D_i]\right)$. From (23), we obtain that, for $j > 1$, $\hat{\psi}_\mu(M^j\omega) = 0$, $\omega \notin \prod_{i=1}^{s}\left[\frac{A_i}{M^2}, \frac{D_i}{M^2}\right]$. Let $G(\omega) = \sum_{E_s - \{0\}}\sum_{j>0}|\hat{\psi}_\mu(M^j\omega)|^2$. Then

$$G(\omega) = \sum_{E_s - \{0\}}|\hat{\psi}_\mu(M\omega)|^2, \quad \omega \notin \prod_{i=1}^{s}\left[\frac{A_i}{M^2}, \frac{D_i}{M^2}\right]$$

and

$$G(\omega) = 0, \quad \omega \notin \prod_{i=1}^{s}\left[\frac{A_i}{M}, \frac{D_i}{M}\right]. \tag{24}$$

Again we noticing that $|\hat{\psi}_\mu|$ is continuous for $\omega \in \partial\left(\prod_{i=1}^{s}[A_i, D_i]\right)$, we obtain that $G(\omega)$ is continuous for $\omega \in \partial\left(\prod_{i=1}^{s}\left[\frac{A_i}{M}, \frac{D_i}{M}\right]\right)$. Using (24), we have

$$G(\omega) = 0, \quad \omega \in \partial\left(\prod_{i=1}^{s}\left[\frac{A_i}{M}, \frac{D_i}{M}\right]\right). \tag{25}$$

Now suppose $D_i - A_i \le 2M\pi$, let

$$L(\omega) = \sum_{\alpha \in Z^s} G(\omega + 2\pi\alpha). \tag{26}$$

Let $H = \frac{A_1}{M} \times \prod_{i=2}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]$. It is clear that

$$H \subset \partial\left(\prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]\right). \tag{27}$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$. When $\alpha_1 = 0$ and $\omega \in H$,

$$\omega + 2\pi\alpha \in \partial\left(\prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]\right) \text{ or } \omega + 2\pi\alpha \notin \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]. \tag{28}$$

Using (24) and (25), we have

$$G(\omega + 2\pi\alpha) = 0, \ \omega \in H \text{ and } G(\omega + 2\pi\alpha) \text{ is continuous for } \omega \in H. \tag{29}$$

Similarly, when $\alpha_1 \neq 0$ and $\omega \in H$, in view of $D_1 - A_1 \leq 2M\pi$, (28) and (29) are still valid.

Therefore

$$L(\omega) = 0, \quad \omega \in H. \tag{30}$$

Since $G(\omega)$ has a compact support, we obtain that for $\omega \in \left[\frac{A_1}{M} - \varepsilon, \frac{A_1}{M} + \varepsilon\right] \times \prod_{i=2}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]$, there are only finite terms in $L(\omega)$. Again noticing that $G(\omega + 2\pi\alpha)$ is continuous for $\omega \in H$, we know that $L(\omega)$ is continuous for $\omega \in H$. By Lemma 2 and Lemma 3(i), we have $A_1 < B_1$. From the definition $L(\omega)$ and $G(\omega)$, noticing that $G(\omega) \geq 0$, we know that

$$L(\omega) \geq G(\omega) \geq \sum_{\mu} |\hat{\psi}_{\mu}(M\omega)|^2.$$

But by (9), we get

$$\sum_{\mu} |\hat{\psi}_{\mu}(M\omega)|^2 > 0 \text{ for a.e. } \omega \in \left[\frac{A_1}{M}, \frac{B_1}{M}\right] \times \prod_{i=2}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right].$$

So

$$L(\omega) > 0 \quad \text{for a.e. } \omega \in \left[\frac{A_1}{M}, \frac{B_1}{M}\right] \times \prod_{i=2}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right].$$

Again because $L(\omega)$ is continuous for $\omega \in H$, using (30), we know that there exist $0 < \delta < \frac{B_1}{M} - \frac{A_1}{M}$ such that

$$0 < |L(\omega)| < \frac{1}{2} \quad \text{for a.e. } \omega \in \left[\frac{A_1}{M}, \frac{A_1}{M} + \delta\right] \times \prod_{i=2}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right].$$

Again noticing the definition of $L(\omega)$, we will see that this is contrary to Proposition 2. So $D_1 - A_1 > 2M\pi$. Similarly, we get $D_i - A_i > 2M\pi$, $i = 1, 2, \ldots, s$.

Finally, we prove (iii).

From (9) and Lemma 3(i), we have

$$\prod_{i=1}^{s} [A_i, D_i] \supset \operatorname{supp}\left(\sum_{\mu \in E_s - \{0\}} |\hat{\psi}_\mu(\omega)|\right) \supset \prod_{i=1}^{s} [A_i, D_i] - \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right].$$

So for $j > 0$,

$$\prod_{i=1}^{s} \left[\frac{A_i}{M^j}, \frac{D_i}{M^j}\right] \supset \operatorname{supp}\left(\sum_{\mu \in E_s - \{0\}} |\hat{\psi}_\mu(M^j\omega)|\right)$$

$$\supset \prod_{i=1}^{s} \left[\frac{A_i}{M^j}, \frac{D_i}{M^j}\right] - \prod_{i=1}^{s} \left[\frac{A_i}{M^{j+1}}, \frac{D_i}{M^{j+1}}\right].$$

Since

$$\bigcup_{j>0} \left(\prod_{i=1}^{s} \left[\frac{A_i}{M^j}, \frac{D_i}{M^j}\right] - \prod_{i=1}^{s} \left[\frac{A_i}{M^{j+1}}, \frac{D_i}{M^{j+1}}\right]\right) = \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right] - \{0\},$$

we have

$$\operatorname{supp} \sum_{\mu \in E_s - \{0\}} \sum_{j>0} |\hat{\psi}_\mu(M^j\omega)|^2 = \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]. \tag{31}$$

So letting $\Omega = \bigcup_{\mu \in E_s/\{0\}} \bigcup_{j<0} (M^j \mathrm{supp}\hat{\psi}_\mu)$, we have $\Omega = \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]$. Again from $D_i - A_i > 2M\pi$, it follows that $\bigcup_{\alpha \in Z^s} (\Omega + 2\pi\alpha) = R^s$. Using Proposition 1, we have $\{\psi_\mu\}_{\mu \in E_s/\{0\}}$ is an M-MRA wavelet. Lemma 3 is proved. □

Proof of Theorem 2. Using Lemma 3, we know that $\{\psi_\mu\}_{\mu \in E_s - \{0\}}$ is an M-MRA wavelet. Here we define the corresponding M-MRA by $\{V_m\}_{m \in Z}$ and the scaling function by $\varphi(t)$.

Using Lemma 2, we have

$$B_i < 0 < C_i, \quad i = 1, 2, ..., s. \tag{32}$$

Under the assumption condition, it is clear that $\mathrm{supp}\hat{\psi}_\mu \subset \prod_{i=1}^{s} [A_i, D_i]$. Again by $A_i < 0$, $D_i > 0$, it follows that $\mathrm{supp}\hat{\psi}_{\mu, k, l} \subset \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]$, $k < 0$. Because $\varphi \in V_0$ and

$$V_0 = \overline{\mathrm{supp}}\{\psi_{\mu, k, l} : \mu \in E_s - \{0\}, k < 0, l \in Z^s\}, \tag{33}$$

we have

$$\mathrm{supp}\hat{\varphi} \subset \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]. \tag{34}$$

By (34), it is clear that $\lim_{\omega \to \infty} \hat{\varphi}(\omega) = 0$. Using Proposition 4, we obtain

$$|\hat{\varphi}(\omega)|^2 = \sum_{\mu \in E_s - \{0\}} \sum_{j>0} |\hat{\psi}_\mu(M^j \omega)|^2 \quad \text{for a.e. } \omega \in R^s.$$

Again from (31), it follows that

$$\mathrm{supp}\hat{\varphi} = \prod_{i=1}^{s} \left[\frac{A_i}{M}, \frac{D_i}{M}\right]. \tag{35}$$

So by Theorem 2 and Lemma 3(ii), the following results hold:

$$2M\pi < D_i - A_i \leq \frac{4M^2}{M+1}\pi, \quad \frac{A_i}{M} + 2M\pi \geq D_i,$$
$$\frac{D_i}{M} - 2M\pi \leq A_i, \quad -2M\pi \leq A_i < 0 < D_i \leq 2M\pi, \tag{36}$$
$$|\hat{\varphi}(\omega)| = 1, \quad \text{for a.e. } \omega \in \prod_{i=1}^{s}\left[\frac{D_i}{M} - 2\pi, \frac{A_i}{M} + 2\pi\right]$$

and

$$0 < |\hat{\varphi}(\omega)| < 1, \quad \text{for a.e. } \omega \in \prod_{i=1}^{s}\left[\frac{A_i}{M}, \frac{D_i}{M}\right] - \prod_{i=1}^{s}\left[\frac{D_i}{M} - 2\pi, \frac{A_i}{M} + 2\pi\right]. \tag{37}$$

From (35) and (37), it follows that

$$\operatorname{supp}\left\{\left|\hat{\varphi}\left(\frac{\omega}{M}\right)\right|^2 - |\hat{\varphi}(\omega)|^2\right\} = \prod_{i=1}^{s}\left[\frac{A_i}{M}, \frac{D_i}{M}\right] - \prod_{i=1}^{s}\left(\frac{D_i}{M} - 2\pi, \frac{A_i}{M} + 2\pi\right).$$

Furthermore, using Proposition 4, we know the whole support

$$\bigcup_{\mu \in E_s - \{0\}} \operatorname{supp}(\hat{\psi}_\mu) = \prod_{i=1}^{s}\left[\frac{A_i}{M}, \frac{D_i}{M}\right] - \prod_{i=1}^{s}\left(\frac{D_i}{M} - 2\pi, \frac{A_i}{M} + 2\pi\right).$$

Comparing it with the assumption condition, we get

$$B_i = \frac{D_i}{M} - 2\pi, \quad C_i = \frac{A_i}{M} + 2\pi. \tag{38}$$

From (38) and (36), we have $MB_i \leq \frac{A_i}{M}, \frac{D_i}{M} \leq MC_i$. By Lemma 3(ii) and (38), we finally get $\frac{A_i}{M} < B_i, \frac{D_i}{M} > C_i$. Again by (32) this completes the proof of (10). So Theorem 1 is proved. □

Proof of Theorem 3. First, we prove that $a_j < 0, \ d_j > 0$, $j = 1, 2, ..., s$.

By Proposition 5, we know that $\hat{\varphi}(0) \neq 0$ and 0 is inner point of $\prod_{j=1}^{s}[a_j, d_j] - \prod_{j=1}^{s}(b_j, c_j)$. Therefore, $a_j < 0, d_j > 0, j = 1, 2, ..., s$.

Next, we prove that $d_j - a_j > 2\pi,\ j = 1, 2, \ldots, s.$

Since $\varphi(x)$ generates an orthonormal M-MRA, we know that

$$\sum_{k \in Z^s} |\hat{\varphi}(\omega + 2\pi k)|^2 = 1, \ \text{a.e. } \omega \in R^s. \tag{39}$$

Suppose that there exists some j such that $d_j - a_j < 2\pi.$ Without loss of generality, we set $j = 1,$ i.e., $d_1 - a_1 < 2\pi.$ For $\forall \omega_1 \in (d_1, a_1 + 2\pi),$ then we have

$$\omega_1 + 2k\pi \notin [a_1, d_1], \quad k \in Z.$$

Therefore

$$\sum_{k \in \mathbf{Z}^s} |\hat{\varphi}(\omega + 2\pi k)|^2 = 0, \quad \omega \in (d_1, a_1 + 2\pi) \times \prod_{i=2}^{s} [a_i, d_i].$$

This is contrary to (39). So, $d_j - a_j \geq 2\pi,\ j = 1, 2, \ldots, s.$

Suppose that there exists some j such that $d_j - a_j = 2\pi.$ Then, when $\omega \in \operatorname{supp}\hat{\varphi},$ we have

$$\sum_{k \in Z^s} |\hat{\varphi}(\omega + 2\pi k)|^2 = |\hat{\varphi}(\omega)|^2 = 1.$$

So $\hat{\varphi}(\omega) = \pm 1,$ it is contrary to the continuous of $\hat{\varphi}.$ So $d_j - a_j \neq 2\pi,$ $j = 1, 2, \ldots, s.$

We obtain $d_j - a_j > 2\pi,\ j = 1, 2, \ldots, s.$

Next, we prove that $a_j < \frac{d_j}{M} \leq b_j, \quad j = 1, 2, \ldots, s.$ The proof is processed in three steps:

(a) We prove that $\frac{c_j}{M} \leq b_j,\ j = 1, 2, \ldots, s.$

Since $\varphi(t)$ generates an orthonormal M-MRA, we know that

$$\hat{\varphi}(M\xi) = m_0(\xi)\hat{\varphi}(\xi), \tag{40}$$

where $m_0(\omega)$ is the mask function. Suppose that there exists some j such that $b_j < \frac{c_j}{M}$. Without loss of generality, we set $j = 1$, i.e., $b_1 < \frac{c_1}{M}$, and take any ξ_1 such that

$$b_1 < \frac{c_1}{M} < \xi_1 < \min\left\{c_1, \frac{d_1}{M}\right\} \leq c_1, \quad c_1 < M\xi_1 < d_1.$$

Let $\xi = (\xi_1, \xi_2, \dots, \xi_s)$, $\xi_j \in \left[\frac{a_j}{M}, \frac{d_j}{M}\right] - \left(\frac{b_j}{M}, \frac{c_j}{M}\right)$, $j = 2, 3, \dots, s$. Then we obtain

$$M\xi \in \operatorname{supp}\hat{\varphi}, \quad \xi \notin \operatorname{supp}\hat{\varphi}.$$

Therefore

$$\hat{\varphi}(M\xi) \neq 0, \quad \hat{\varphi}(\xi) = 0.$$

It is contrary to (40). Thus, we have $\frac{c_j}{M} \leq b_j$, $j = 1, 2, \dots, s$.

(b) We prove that $\frac{d_j}{M} \leq c_j$, $j = 1, 2, \dots, s$.

Suppose that there exists some j such that $b_j < \frac{c_j}{M}$. Without loss of generality, we set $j = 1$, i.e., $c_1 < \frac{d_1}{M}$. We show that it is impossible in two cases. When $d_1 \geq M^2 b_1$, take ξ_1 with $Mb_1 - c_1 < \varepsilon_1 < \min\left\{\frac{d_1}{M} - c_1, c_1\right\}$, thus

$$c_1 < c_1 + \varepsilon_1 = \xi_1 < \frac{d_1}{M} < d_1, \, b_1 < \frac{\xi_1}{M} < c_1.$$

Let $\xi = (\xi_1, \xi_2, \dots, \xi_s)$, $\xi_j \in [a_j, d_j] - (b_j, c_j)$, $j = 2, 3, \dots, s$. Then we obtain

$$\xi \in \operatorname{supp}\hat{\varphi}, \quad \frac{\xi}{M} \notin \operatorname{supp}\hat{\varphi}.$$

Therefore

$$\hat{\varphi}(\xi) \neq 0, \quad \hat{\varphi}\left(\frac{\xi}{M}\right) = 0.$$

It is contrary to (40). When $d_1 < M^2 b_1$, take ε_1 with $Mb_1 - \frac{d_1}{M} < \varepsilon_1 < \min\left\{\frac{d_1}{M}, Mc_1 - \frac{d_1}{M}\right\}$, then

$$c_1 < \frac{d_1}{M} < \frac{d_1}{M} + \varepsilon_1 = \xi_1 < d_1, \quad b_1 < \frac{\xi_1}{M} < c_1.$$

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_s)$, $\xi_j \in [a_j, d_j] - (b_j, c_j)$, $j = 2, 3, \ldots, s$. Then we obtain

$$\xi \in \operatorname{supp}\hat{\varphi}, \quad \frac{\xi}{M} \notin \operatorname{supp}\hat{\varphi}.$$

Therefore

$$\hat{\varphi}(\xi) \neq 0, \quad \hat{\varphi}\left(\frac{\xi}{M}\right) = 0.$$

It is contrary to (40). Consequently, we have $\forall j = 1, 2, \ldots, s$, $\frac{d_j}{M} \leq c_j$.

(c) We prove that $\frac{d_j}{M} \leq b_j$, $j = 1, 2, \ldots, s$.

Suppose that there exists some j such that $b_j < \frac{d_i}{M}$. Without loss of generality, we set $j = 1$, i.e., $b_1 < \frac{d_1}{M}$. Let ξ with $b_1 < \xi_1 < \frac{d_1}{M} \leq c_1$. Then

$$c_1 \leq Mb_1 < M\xi_1 < d_1.$$

Let $\xi = (\xi_1, \xi_2, \ldots, \xi_s)$, $\xi_j \in \left[\frac{a_j}{M}, \frac{d_j}{M}\right] - \left(\frac{b_j}{M}, \frac{c_j}{M}\right)$, $j = 2, 3, \ldots, s$. Then we obtain

$$M\xi \in \operatorname{supp}\hat{\varphi}, \quad \xi \notin \operatorname{supp}\hat{\varphi}.$$

Therefore

$$\hat{\varphi}(\xi) \neq 0, \quad \hat{\varphi}\left(\frac{\xi}{M}\right) = 0$$

a contradiction is obtained. Thus, $\frac{d_j}{M} \leq b_j$, $j = 1, 2, \ldots, s$.

Using above results (a), (b) and (c), we obtain the inequality

$$a_j < \frac{d_j}{M} \le b_j, \quad j = 1, 2, \ldots, s. \tag{41}$$

Finally, we prove that $d_j - a_j \le 4M\pi,\ j = 1, 2, \ldots, s.$

When the inequality (41) holds, we have that

$$\begin{cases} m_0(\omega) = 0, & \text{otherwise}, \\ m_0(\omega) \ne 0, & \omega \in \prod_{j=1}^{s}\left[\frac{a_j}{M}, \frac{d_j}{M}\right] - \prod_{j=1}^{s}\left(\frac{b_j}{M}, \frac{c_j}{M}\right). \end{cases}$$

By the periodicity of $m_0(\omega)$, we obtain that

$$\frac{b_j}{M} - a_j \le 2\pi, \quad \frac{c_j}{M} - \frac{a_j}{M} \le 2\pi, \quad \frac{d_j}{M} - \frac{b_j}{M} \le 2\pi, \quad b_j - \frac{c_j}{M} \le 2\pi$$

or

$$\frac{d_j}{M} - a_j \le 2\pi, \quad b_j - \frac{a_j}{M} \le 2\pi, \quad \frac{d_j}{M} - \frac{b_j}{M} \le 2\pi, \quad b_j - \frac{c_j}{M} \le 2\pi.$$

Consequently,

$$d_j - a_j \le 4M\pi, \quad j = 1, 2, \ldots, s. \tag{42}$$

Therefore, we have

$$a_j < 0, \quad d_j > 0, \quad 2\pi < d_j - a_j \le 4M\pi, \quad a_j < \frac{d_j}{M} \le b_j.$$

The proof of Theorem 3 is completed. □

4. Conclusion

Suppose $\{\psi_\mu\}$ is an orthonormal wavelet of $L^2(R^s)$ and the whole support of its Fourier transform is

$$\bigcup_{\mu} \operatorname{supp}\{\hat{\psi}_\mu\} = \prod_{i=1}^{s}[A_i, D_i] - \prod_{i=1}^{s}(B_i, C_i),$$

$$A_i \le B_i \le C_i \le D_i, \quad i = 1, 2, \ldots, s.$$

A characterization of the above whole support is given by some equalities and inequalities. We have improved completely Long's results and generalized Zhang's results.

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