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SINGULARITIES OF DEFINABLE MAPS

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Abstract

Let $f : X \to Y$ be a definable map between definable sets X, Y and k be a positive integer. We prove that the C^k singular set of f is a definable subset of X with codimension at least 1.

1. Introduction

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. In this paper "definable" means "definable with parameters in $\mathcal{M}$", everything is considered in $\mathcal{M}$ and a definable map means a map with definable graph.

Any definable category is a generalization of the semialgebraic category. Many results in semialgebraic geometry hold true in the more general setting of an o-minimal expansion $\mathcal{M}$. There are other examples and constructions of them ([2], [4], [7]). General references on o-minimal structures are ([1], [3], [9]), and there exist uncountably many o-minimal expansions of $\mathcal{R}$ [8]. Definable sets and definable continuous maps are studied in [5].

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Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be definable set and $f : X \to Y$ be a definable map. For any positive integer k, ∞ or ω, we define the C^k *singular set* $\sum_k (f)$ of f to be $\{x \in X \mid f^{-1}(f(x))$ is not a C^k submanifold of $\mathbb{R}^n$ at $x\}$. Then by definition $\sum_1 (f) \subset \sum_2 (f) \subset \cdots \subset \sum_\infty (f) \subset \sum_\omega (f)$.

Theorem 1.1. *For any positive integer* k, $\sum_k (f)$ *is a definable subset of X with codimension at least* 1. *Here* $\dim \varnothing = -\infty$.

Problem 1.2. Are $\sum_\infty (f)$ and $\sum_\omega (f)$ definable? If they are definable and $\dim X \geq 2$, then do they have at least codimension 2?

Koike and Shiota [6] solved Problem 1.2 affirmatively if $\mathcal{M} = \mathcal{R}$. They also constructed a semialgebraic map such that for any positive integer k its C^k singular set has codimension 1 [6]. Thus in Theorem 1.1 we cannot state that $\sum_k (f)$ is a definable subset of X with codimension at least 2.

2. Proof of Theorem 1.1

The following two results are the definable triangulation theorem (8.2.9 [1]) and piecewise triviality (9.1.7 [1]).

Theorem 2.1 (8.2.9 [1]). (*Definable triangulation theorem*) *Let X be a definable set and* $X_1, \ldots, X_k$ *be definable subsets of X. Then there exists a definable triangulation* (M, τ) *of X compatible with* $X_1, \ldots, X_k$, *namely M is a simplicial complex and* τ *is a definable homeomorphism from X to a union of open simplexes of M such that each* $\tau(X_i)$ *is a union of open simplexes of M. In particular, if X is compact, then* $\tau(X) = M$.

Theorem 2.2 (9.1.7 [1]). (*Piecewise triviality*) *Let X, Y be definable sets and* $f : X \to Y$ *be a definable map. Then there exist a finite partition* $\{V_j\}_{j=1}^u$ *of Y into definable sets and a family of definable homeomorphisms* $\{\phi_j : f^{-1}(V_j) \to V_j \times f^{-1}(a_j)\}_{j=1}^u$ *such that for each* j $proj_{V_j} \circ \phi_j =$

$f|f^{-1}(V_j)$, *where* $proj_{V_j} : V_j \times f^{-1}(a_j) \to V_j$ *denotes the projection and* a_j *is some point of* V_j.

Since we can write the set of points at which f is continuous (resp. locally injective, locally surjective) by some sentence, it is definable.

Proof of definability of $\sum_k(f)$. Under assumption of Theorem 1.1, for a non-negative integer k, let A_k denote the set of points $x \in X$ at which $f^{-1}(f(x))$ is a C^k submanifold of $\mathbb{R}^n$, where a C^0 manifold of $\mathbb{R}^n$ means a topological manifold with the relative topology induced from $\mathbb{R}^n$.

To prove that $\sum_k(f)$ is definable, we first prove A_0 is definable.

Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the canonical projection. Replacing f by $\pi|\Gamma(f) : \Gamma(f) \to Y$, we may assume that f is continuous, where $\Gamma(f)$ denotes the graph of f. By Theorem 2.2, there exist a finite partition $\{V_j\}_{j=1}^u$ of Y into definable sets and a family of definable homeomorphisms $\{\phi_j : f^{-1}(V_j) \to V_j \times f^{-1}(a_j)\}_{j=1}^u$ such that for each j $proj_{V_j} \circ \phi_j = f|f^{-1}(V_j)$, where $proj_{V_j} : V_j \times f^{-1}(a_j) \to V_j$ denotes the projection and a_j is some point of V_j. To prove A_0 is definable, we may assume that $X = Y_1 \times Y$ for a definable set $Y_1 \subset \mathbb{R}^n$ and $f : Y_1 \times Y \to Y$ is the projection. Let A_0^1 be the set of points of Y_1 at which Y_1 is a topological submanifold of $\mathbb{R}^n \subset S^n \subset \mathbb{R}^{n+1}$. By Theorem 2.1, there exists a definable triangulation (M, τ) of S^n such that $\tau(Y_1)$ is a union of open simplexes of M. For $a_1, a_2 \in Int\ \sigma$, $a_1 \in A_0^1$ if and only if $a_2 \in A_0^1$, where $\sigma \in M$. Thus A_0 is definable.

For $0 \le j \le n$, let $A_{k,j} \subset A_k$ be the set of points $x \in X$ at which $p|f^{-1}(f(x)) : f^{-1}(f(x)) \to \mathbb{R}^j$ is a C^k diffeomorphism (homeomorphism if $k = 0$) locally at x, where $p : \mathbb{R}^n \to \mathbb{R}^j$ denotes the projection. Let

$p_1, p_2, \dots$ denote all the projections $\mathbb{R}^n \to \mathbb{R}^j$ forgetting some factors. If $k > 0$, then it suffices to show that each $A_{k,j}$ is definable because $\sum_k(f)$ is the complement of the union of $A_{k,j}$'s for some $p = p_l$, $0 \le i \le n$ in X.

We now consider $A_{0,j}$. Clearly $A_{0,j}$ is the set of points $x \in X$ such that $p \mid f^{-1}(f(x)) : f^{-1}(f(x)) \to \mathbb{R}^j$ is injective and surjective locally at x and $(p \mid f^{-1}(f(x)))^{-1}$ is continuous at $p(x)$. Thus

$A_{0,j} = \{x \in X \mid \exists \varepsilon > 0\ \forall x', x'' \in X$ if $|x - x'| < \varepsilon$, $|x - x''| < \varepsilon$, $x' \neq x''$ and $f(x) = f(x') = f(x'')$, then $p(x') \neq p(x'')$; $\forall \varepsilon > 0\ \exists \delta > 0\ \forall a' \in \mathbb{R}^j$ if $|p(x) - a'| < \delta$, then $\exists x' \in X$ such that $|x - x'| < \varepsilon$, $f(x) = f(x')$, $p(x') = a$; $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \varepsilon' > 0\ \exists \delta' > 0\ \forall a', a'' \in \mathbb{R}^j$ if $|p(x) - a'| < \delta$, $|p(x) - a''| < \delta$, $|a' - a''| < \delta'$, then $\exists x', x'' \in X$ such that $|x - x'| < \varepsilon$, $|x - x''| < \varepsilon$, $f(x) = f(x') = f(x'')$, $p(x') = a$, $p(x'') = a'\}$. Therefore $A_{0,j}$ is definable.

We next consider $A_{1,j}$. Let $B = \{(x, x') \in A_{0,j} \times A_{0,j} \mid f(x) = f(x')\}$ and $B_x = B \cap (\{x\} \times A_{0,j})$. Then B, B_x are definable and for each $x \in A_{0,j}$ the map $B_x \to \mathbb{R}^j$ defined by $(x, x') \mapsto p(x')$ is a local homeomorphism. Hence there exists a definable open neighborhood U of the diagonal of $A_{0,j}$ in B such that the map $U \cap B_x \to \mathbb{R}^j$ defined by $(x, x') \mapsto p(x')$ is a homeomorphism onto an open set V_x in $\mathbb{R}^j$. For any $x \in A_{0,j}$, let $q_x : V_x \to \mathbb{R}^n$ be the composition of the inverse map $V_x \to B_x$ and the projection $B_x \ni (x, x') \to x' \in A_{0,j} \subset \mathbb{R}^n$. Let $V = \cup_x \{x\} \times V_x \subset A_{0,j} \times \mathbb{R}^j$, where the union is taken over $A_{0,j}$ and $q(x, a) = q_x(a)$ for $(x, a) \in V$. Then V and $q : V \to \mathbb{R}^n$ are definable, q_x is a homeomorphism onto its image containing x, and $A_{1,j} = \{x \in A_{0,j} \mid q_x$ is a C^1 imbedding at $p(x)\}$.

Hence it suffices to prove the following assertion.

Assertion $*_1$**.** Let C, $D \subset C \times \mathbb{R}^j$ be definable sets and $\phi : D \to \mathbb{R}^n$ be a definable map. If for each $x \in C$, $D_x = D \cap (\{x\} \times \mathbb{R}^j)$ is open in $\{x\} \times \mathbb{R}^j$ and $\phi | D_x$ is a homeomorphism onto its image, then $D^1 = \{(x, y) \in D | \phi | D_x$ is a C^1 imbedding at $(x, y)\}$ is definable.

Let

$$\widetilde{D} = \{(x, y, y', t) \in D \times \mathbb{R}^j \times (0, 1] | \forall s \in [0, 1], (x, y + sy') \in D\},$$

$$\widetilde{\phi}(x, y, y', t) = (\phi(x, y + ty') - \phi(x, y))/t \text{ for } (x, y, y', t) \in \widetilde{D},$$

$$G = (D \times \mathbb{R}^j \times \{0\} \times \mathbb{R}^n) \cap \overline{graph\, \widetilde{\phi}},$$

$G_{x, y} = (\{(x, y)\} \times \mathbb{R}^j \times \{0\} \times \mathbb{R}^n) \cap G$ for $(x, y) \in D$, and let $\rho_1 : G \to \mathbb{R}^j$ and $\rho_2 : G \to \mathbb{R}^n$ be the projections, where $\overline{graph\, \widetilde{\phi}}$ denotes the closure of *graph* $\widetilde{\phi}$. Then $\widetilde{D}$, $\widetilde{\phi} : \widetilde{D} \to \mathbb{R}^n$, G, $G_{x, y}$, ρ_1 and ρ_2 are definable and $D^1 = \{(x, y) \in D | \rho_1 | G_{x, y}$ and $\rho_2 | G_{x, y}$ are homeomorphism onto $\mathbb{R}^j\}$. As in the first argument, D^1 is definable.

Let $k \geq 2$. By the above argument, Assertion $*_k$ which is similarly defined by replacing D^1 in Assertion $*_1$ by $D^k = \{(x, y) \in D^{k-1} | \phi | D^{k-1}$ is a C^k imbedding at $(x, y)\}$ implies that $A_{k, j}$ is definable.

Assertion $*_2$ is proved as follows. Let $E = D^1 \times \mathbb{R}^j$, $\psi : D \times \mathbb{R}^j \to \mathbb{R}^n \times \mathbb{R}^n$, $\psi(x, y, y') = (\phi(x, y), d(\phi | D^1_x)_y y')$, where d denotes the differential operator. Then E and $\psi : E \to \mathbb{R}^n \times \mathbb{R}^n$ are definable and for any $x \in C$, $E_x = E \cap (\{x\} \times \mathbb{R}^j \times \mathbb{R}^j)$ is open in $\{x\} \times \mathbb{R}^j \times \mathbb{R}^j$ and $\psi | E_x$ is a homeomorphism onto its image. Thus by Assertion $*_1$, $E^1 = \{(x, y, y') \in E | \psi | E_x$ is a C^1 imbedding at $(x, y, y')\}$ is definable. Since $D^2 = \{(x, y) \in D^1 | \forall y \in \mathbb{R}^j, \psi | E_x$ is a C^1 imbedding at $(x, y, y')\}$, D^2 is definable.

Using induction on k, we have Assertion $*_k$.

Therefore $\sum_k (f)$ is definable. □

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be definable sets, $A \subset X$ be a definable subset of X and $f : X \to Y$ be a definable map. For any $y \in f(A)$, let $\sum_k (f^{-1}(y)) = \{x \in f^{-1}(y) | f^{-1}(y)$ be a definable C^k manifold in $\mathbb{R}^n$ at $x\}$. Then $\sum_k (f^{-1}(y)) = \sum_k (f) \cap f^{-1}(y)$.

By Theorem 2.1, we have the following lemma.

Lemma 2.3. *For any* $y \in f(A)$, $\dim \sum_k (f^{-1}(y)) < \dim f^{-1}(y)$.

Lemma 2.4. *Let* $f : X \to Y$ *be a definable continuous map and* b *be a positive integer. If* $\dim(A \cap f^{-1}(y)) + b \leq \dim f^{-1}(y)$ *for any* $y \in f(A)$, *then* $\dim A + b \leq \dim X$.

Proof. By Theorem 2.2, there exists a finite partition of $f(X)$ into definable sets R_i, and for any i there exist a definable set $D_i \subset \mathbb{R}^n$ and a definable homeomorphism $\phi_i : D_i \times R_i \to f^{-1}(R_i)$ compatible with the projection onto R_i. Moreover there exists a finite partition of $f(A)$ into definable sets S_j, and for any j there exist a definable set $E_j \subset \mathbb{R}^n$ and a definable homeomorphism $\psi_j : E_j \times S_j \to A \cap f^{-1}(S_j)$ compatible with the projection onto S_j. By Theorem 2.1, we have a finite partition of $f(A)$ into Nash manifolds N_k compatible with R_i's and S_j's. Namely for any k, there exist some $i(k)$, $j(k)$ such that $\phi_{i(k)} | D_{i(k)} \times N_k : D_{i(k)} \times N_k \to f^{-1}(N_k)$, $\psi_{j(k)} | E_{j(k)} \times N_k : E_{j(k)} \times N_k \to A \cap f^{-1}(N_k)$ are definable homeomorphisms compatible with the projections. Thus $\dim f^{-1}(y) + \dim N_k = \dim D_{i(k)} + \dim N_k = \dim f^{-1}(N_k) \leq \dim X$, $y \in N_k$. Moreover there exists k_0 such that $\dim(A \cap f^{-1}(y)) + \dim N_{k_0} = \dim E_{j(k_0)} + \dim N_{k_0}$

$= \dim(A \cap f^{-1}(N_{k_0})) = \dim A,\ y \in N_{k_0}$. Assume that $\dim A + b > \dim X$. Then $\dim(A \cap f^{-1}(y)) + \dim N_{k_0} + b > \dim f^{-1}(y) + \dim N_{k_0},\ y \in N_{k_0}$. Hence we have $\dim(A \cap f^{-1}(y)) + b > \dim f^{-1}(y),\ y \in N_{k_0}$. This contradiction proves the result. □

Proof of $\sum_k (f)$ with codimension at least 1. Definability of $\sum_k (f)$, Lemma 2.3 and Lemma 2.4 prove that $\sum_k (f)$ has codimension at least 1. □

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