Far East Journal of Mathematical Sciences (FJMS)
Volume 31, Issue 2, 2008, Pages 311-318
Published online: December 31, 2008
This paper is available online at http://www.pphmj.com

ON NONSINGULARITY AND RIGHT EIGENVALUES OF A QUATERNION CIRCULANT MATRIX

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Abstract

Let A be an $n \times n$ quaternion circulant matrix. Then the complex representation A_c of A is similar to a complex matrix $\operatorname{diag}(M_0^A, M_1^A, \ldots, M_{n-1}^A)$. By using this property, a necessary and sufficient condition of A being nonsingular is given, and a system of right eigenvalues of A is determined.

1. Introduction

Let R be the *real field*, $C = \{a + bi \mid a, b \in R\}$ be the *complex field*, and $H = \{x + yj \mid x, y \in C\}$ be the *quaternion field*, where $i^2 = j^2 = -1$ and $ij = -ji$. By custom, the *conjugate* of a quaternion α is denoted by $\overline{\alpha}$. That is to say, $\overline{\alpha} = a - bi$ if $\alpha = a + bi \in C$, and $\overline{\alpha} = \overline{x} - yj$ if $\alpha = x + yj \in H$. The *module* $\sqrt{\alpha\overline{\alpha}}$ of a quaternion α is denoted by $|\alpha|$.

2000 Mathematics Subject Classification: 15A33, 15A18.

Keywords and phrases: quaternion circulant matrix, complex representation, nonsingularity, system of right eigenvalue.

Received October 1, 2008

The matrices with entries in H are called *quaternion matrices*. Especially, the matrices with entries in C are called *complex matrices*. The researches of complex matrices have made giant development. Since the multiplication of H loss commutativity, the consideration of quaternion matrices is much more difficult than that of complex matrices. However, some important natures of quaternion matrices have been explored. For example, Huang [2] gave a necessary and sufficient condition for a quaternion matrix being nonsingular; Zhang [6] described the characteristics of the set of all right eigenvalues of a quaternion matrix. But Huang's and Zhang's results are abstract and unfeasible in algorithm. In this paper, we improve these results in quaternion circulant matrices.

2. Nonsingularity of a Quaternion Circulant Matrix

Let $A = (\alpha_{ij})$ be an $n \times n$ quaternion matrix. We customarily denote the *conjugate*, the *transpose* and the *conjugate transpose* of A by $\overline{A}$, A^T and A^*, respectively. That is to say,

$$\overline{A} = (\overline{\alpha_{ij}}), \quad A^T = (\alpha_{ji}), \quad A^* = (\overline{\alpha_{ji}}).$$

A *quaternion circulant matrix* is a quaternion matrix of the form

$$\operatorname{Circ}(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_0 \end{pmatrix}.$$

The set of $n \times n$ quaternion circulant matrices is denoted by $\operatorname{Circ}_n(H)$. The $n \times n$ quaternion circulant matrix

$$P = \operatorname{Circ}(0, 1, 0, 0, \ldots, 0)$$

is called the $n \times n$ *basic circulant matrix*. Quaternion circulant matrices with entries in C are called *complex circulant matrices*. Since were raised by Good [1], complex circulant matrices have been systematically investigated and widely used in coding, statistics, theoretical physics, structural analysis, digital image processing and so on [5].

It is evident that an $n \times n$ quaternion matrix A is a quaternion circulant matrix if and only if there exist $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in H$ such that

$$A = \sum_{k=0}^{n-1} \alpha_k P^k.$$

Furthermore, if A, B are $n \times n$ quaternion circulant matrices, then so do $\overline{A}$, A^T, A^*, $\alpha A \beta$, $A + B$ and AB, where α, β are arbitrary quaternions.

Let

$$\omega_k = \exp\left(\frac{2k\pi i}{n}\right), \quad (k = 0, 1, \ldots, n-1),$$

be the *unity roots of order n*, and let

$$F_n = n^{-\frac{1}{2}} (\omega_{k-1}^{l-1})_{n \times n}$$

be the $n \times n$ *Fourier matrix*. If $f(\lambda) = \sum_{l=0}^{m} a_l \lambda^l$ is a polynomial of complex coefficients, then, for each $k = 0, 1, \ldots, n-1$, we have

$$\overline{f(\omega_k)} = \overline{\sum_{l=0}^{m} a_l \omega_k^l} = \sum_{l=0}^{m} \overline{a_l}\, \overline{\omega_k}^l = \sum_{l=0}^{m} \overline{a_l}\, \omega_{n-k}^l = \bar{f}(\omega_{n-k}).$$

This leads to the following

Lemma 2.1. *If* $f(\lambda)$ *is a polynomial of complex coefficients*, *then*

$$\overline{f(\omega_k)} = \bar{f}(\omega_{n-k}) \quad (k = 0, 1, \ldots, n-1). \qquad \square$$

Recall that the *complex representation of a quaternion* $\alpha = a + bj$ is defined by the 2×2 complex matrix

$$(\alpha)_c = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

Huang [2] advised the *complex representation* of an $n \times n$ quaternion matrix A by the $2n \times 2n$ complex matrix

$$A_c = ((\alpha_{ij})_c)$$

and then established the following lemma.

Lemma 2.2 [2]. *An $n \times n$ quaternion matrix A is nonsingular if and only if A_c is nonsingular, and if and only if* $\det A_c \neq 0$. □

If $A = \mathrm{Circ}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathrm{Circ}_n(H)$, where $\alpha_k = a_k + b_k j \in H$ for each $k = 0, 1, \ldots, n-1$, we define

$$f^A(\lambda) = \sum_{k=0}^{n-1} a_k \lambda^k, \quad \bar{f}^A(\lambda) = \sum_{k=0}^{n-1} \overline{a_k} \lambda^k, \quad g^A(\lambda) = \sum_{k=0}^{n-1} b_k \lambda^k, \quad \bar{g}^A(\lambda) = \sum_{k=0}^{n-1} \overline{b_k} \lambda^k,$$

and let

$$M_k^A = \begin{pmatrix} f^A(\omega_k) & g^A(\omega_k) \\ -\bar{g}^A(\omega_k) & \bar{f}^A(\omega_k) \end{pmatrix}.$$

Then, we have

Theorem 2.3. *An $n \times n$ quaternion circulant matrix A is nonsingular if and only if*

$$\det M_k^A \neq 0 \quad \left(k = 0, 1, \ldots, \left[\frac{n}{2}\right]\right).$$

Proof. Suppose that $A = \mathrm{Circ}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathrm{Circ}_n(H)$, where $\alpha_k = a_k + b_k j \in H$ for each $k = 0, 1, \ldots, n-1$. Note that

$$A_c = \sum_{k=0}^{n-1} P^k \otimes (\alpha_k)_c,$$

here $\otimes$ is Kronecker product of matrices. By direct calculations we can obtain that

$$(F_n \otimes I_2)^* A_c (F_n \otimes I_2) = \mathrm{diag}\left(\sum_{k=0}^{n-1} \omega_0^k (\alpha_k)_c, \sum_{k=0}^{n-1} \omega_1^k (\alpha_k)_c, \ldots, \sum_{k=0}^{n-1} \omega_{n-1}^k (\alpha_k)_c \right).$$

Furthermore, since for each $l = 0, 1, \ldots, n-1$,

$$\sum_{k=0}^{n-1} \omega_l^k (\alpha_k)_c = \begin{pmatrix} \sum_{k=0}^{n-1} a_k \omega_l^k & \sum_{k=0}^{n-1} b_k \omega_l^k \\ -\sum_{k=0}^{n-1} \overline{b_k} \omega_l^k & \sum_{k=0}^{n-1} \overline{a_k} \omega_l^k \end{pmatrix} = M_l^A,$$

we have

$$(F_n \otimes I_2)^* A_c (F_n \otimes I_2) = \operatorname{diag}(M_0^A, M_1^A, \ldots, M_{n-1}^A), \tag{1}$$

and whence

$$\det A_c = \prod_{l=0}^{n-1} (\det M_l^A).$$

By Lemma 2.1, we can see that

$$\begin{aligned}\det M_l^A &= f^A(\omega_l)\bar{f}^A(\omega_l) + g^A(\omega_l)\bar{g}^A(\omega_l)\\ &= \overline{f^A(\omega_{n-l})\bar{f}^A(\omega_{n-l}) + g^A(\omega_{n-l})\bar{g}^A(\omega_{n-l})}\\ &= \overline{\det M_{n-l}^A}.\end{aligned}$$

Therefore

$$\det A_c = \begin{cases} (\det M_0^A)\prod_{l=1}^{p}[(\det M_l^A)\overline{(\det M_l^A)}] & \text{if } n = 2p+1,\\ (\det M_0^A)(\det M_p^A)\prod_{l=1}^{p-1}[(\det M_l^A)\overline{(\det M_l^A)}] & \text{if } n = 2p. \end{cases}$$

Now, by Lemma 2.2, the required result holds. □

Corollary 2.4. *Let* $A = \operatorname{Circ}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \operatorname{Circ}_n(H)$. *If* $\sum_{k=0}^{n-1} \alpha_k = 0$, *then* A *is singular*.

Proof. This claim follows by Theorem 2.3 directly since $\det M_0^A = \left|\sum_{k=0}^{n-1} \alpha_k\right|^2$. □

3. Right Eigenvalues of a Quaternion Circulant Matrix

Let $\sim$ be the *similar relation* on H defined by $\alpha \sim \beta$ if $\alpha = \gamma\beta\gamma^{-1}$ for some nonzero quaternion γ. Then, of course, this relation is an equivalence relation. For each $\alpha \in H$, we denote the $\sim$-class of α by $\tilde{\alpha}$. Observe that $\tilde{\alpha}$ contains exactly one pair of complex numbers, say x, y, with $x = \bar{y}$ [6].

Let A be an $n \times n$ quaternion matrix. It is shown by [3, 4] that A must have a right eigenvalue $\lambda \in C$ and, in this case, each one of the ~-class $\widetilde{\lambda}$ is also a right eigenvalue of A. Accordingly, counting multiplicities, A has n complex right eigenvalues say $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ such that the set of all right eigenvalues of A is $\bigcup_{k=0}^{n-1} \widetilde{\lambda_k}$. We call the set $\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ a *system of right eigenvalues of A*. To determine a system of right eigenvalues of a quaternion circulant matrix, the following lemma is needed.

Lemma 3.1 [6]. *Let* α *be a quaternion. Then* $(\alpha)_c$ *has two complex eigenvalues which are conjugate each other.* □

Lemma 3.2 [3]. *Let* $A = A_1 + A_2 j$ *be an* $n \times n$ *quaternion matrix, where* A_1, A_2 *are* $n \times n$ *complex matrices, and let*

$$A_{cr} = \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}.$$

If $\lambda_0, \overline{\lambda_0}, \lambda_1, \overline{\lambda_1}, \ldots, \lambda_{n-1}, \overline{\lambda_{n-1}}$ *are all complex eigenvalues of the matrix* A_{cr}, *then* $\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ *is a system of right eigenvalues of A.* □

It is evident that, for any $n \times n$ quaternion matrix A, the matrices A_c and A_{cr} are similar. Therefore, we have

Corollary 3.3. *Let A be an* $n \times n$ *quaternion matrix. If* $\lambda_0, \overline{\lambda_0}, \lambda_1, \overline{\lambda_1}, \ldots, \lambda_{n-1}, \overline{\lambda_{n-1}}$ *are all complex eigenvalues of the matrix* A_c, *then* $\{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ *is a system of right eigenvalues of A.* □

The following theorem is the main result of this section.

Theorem 3.4. *Let* $A = \mathrm{Circ}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \mathrm{Circ}_n(H)$, *where* $\alpha_k = a_k + b_k j \in H$ *for each* $k = 0, 1, \ldots, n-1$. *For every* $\varepsilon = 1, 2$ *and every* $l = 0, 1, \ldots, n-1$, *put*

$$\lambda_l^{(\varepsilon)} = \sum_{k=0}^{n-1} \frac{a_k + \overline{a_k}}{2} \omega_l^k + (-1)^{\varepsilon} \sqrt{\left(\sum_{k=0}^{n-1} \frac{a_k - \overline{a_k}}{2} \omega_l^k \right)^2 - \sum_{k=0}^{n-1} \left(\sum_{s+t=k} b_s \overline{b_t} + \sum_{s+t=n+k} b_s \overline{b_t} \right) \omega_l^k}.$$

Let

$$\mathscr{RE} = \begin{cases} \{\lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_p^{(1)}, \lambda_p^{(2)}\} & \text{if } n = 2p+1, \\ \{\lambda_0^{(1)}, \lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_{p-1}^{(1)}, \lambda_{p-1}^{(2)}, \lambda_p^{(1)}\} & \text{if } n = 2p. \end{cases}$$

Then $\mathscr{RE}$ *is a system of right eigenvalues of* A.

Proof. By equality (1), we can see that the set of all complex right eigenvalues of A_c is exactly the union of those of M_l^A's. For each $l = 0, 1, \ldots, n-1$, since

$$\det(\lambda I - M_l^A) = \lambda^2 - [f^A(\omega_l) + \bar{f}^A(\omega_l)]\lambda + f^A(\omega_l)\bar{f}^A(\omega_l) + g^A(\omega_l)\bar{g}^A(\omega_l),$$

the complex eigenvalues of M_l^A are

$$\lambda_l^{(\varepsilon)} = \frac{1}{2}[f^A(\omega_l) + \bar{f}^A(\omega_l)]$$

$$+ (-1)^{\varepsilon}\frac{1}{2}\sqrt{[f^A(\omega_l) + \bar{f}^A(\omega_l)]^2 - 4[f^A(\omega_l)\bar{f}^A(\omega_l) + g^A(\omega_l)\bar{g}^A(\omega_l)]}$$

$$= \frac{1}{2}[f^A(\omega_l) + \bar{f}^A(\omega_l)] + (-1)^{\varepsilon}\frac{1}{2}\sqrt{[f^A(\omega_l) - \bar{f}^A(\omega_l)]^2 - 4g^A(\omega_l)\bar{g}^A(\omega_l)}$$

$$= \sum_{k=0}^{n-1}\frac{a_k + \overline{a_k}}{2}\omega_l^k$$

$$+ (-1)^{\varepsilon}\sqrt{\left(\sum_{k=0}^{n-1}\frac{a_k - \overline{a_k}}{2}\omega_l^k\right)^2 - \sum_{k=0}^{n-1}\left(\sum_{s+t=k} b_s\overline{b_t} + \sum_{s+t=n+k} b_s\overline{b_t}\right)\omega_l^k}$$

$(\varepsilon = 1, 2)$.

Furthermore, by Lemma 2.1, we have

$$\overline{M_l^A} = \begin{pmatrix} \bar{f}^A(\omega_{n-l}) & \bar{g}^A(\omega_{n-l}) \\ -g^A(\omega_{n-l}) & f^A(\omega_{n-l}) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M_{n-l}^A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}.$$

Hence the Jordan canonical forms of $\overline{M_l^A}$ and M_{n-l}^A coincide. It follows that the Jordan canonical forms of M_l^A and M_{n-l}^A are conjugate each other, so that

$$\{\overline{\lambda_l^{(1)}}, \overline{\lambda_l^{(2)}}\} = \{\lambda_{n-l}^{(1)}, \lambda_{n-l}^{(2)}\}.$$

Since $M_0^A = (f^A(1) + g^A(1))_c$, by Lemma 3.1, we claim that

$$\lambda_0^{(2)} = \overline{\lambda_0^{(1)}}.$$

Further, if $n = 2p$, then $M_p^A = (f^A(\omega_p) + g^A(\omega_p))_c$, thus it follows by Lemma 3.1 that

$$\lambda_p^{(2)} = \overline{\lambda_p^{(1)}}.$$

By virtue of Corollary 3.3 and summarizing the above equalities, we can obtain the required conclusion. □

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