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ON THE DIOPHANTINE EQUATION $5x^2 + 1 = 6^n$

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Abstract

Let x and n be positive integers. In this paper, we solve the title equation completely.

Introduction

Let a, b, x, y, n be positive integers, where $\gcd(a, b) = 1$. The history of the diophantine equation $ax^2 + b = y^n$, $n \geq 1$ is very rich. Lebesgue [5] was the first to obtain a non-trivial result, he proved that the above equation has no solutions if $a = b = 1$. In 1993, Cohn [4] solved this equation for $a = 1$ and a several values of the parameter b in the range $1 \leq b \leq 100$. Recently the first author proved that the diophantine equation $ax^2 + 4d = y^n$ has no solution under some conditions [1], and in 2008 she solved the diophantine equation $px^2 + q^{2m} = y^n$ completely [2]. Also she found with Luca and Togbe all the solutions of the diophantine equation $x^2 + 5^a 13^b = y^n$ [3].

In this paper we study the equation $5x^2 + 1 = 6^n$, $n \geq 1$, we apply the unique factorization in the imaginary quadratic field $\mathbb{Q}(\sqrt{-5})$ to reduce the problem to a question about a Fibonacci-type integer sequence.

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We need the following lemma to prove our main result:

Lemma. *Let* $\theta = 1 + \sqrt{-5}$ *and* $\alpha = \sqrt{-5}$. *Then the only rational integral solutions* $n \geq 1$ *of the equation* $\alpha\theta^n - \overline{\alpha}\overline{\theta}^n = \alpha - \overline{\alpha}$ *or* $-(\alpha - \overline{\alpha})$ *is* $n = 1$, *where* $\overline{\theta}$ *and* $\overline{\alpha}$ *denote the algebraic conjugates of* θ *and* α *respectively.*

Proof. It is easy to check that the equation

$$\sqrt{-5}(1 + \sqrt{-5})^n + \sqrt{-5}(1 - \sqrt{-5})^n = 2\sqrt{-5} \text{ or } -2\sqrt{-5},$$

is true only if $n = 1$. □

Now we give the main result:

Theorem. *The diophantine equation*

$$5x^2 + 1 = 6^n, \tag{1}$$

has a unique solution in positive integers (x, n) *given by* $(1, 1)$.

Proof. Let $\mathbb{Z}$ be the set of rational integers and R denotes the ring of algebraic integers in the quadratic field $\mathbb{Q}(\sqrt{-5})$. Then $R = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$, which is a unique factorization domain [6].

Let $\theta = 1 + \sqrt{-5}$ and $\overline{\theta} = 1 - \sqrt{-5}$. Then $\theta\overline{\theta} = 6$ and $\theta^2 = 2\theta - 6$.

We can factorize equation (1) in the ring R as

$$(1 + x\sqrt{-5})(1 - x\sqrt{-5}) = \theta^n\overline{\theta}^n.$$

Note that θ and $\overline{\theta}$ are irreducible in R. If $\theta\overline{\theta} \mid 1 + x\sqrt{-5}$, then there exist rational integers i, j, $1 \leq i$, $j \leq n$ such that

$$1 + x\sqrt{-5} = \pm\theta^i\overline{\theta}^j.$$

Suppose $i \leq j$, then

$1 + x\sqrt{-5} = \pm(\theta\overline{\theta})^i\overline{\theta}^{j-i}$, $\qquad (\overline{\theta} = 2 - \theta)$

$$= \pm 6^i (2 - \theta)^{i-j}$$

$$= \pm 6^i \left(2^{i-j} - \begin{pmatrix} i-j \\ 1 \end{pmatrix} 2^{i-j-1}\theta + \begin{pmatrix} i-j \\ 1 \end{pmatrix} 2^{i-j-2}\theta^2 + \cdots + (-1)^{i-j}\theta^{i-j} \right)$$

$$= \pm 6^i (A + B\sqrt{-5}),$$

where A and B are rational integers.

Since $\{1, \sqrt{-5}\}$ is an integral basis of R, the equality $1 + x\sqrt{-5} =$ $= \pm 6^i (A + B\sqrt{-5})$ is impossible. Similarly if $j < i$, we get a contradiction.

This implies that $\theta\overline{\theta}$ does not divide $1 + x\sqrt{-5}$, similarly $\theta\overline{\theta}$ does not divide $1 - x\sqrt{-5}$, hence we have

$$\theta^n = \pm(1 + x\sqrt{-5}) \text{ or } \theta^n = \pm(1 - x\sqrt{-5}).$$

So

$$\sqrt{-5}\theta^n = \pm(\sqrt{-5} - 5x) \text{ or } \sqrt{-5}\theta^n = \pm(\sqrt{-5} + 5x)$$

which can be written as

$$\sqrt{-5}\theta^n = a + \theta \text{ or } \sqrt{-5}\theta^n = a - \theta.$$

Now

$$\sqrt{-5}\theta^m = a + \theta,$$

$$a + 1 = \sqrt{-5}\theta^m - \sqrt{-5},$$

$$(a+1)^2 + 5 = 56^m.$$

So $\left(\frac{a+1}{5}, m\right)$ is a solution of $5x^2 + 1 = 6^n$. This means that equation (1) has a positive rational solution (x, n) for $n = m$ if and only if

$$\sqrt{-5}\theta^m = a + \theta \text{ or } \sqrt{-5}\theta^m = a - \theta \text{ for } a \in \mathbb{Z}.$$

The problem now is to determine exactly those powers n such that $\sqrt{-5}\theta^m$ can be written either in the form $a + \theta$ or $a - \theta$ for $a \in \mathbb{Z}$.

Since $\{1, \sqrt{-5}\}$ is also an integral basis of R, then we can write

$$\sqrt{-5}\theta^n = a_n + b_n\sqrt{-5}, \quad a_n, b_n \in \mathbb{Z}.$$

We have

$$\begin{aligned} a_{n+1} + b_{n+1}\theta &= \sqrt{-5}\theta^{n+1} \\ &= (\sqrt{-5}\theta^n)\theta \\ &= (a_n + b_n\theta)\theta \\ &= -6b_n + (a_n + 2b_n)\theta, \end{aligned}$$

which implies that

$$b_{n+2} = 2b_{n+1} - 6b_n.$$

The first terms of the sequence b_n are

$$1, -4, -14, -4, 76, 176, \ldots.$$

Since $b_1 = 1$ we find one solution $(x, n) = (1, 1)$. Now we will prove there are no further occurrences of 1, -1 in the sequence $\{b_n\}_{n=1}^{\infty}$. Now for $\alpha = \sqrt{-5}$, $\theta = 1 + \sqrt{-5}$, we have $\alpha\theta = \sqrt{-5}(1 + \sqrt{-5}) = b_2 - b_1\overline{\theta}$.

Suppose for all rational integers k, $1 \leq k \leq n$, that $\alpha\theta^k = b_{k+1} - b_k\overline{\theta}$. Then we have

$$\begin{aligned} \alpha\theta^{n+1} &= (\alpha\theta^n)\theta \\ &= (b_{k+1} - b_k\overline{\theta})(2 - \overline{\theta}) \\ &= 2b_{k+1} - 2b_k\overline{\theta} - b_{k+1}\overline{\theta} + 2b_k\overline{\theta} - 6b_k \\ &= (2b_{k+1} - 6b_k) - b_{k+1}\overline{\theta} \\ &= b_{k+2} - b_{k+1}\overline{\theta}. \end{aligned}$$

So $\alpha\theta^n = b_{n+1} - b_n\overline{\theta}$, for $n \geq 1$. Similarly $\overline{\alpha}\overline{\theta}^n = b_{n+1} - b_n\theta$, it follows that

$$\alpha\theta^n - \overline{\alpha}\overline{\theta}^n = b_n(\theta - \overline{\theta}) = b_n(\alpha - \overline{\alpha}),$$

which implies that

$$b_n = \frac{\alpha\theta^n - \overline{\alpha}\overline{\theta}^n}{\alpha - \overline{\alpha}}.$$

By the Lemma $b_n = 1$ or -1 only for $n = 1$, so the equation (1) has a unique positive solution $(1, 1)$. □

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