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ON PROJECTIVE GROUP RINGS WITH AN INNER AUTOMORPHISM GROUP

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Abstract

Let RG_f be a projective group ring of a finite group G over a ring R with 1 with a factor set $f : G \times G \to$ units of the center of R and $\overline{G}$ be the inner automorphism group induced by the generators of RG_f. Characterizations of a Galois RG_f with inner Galois group $\overline{G}$ in terms of the center of RG_f are given.

1. Introduction

Galois extensions with an inner Galois group have been intensively investigated [1, 4-7]. In [1], it was shown that any central Galois algebra A with an inner Galois group G is a projective group algebra RG_f over R; that is, $A = RG_f = \oplus\sum_{g\in G} RU_g$, where $\{U_g \mid g \in G\}$ are free generators such that $U_gU_{g'} = U_{gg'}f(g, g')$, $f : G \times G \to$ units of R is a factor set, and $A^G = R$ [1, Theorem 6]. The converse also holds: If $A = RG_f = \oplus\sum_{g\in G} RU_g$ is an Azumaya R-algebra, then A is a central Galois

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algebra over R with an inner Galois group $\overline{G}$ induced by $\{U_g \mid g \in G\}$ [2, Theorem 3]. In the present paper, we consider two general cases: (1) the center of RG_f is not necessarily R, and (2) R is noncommutative. We shall give equivalent conditions for a Galois projective group algebra and for a Galois projective group ring respectively. Let Z be the center of G, $K = \{g \in Z \mid f(g, g') = f(g', g)$ for all $g' \in G\}$, and C be the center of RG_f. Then it will be shown that a projective group algebra RG_f is a central Galois algebra over C with an inner Galois group $\overline{G}$ if and only if $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over C and $C = \oplus\sum_{g\in K} RU_g$, where $U_{\overline{g}} = U_g$ for each $g \in G$. In particular, when $K = \langle 1\rangle$, this result recovers Theorem 3 in [2]. Moreover, characterizations for a Galois projective group ring and examples are given in Section 3.

2. Galois Projective Group Algebras

In this section, let $RG_f = \oplus\sum_{g\in G} RU_g$ be a projective group algebra over a commutative ring R with 1 and $f : G \times G \to$ units of R be a factor set, C be the center of RG_f, Z be the center of G, and $K = \{g \in Z \mid f(g, g') = f(g', g)$ for all $g' \in G\}$. We shall characterize a Galois projective group algebra RG_f with an inner Galois group $\overline{G}$ induced by $\{U_g \mid g \in G\}$ in terms of C. We begin to describe $\overline{G}$ and C.

Lemma 2.1. *Let $\overline{G} = \{\overline{g} \mid \overline{g}(x) = U_g x U_g^{-1}$ for all $x \in RG_f\}$. Then $\overline{G}$ is an inner automorphism group of RG_f.*

Proof. Since $\{\overline{g} \mid g \in G\}$ is a finite set, it suffices to show that $\overline{g} \cdot \overline{g'} = \overline{gg'}$ for $g, g' \in G$. In fact, for each $x \in RG_f$, $(\overline{g} \cdot \overline{g'})(x) = U_g U_{g'} x U_{g'}^{-1} U_g^{-1} = U_{gg'} f(g, g') x U_{gg'}^{-1} f(g, g')^{-1} = U_{gg'} x U_{gg'}^{-1} = \overline{gg'}(x)$. Thus $\overline{g} \cdot \overline{g'} = \overline{gg'}$. Also $(\overline{G}, \cdot)$ is associative, so $\overline{G}$ is a group.

Lemma 2.2. *Let* $\pi : G \to \overline{G}$ *by* $\pi(g) = \overline{g}$ *for each* $g \in G$. *Then* π *is a group homomorphism from* G *onto* $\overline{G}$ *with kernel* $K = \{g \in Z \mid f(g, g') = f(g', g)$ *for all* $g' \in G\}$.

Proof. For $g, g' \in G$, $\pi(gg') = \overline{gg'} = \overline{g} \cdot \overline{g'} = \pi(g) \cdot \pi(g')$, so π is a group homomorphism from G onto $\overline{G}$. Next, let $\overline{g} = \overline{1}$ in $\overline{G}$. Then $\overline{g}(x) = x$ for all $x \in RG_f$. Hence $U_g x U_g^{-1} = x$, and so, $U_g x = x U_g$. In particular, let $x = U_{g'}$ for each $g' \in G$, we have that $U_g U_{g'} = U_{g'} U_g$. Thus $U_{gg'} f(g, g') = U_{g'g} f(g', g)$. This is equivalent to that $gg' = g'g$ and $f(g, g') = f(g', g)$ for each $g' \in G$, and so $g \in K$. Therefore the kernel of $\pi = K$.

Theorem 2.3. *Let* C *be the center of* RG_f. *Then* (1) $(RG_f)^{\overline{G}} = C$, *and* (2) RG_f *is a Galois* C*-algebra with an inner Galois group* $\overline{G}$ *if and only if* $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ *are free over* C *and* $C = \oplus \sum_{g \in K} RU_g$, *where* $U_{\overline{g}} = U_g$ *for each* $g \in G$.

Proof. (1) Since R is commutative and $rU_g = U_g r$ for each $r \in R$ and $g \in G$, $(RG_f)^{\overline{G}} = C$.

(2) ($\Rightarrow$) Since $(RG_f)^{\overline{G}} = C$ by part (1), the Galois algebra RG_f is a central Galois algebra over C with an inner Galois group $\overline{G}$ by Lemma 2.1. Hence by Theorem 6 in [1], $RG_f = C\overline{G}_f$, a projective group algebra of $\overline{G}$ over C with a factor set $f : \overline{G} \times \overline{G} \to$ units of the center of C induced by $f : G \times G \to$ units of R. This implies that $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over C and $RG_f (= C\overline{G}_f)$ is an Azumaya C-algebra. Thus C is a direct summand of RG_f as C-bimodule [3, Lemma 3.1, page 51]. Since $\text{rank}_R(RG_f) = |G|$, the order of G, and $\text{rank}_C(C\overline{G}_f) = |\overline{G}|$, we have that $\text{rank}_R(RG_f) = \text{rank}_C(RG_f) \cdot \text{rank}_R(C)$ so that $\text{rank}_R(C)$ is well defined and

$\mathrm{rank}_R(C) = |G|/|\overline{G}| = |K|$. Moreover, since $\oplus \sum_{g\in K} RU_g \subset C$ and $\mathrm{rank}_R\left(\oplus \sum_{g\in K} RU_g\right) = |K| = \mathrm{rank}_R(C)$, we have that $C = \oplus\sum_{g\in K} RU_g$.

($\Leftarrow$) By hypothesis, $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over C, so $C\overline{G}_f = \oplus \sum_{\overline{g}\in\overline{G}} CU_{\overline{g}}$ is a projective group algebra of $\overline{G}$ over C with factor set $f : \overline{G}\times\overline{G} \to$ units of C induced by $f : G\times G \to$ units of R. Also, since $C = \oplus\sum_{g'\in K} RU_{g'}$,

$$C\overline{G}_f = \sum_{\overline{g}\in\overline{G}}\left(\sum_{g'\in K} RU_{g'}\right)U_{\overline{g}} = \sum_{i=1}^{k}\left(\sum_{g'\in K} RU_{g'}\right)U_{g_i}$$

$$= \sum_{i=1}^{k}\sum_{g'\in K} RU_{g'g_i} = \sum_{g\in G} RU_g = RG_f,$$

where $G = \sum_{i=1}^{k} Kg_i$. Thus $RG_f (= C\overline{G}_f)$ is an Azumaya C-algebra; and so RG_f is a central Galois C-algebra with an inner Galois group $\overline{G}$ [2, Theorem 3].

We note that Theorem 3 in [2] is a special case of Theorem 2.3.

Corollary 2.4. *Let RG_f be a projective group algebra of a finite group G over a commutative ring R with a factor set $f : G\times G \to$ units of R. Then the following are equivalent*: (1) *RG_f is a Galois R-algebra with an inner Galois group $\overline{G}$*; (2) *RG_f is an Azumaya R-algebra*; *and* (3) $K = \langle 1\rangle$.

3. Galois Projective Group Rings

Let R be a ring with 1 with center R_0, G be a finite group, RG_f be a projective group ring of G over R with a factor set $f : G\times G \to$ units of R_0, and C be the center of RG_f. We shall characterize a Galois projective group ring RG_f in terms of C. As given in Section 2, the center of G is denoted by Z and $K = \{g \in Z \mid f(g, g') = f(g', g)$ for all $g' \in G\}$. Clearly, the projective group algebra $R_0K_f \subset C$.

Lemma 3.1. *If RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$, then $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over RC where $U_{\overline{g}} = U_g$ for each $g \in G$.*

Proof. Since RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$, there exists a $\overline{G}$-Galois system for RG_f, $\{x_i, y_i \in RG_f$, $i = 1, 2, \ldots, m$ for some integer $m\}$ such that $\sum_{i=1}^{m} x_i \overline{g}(b_i) = \delta_{\overline{1}, \overline{g}}$ for each $\overline{g} \in \overline{G}$. Let $\sum_{\overline{g} \in \overline{G}} a_{\overline{g}} U_{\overline{g}} = 0$ for some $a_{\overline{g}} \in C$. Then for each $\overline{h} \in \overline{G}$,

$$0 = \sum_{i=1}^{m} x_i \left(\sum_{\overline{g} \in \overline{G}} a_{\overline{g}} U_{\overline{g}} \right) \overline{h}^{-1}(y_i) = \sum_{\overline{g} \in \overline{G}} a_{\overline{g}} \sum_{i=1}^{m} x_i U_{\overline{g}} \overline{h}^{-1}(y_i)$$

$$= \sum_{\overline{g} \in \overline{G}} a_{\overline{g}} \sum_{i=1}^{m} x_i \overline{g}(\overline{h}^{-1}(y_i)) U_{\overline{g}} = \sum_{\overline{g} \in \overline{G}} a_{\overline{g}} \left(\sum_{i=1}^{m} x_i \,(\overline{g} \cdot \overline{h}^{-1})(y_i) \right) U_{\overline{g}}$$

$$= \sum_{\overline{g} \in \overline{G}} a_{\overline{g}} \delta_{\overline{1}, \overline{g} \cdot \overline{h}^{-1}} U_{\overline{g}} = a_{\overline{h}} U_{\overline{h}}.$$

Thus $a_{\overline{h}} = 0$ for each $\overline{h} \in \overline{G}$. This proves that $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over C. But then $CU_{\overline{1}} \cap \sum_{\overline{g} \neq \overline{1}} CU_{\overline{g}} = \{0\}$. Thus $RCU_{\overline{1}} \cap \sum_{\overline{g} \neq \overline{1}} RCU_{\overline{g}} = \{0\}$. Therefore $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over RC.

Theorem 3.2. *Let RG_f be a Galois projective group ring of G over a ring R. Then the following are equivalent*: (1) *RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$*; (2) *$C\overline{G}_{\overline{f}}$ is a central Galois projective group algebra of $\overline{G}$ over C with factor set $\overline{f} : \overline{G} \times \overline{G} \to$ units of C induced by $f : G \times G \to$ units of R_0*; *and* (3) *$\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over RC and $RC = \oplus \sum_{g \in K} RU_g$, where $U_{\overline{g}} = U_g$ for each $g \in G$.*

Proof. (1) $\Rightarrow$ (2) Since RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$, $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over RC by Lemma 3.1. Let R_0 be the center R. Then $R_0 \subset C$. Noting that $\bar{f} : \overline{G} \times \overline{G} \to$ units of R_0, we have that $C\overline{G}_{\bar{f}}$ is a projective group algebra of $\overline{G}$ over C with factor set $\bar{f} : \overline{G} \times \overline{G} \to$ units of C where $\bar{f}$ is induced by $f : G \times G \to$ units of R_0. Moreover, since $R_0K_f \subset C$, $\sum_{\overline{g} \in \overline{G}} (R_0K_f)U_{\overline{g}} \subset C\overline{G}_{\bar{f}}$. But $\overline{G} = G/K$ by Lemma 2.2, so

$$RG_f = \sum_{g \in G} RU_g = R(R_0G_f) \subset R\left(\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}}\right) = R(C\overline{G}_{\bar{f}}) \subset RG_f.$$

Hence $RG_f = R(C\overline{G}_{\bar{f}})$. Thus $\overline{G}\,|_{C\overline{G}_{\bar{f}}} \cong \overline{G}$. Next we claim that C is also the center of $\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}} (= C\overline{G}_{\bar{f}})$. In fact, clearly, C is contained in the center of $C\overline{G}_{\bar{f}}$. Conversely, for any $x \in$ the center of $C\overline{G}_{\bar{f}}$, x is in the center of $\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}}$. Also, for any $r \in R$, $rx = xr$, so x is in the center of $R\left(\sum_{\overline{g} \in \overline{G}} CU_{\overline{g}}\right)$ which is RG_f. Thus $x \in C$. Therefore $C\overline{G}_f$ is an Azumaya C-algebra; and so $C\overline{G}_f$ is a central Galois C-algebra with an inner Galois group $\overline{G}\,|_{C\overline{G}_{\bar{f}}} \cong \overline{G}$ [2, Theorem 3].

(2) $\Rightarrow$ (1) Since $C\overline{G}_{\bar{f}} \subset R(C\overline{G}_{\bar{f}}) = RG_f$ and $R \subset (RG_f)^{\overline{G}}$ such that $\overline{G}\,|_{C\overline{G}_{\bar{f}}} \cong \overline{G}$, a $\overline{G}$-Galois system for $C\overline{G}_{\bar{f}}$ can be taken as a $\overline{G}$-Galois system for RG_f. Thus RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$.

(2) $\Rightarrow$ (3) Since $C\overline{G}_{\bar{f}}$ is a central Galois algebra over C with Galois group $\overline{G}$ and since $rx = xr$ for each $r \in R$ and $x \in C\overline{G}_{\bar{f}}$, $\{U_{\overline{g}} \mid \overline{g} \in \overline{G}\}$ are free over RC by Lemma 3.1. Moreover, since $C\overline{G}_{\bar{f}}$ is a central Galois

algebra over C with Galois group $\overline{G}\,|_{C\overline{G}_{\bar{f}}} \cong \overline{G}$ again, $C = \oplus\sum_{g\in K} R_0U_g$ by Theorem 2.3. Thus $RC = R\left(\oplus \sum_{g\in K} R_0U_g\right) = \oplus\sum_{g\in K} RU_g$.

(3) $\Rightarrow$ (2) By hypothesis, $\{U_{\bar{g}}\,|\,\bar{g} \in \overline{G}\}$ are free over RC, so $\{U_{\bar{g}}\,|\,\bar{g} \in \overline{G}\}$ are free over C. Also since $RC = \oplus\sum_{g\in K} RU_g$, $C = \oplus \sum_{g\in K} R_0U_g$, where R_0 is the center of R. Thus $C\overline{G}_f$ is a central Galois C-algebra with an inner Galois group $\overline{G}\,|_{C\overline{G}_{\bar{f}}} \cong \overline{G}$ by Theorem 2.3.

We conclude the present paper with three kinds of projective group rings.

Example 1. Let $R[i, j, k]$ be the real quaternion algebra over real field R with inner automorphism group $G = \{1, \bar{i}, \bar{j}, \bar{k}\}$, where $\bar{i}(x) = ixi^{-1}$, $\bar{j}(x) = jxj^{-1}$, and $\bar{k}(x) = kxk^{-1}$ for $x \in R[i, j, k]$. Then $R[i, j, k] = R \oplus Ri \oplus Rj \oplus Rk$, a projective group algebra RG_f with center R; and so it is a central Galois algebra over R with an inner Galois group G.

Example 2. Let $T = R[i] \subset R[i, j, k]$ as given in Example 1 and $H_i = \{1, \bar{i}\} \subset G$. Then $(R[i, j, k])^{H_i} = R[i]$ and $R[i, j, k]$ is a noncommutative Galois extension of $R[i]$ with a cyclic Galois group H_i. We note that any Galois algebra with a cyclic Galois group is commutative [1, Theorem 11].

Example 3. Let $M_{2\times 2}(R[i, j, k])$ be the 2×2 matrix ring over $R[i, j, k]$ with inner automorphism group $\overline{G}$ induced by i, j, k. Then $M_{2\times 2}(R[i, j, k]) \cong M_{2\times 2}(R) \otimes_R R[i, j, k] \cong M_{2\times 2}(R)\overline{G}_f$, a Galois projective group ring of $\overline{G}$ over $M_{2\times 2}(R)$.

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