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LATTICES AND GROUPS OF DIVISIBILITY

A. DE A. E SILVA, G. G. BASTOS and S. O. JURIAANS

Departamento de Matemática
Universidade Federal da Paraíba
CEP 58.051-970 João Pessoa
Brazil
e-mail: andrade@mat.ufpb.br

Departamento de Matemática
Universidade Federal do Ceará
CEP 60.455-760 Fortaleza
Brazil

Instituto de Matemática e Estatística
Universidade Estadual de São Paulo
Caixa Postal 66281
CEP 05315-970, São Paulo
Brazil

Abstract

We give a complete classification of the sublattices of $(\mathbb{Z}^n, +, \geq)$ which are not groups of divisibility. Doing so we provide a new class of ordered filtered groups which are not groups of divisibility. These examples generalize those due to P. Jaffard and G. G. Bastos.

1. Introduction

A lattice in $\mathbb{R}^n$ is a discrete additive subgroup and is complete if it contains a basis of $\mathbb{R}^n$. Lattices are known to have an important role in

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algebraic number theory. Here we study their connection with groups of divisibility.

An example of a filtered ordered group which is not a group of divisibility, was first described by Jaffard [4]. Ohm [6] gives a procedure for the construction of a class of groups which are not groups of divisibility. Bastos in [1] generalizes Jaffard's example by defining what he called a *Jaffard group*. A Jaffard group turns out never to be a group of divisibility.

We present a class of filtered ordered lattices which are not groups of divisibility, thus generalizing the examples due to Jaffard and Bastos. This class of groups is very natural and points out a connection between groups of divisibility and the spherical packing problem.

The paper is build up as follows. In Section 2, we present some basic definitions related to ordered groups, which are necessary for a better understanding of the paper. In Section 3, we study the connection between lattices and groups of divisibility and give a complete classification of those lattices that are groups of divisibility. We also give criteria to determine when a given lattice, Λ say, is proper knowing an upper triangular matrix whose rows form a basis of Λ. An algorithm to determine such a matrix is also given. We finish the paper giving an example of a lattice which is not a group of divisibility and which is a generalization of the well known checkerboard lattice D_n of $\mathbb{Z}^n$.

2. Basic Results

We shall briefly recall some basic facts concerning ordered groups. An ordered group is a commutative group with a partial order such that

$$a \leq b \Rightarrow a + c \leq b + c, \quad \forall a, b, c \in G.$$

G is *totally ordered* if it is a totally ordered set. The *positive cone* of G is denoted by $G_+ = \{a \in G : a \geq 0\}$. An *ordered subgroup* H of G is an ordered group contained in G such that $H_+ = G_+ \cap H$.

An ordered group G is called *filtered* if for any $a, b \in G$, there exists $c \in G$ such that $c \leq a, b$, and is an *l-group* if the infimum (supremum) of a and b exists for each $a, b \in G$. We shall denote the infimum (supremum) of a and b by $\inf\{a, b\}$ $(\sup\{a, b\})$. In this case we call the partial order on G an *l-order*. If G is ordered and $a_0, a_1, \ldots, a_n \in G$, then the expression

$$a_0 \geq \inf_G\{a_1, a_2, \ldots, a_n\}$$

means:

$$a_0 \geq \inf_G\{a_1, a_2, \ldots, a_n\} \Leftrightarrow a_0 \geq a, \quad \forall a \in G \text{ with } a \leq a_1, a_2, \ldots, a_n.$$

A homomorphism $\sigma : G \to G'$ between ordered groups is an *order homomorphism* if $\sigma(G_+) \subseteq G'_+$. It is an *order isomorphism* if it is a group isomorphism and $\sigma(G_+) = G'_+$. It is a *V-homomorphism* if for any $a_0, a_1, \ldots, a_n \in G$, we have that $a_0 \geq \inf_G\{a_1, a_2, \ldots, a_n\}$ implies $\sigma(a_0) \geq \inf_{G'}\{\sigma(a_1), \ldots, \sigma(a_n)\}$.

A *V-isomorphism* is an isomorphism such that it and its inverse are V-homomorphisms. A *V-embedding* of G in G' is a V-homomorphism which is injective. A subgroup H of G is a *V-subgroup* if the inclusion map is a V-homomorphism.

Here $\mathbb{Z}$ denotes the additive group of integers with its usual order. Let $\mathbb{I} \subset \mathbb{Z}$ has cardinality at least two and let G_i, $i \in \mathbb{I}$ be ordered groups. The *ordered direct product* of the G_i's is the group $W = \prod_{i \in \mathbb{I}} G_i$ with positive cone $W_+ = \{(a_i)_{i \in \mathbb{I}} \in W : a_i \geq 0, \forall i \in \mathbb{I}\}$. The *ordered direct sum* of the G_i's, denoted by $\sum_{i \in \mathbb{I}} G_i$, is the subgroup of W, of elements $\mathbf{a} = (a_i)_{i \in \mathbb{I}}$ such that $a_i = 0$ for all but a finite number of $i \in \mathbb{I}$.

Lemma 2.1. *Let $W = \prod_{i \in \mathbb{I}} G_i$ be the ordered direct product. Then p_i is a V-homomorphism of W onto G_i, for each $i \in \mathbb{I}$.*

Let $\mathbb{K}$ be a field and let $\mathbb{K}^* = \mathbb{K} - \{0\}$ be the multiplicative group of $\mathbb{K}$. A *semi-valuation* of $\mathbb{K}$ is a map $\nu : \mathbb{K}^* \to (G, +, \geq)$ such that, for all $x, y \in \mathbb{K}^*$,

(i) $\nu(xy) = \nu(x) + \nu(y)$.

(ii) $\nu(x + y) \geq \inf_{\nu(\mathbb{K}^*)} \{\nu(x), \nu(y)\}$, with $x + y \neq 0$.

(iii) $\nu(-1) = 0$.

$\nu(\mathbb{K}^*)$ is called the *semi-value group* of ν. The function ν can be extended to $\mathbb{K}$ defining $\nu(0) = \infty$, where $\infty > g$, $\forall g \in G$ and

$$g + \infty = \infty + g = \infty + \infty = \infty, \quad \forall g \in G \text{ e } \infty \geq \infty.$$

In this case, (ii) and (iii) can be replaced by the single axiom

(ii)′ $\nu(x - y) \geq \inf_{\nu(\mathbb{K})} \{\nu(x), \nu(y)\}$.

It is easily proved that, if ν is a semi-valuation of $\mathbb{K}$ with semi-value group G and φ is a V-homomorphism of G into an ordered group H, then $\varphi \circ \nu$ is a semi-valuation of $\mathbb{K}$.

Let D be an integral domain with quotient field $\mathbb{K}$ and with group of units $U(D)$. The group $G(D) = \dfrac{\mathbb{K}^*}{U(D)}$ may be considered to be filtered with the order relation $\omega(x) \leq \omega(y)$ iff there exists $d \in D$ such that $y = dx$. The canonical map $\omega : \mathbb{K}^* \to G(D)$ written additively is a semi-valuation. In this case $G(D)_+ = \omega(D - \{0\})$.

The ordered group $G(D)$ is called the *group of divisibility* of D. More generally, an ordered group G is called a *group of divisibility* if there exists an integral domain D such that G is order isomorphic to $G(D)$.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. A lattice Λ in $\mathbb{R}^n$ is a discrete additive subgroup of $\mathbb{R}^n$ or, equivalently,

$$\Lambda = \left\{ \sum_{j=1}^{m} x_j \alpha_j \; : \; x_j \in \mathbb{Z} \right\},$$

where the vectors $\alpha_1, \ldots. \alpha_m$ are linearly independent over $\mathbb{R}$ and hence $m \leq n$. We denote by $\mathcal{B} = \{\alpha_1, \ldots, \alpha_m\}$ a *basis* for the lattice. The $m \times n$

matrix $\Lambda_{\mathcal{B}}$ whose i-th row is the vector α_i is called a *generating matrix* for the lattice Λ.

We consider $\mathbb{Z}^n$ with the canonical basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and the natural order. Let Λ be a sublattice of $\mathbb{Z}^n$ and $\mathcal{B} = \{\alpha_1, \ldots, \alpha_m\}$ be a $\mathbb{Z}$-basis of Λ. We say that $\mathcal{B}$ is *positive* if $\alpha_i > \mathbf{0}$, for all $i = 1, \ldots, m$ and Λ is complete if it contains an $\mathbb{R}$-basis of $\mathbb{R}^n$.

3. Main Results

In this section, we prove our main results and give criteria for a sublattice to be proper. We finish with some concrete examples.

We start with the following results which should be known but we include a proof for the sake of completeness.

Lemma 3.1. *Let $\Lambda \subset \mathbb{Z}^n$ be a complete lattice. Then there exists a basis $\mathcal{B}$ of Λ such that*

(1) *$\mathcal{B}$ is positive.*

(2) *$\Lambda_{\mathcal{B}} = (x_{ij})$ is lower triangular.*

(3) *$0 \le x_{ij} < x_{jj}$, if $1 \le j < i \le n$.*

Proof. We use induction on n. Consider $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ be the projection on the last n coordinates and $\hat{\Lambda} := \pi(\Lambda)$. Since Λ is complete, there exists $m = \det(\Lambda) > 0$ such that $me_i \in \Lambda$, for all $i \ge 2$. From this it follows that $\hat{\Lambda}$ is a complete lattice of $\mathbb{R}^n$. By induction hypothesis $\hat{\Lambda}$ has a basis $\hat{\mathcal{B}}$ such that $\hat{\Lambda}_{\hat{\mathcal{B}}}$ is upper triangular. Choose $\alpha_1, \ldots, \alpha_{n+1}$ such that

$$\Lambda \bigcap span_{\mathbb{Z}}[e_1] = span_{\mathbb{Z}}[\alpha_1], \quad \hat{\mathcal{B}} = \{\pi(\alpha_2), \ldots, \pi(\alpha_{n+1})\}$$

and let $\mathcal{B} := \{\alpha_1, \ldots, \alpha_{n+1}\}$. Then, we can see that $\mathcal{B}$ is a basis for Λ and $\Lambda_{\mathcal{B}}$ is upper triangular.

We now show that we can choose $\mathcal{B}$ to be positive. First choose $x_{11} > 0$, this can be done just changing signs. Now choose α_2, just change its sign if necessary, such that $x_{22} > 0$ and use x_{11} to make $x_{21} \geq 0$. Then choose α_3, just change its sign if necessary, such that $x_{33} > 0$ and then use α_2 to make $x_{31} \geq 0$ and $x_{32} \geq 0$. Thus one can repeat this process until one gets a positive basis.

Now start with row n and use row $n-1$ to make $0 \leq x_{n(n-1)} < x_{(n-1)(n-1)}$. Secondly use row $n-2$ to make $0 \leq x_{(n-1)(n-2)} < x_{(n-2)(n-2)}$ and $0 \leq x_{n(n-2)} < x_{(n-2)(n-2)}$. Continue this strategy to get the desired basis.

We shall call a basis which satisfies the properties of the previous lemma a *triangular positive basis*. We shall say that Λ is a *non-proper lattice* if it has a basis $\mathcal{B}$ such that $\Lambda_{\mathcal{B}}$ is diagonal. Otherwise we shall say that Λ is proper.

Proposition 3.2. *Let* $\Lambda \subset \mathbb{Z}^n$ *be a complete lattice,* $\mathcal{B} \subset \Lambda$ *be a positive basis and* $1 \leq j_0 \leq n-1$. *Suppose that*

(1) $e_{j_0} \notin \Lambda$.

(2) $\Lambda_{\mathcal{B}} = [x_{ij}]$ *is under triangular and there exists* $1 < l < k \leq n$ *such that* $gcd(x_{lj_0}, x_{kj_0}) = 1$.

Then Λ *is proper.*

Proof. Suppose that Λ is non-proper then, since it is complete, it has a basis of the form $\hat{\mathcal{B}} = \{a_1 e_1, \ldots, a_n e_n\}$, where $a_i \in \mathbb{N}_+$. We also have that $\forall 1 \leq k \leq n$, $\sum_{j=1}^{k} x_{kj} e_j \in span_{\mathbb{Z}}[\hat{\mathcal{B}}]$ and hence $a_j \mid x_{kj}$, $\forall j \leq k \leq n$, $\forall 1 \leq j < n$. Also since $|\det(\Lambda_{\mathcal{B}})|$ does not depend on the basis one sees easily that $x_{jj} = a_j$ and hence $x_{jj} \mid x_{kj}$, $\forall j \leq k \leq n$. From this and the fact that $a_{j_0} \neq 1$, it follows that $mdc(x_{lj_0}, x_{kj_0}) \neq 1$, a contradiction. Hence Λ must be proper.

The proof of the proposition also tells us that if Λ is non-proper and $\mathcal{B}$ is a triangular positive basis of Λ, then $\Lambda_{\mathcal{B}}$ is a diagonal matrix.

We can also use the proof of the proposition to decide whether a given upper triangular matrix corresponds to a proper lattice. For example consider the matrix

$$\mathbf{A} = \begin{bmatrix} 6 & 0 \\ 4 & 2 \end{bmatrix}$$

and let Λ be the lattice associated to $\mathbf{A}$. The proof of the proposition tells us that if Λ was non-proper, then $\{6e_1, 2e_2\}$ would be a diagonal basis. But $6 \nmid 4$ and so Λ must be proper.

Theorem 3.3. *Let Λ be a sublattice of $\mathbb{Z}^n$ with $n \geq 2$.*

(1) *If Λ is non-proper, then it is an l-group.*

(2) *If Λ is proper, then it is filtered but is never an l-group.*

Proof. If Λ is non-proper, then it is obviously order isomorphic to $\mathbb{Z}^k$ for some $1 \leq k \leq n$ and so it is an l-group. So suppose that Λ is a proper sublattice of $\mathbb{Z}^n$. First use the standard basis of $\mathbb{Z}^n$ to obtain a complete sublattice $\hat{\Lambda}$ of $\mathbb{Z}^n$ which contains Λ. It is clear that $\hat{\Lambda}$ is also a proper sublattice of $\mathbb{Z}^n$ and that in order to prove the theorem we should prove it for $\hat{\Lambda}$. So we may suppose that Λ is a complete proper sublattice of $\mathbb{Z}^n$ and that $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ is a triangular positive basis of Λ. It is easy to see that $\left\{\sum a_i\alpha_i \in \Lambda : a_i \in \mathbb{Z}_+\right\} \subset \Lambda_+$, the positive cone of Λ. For $\alpha \in \Lambda$, define $\alpha = \sum b_i\alpha_i$, $b_i \in \mathbb{Z}$. Choose $r_i \in \mathbb{Z}_+$ such that $b_i + r_i \geq 0$, $i = 1, 2, \ldots, k$. Then $\alpha = \sum(b_i + r_i)\alpha_i - \sum r_i\alpha_i \in \Lambda_+ - \Lambda_+$ and hence $\Lambda = \Lambda_+ - \Lambda_+$, that is, Λ is filtered.

Since Λ is proper there exists $1 \leq i_0 \leq n$ such that $\alpha_{i_0-1} \in span_{\mathbb{Z}}[e_{i_0-1}]$ and $\alpha_{i_0} \notin span_{\mathbb{Z}}[e_{i_0}]$. Let $\alpha = \inf_{\mathbb{Z}^n}\{\alpha_{i_0-1}, \alpha_{i_0}\}$ and $\beta = \inf_{\Lambda}\{\alpha_{i_0-1}, \alpha_{i_0}\}$. By the choice of $\mathcal{B}$ and i_0, we have that $\alpha = (0, \ldots, x_{i_0(i_0-1)}, \ldots, 0)$ and $0 \leq$

$\alpha \leq \beta \leq \alpha_{(i_0-1)}$. Hence, by definition, $\beta = (0, \ldots, r, \ldots, 0)$ with $x_{i_0(i_0-1)} \leq r$ and so $\beta = \alpha$. Observing that $\Lambda \bigcap span_{\mathbb{Z}}[e_{i_0-1}] = span_{\mathbb{Z}}[\alpha_{i_0-1}]$ and $0 \leq x_{i_0(i_0-1)} < x_{(i_0-1)(i_0-1)}$, we must have that $\beta = 0$ and so $\alpha_{i_0} \in span_{\mathbb{Z}}[e_{i_0}]$, a contradiction. It follows that Λ is not an l-group.

Theorem 3.4. *Let Λ be a proper sublattice of $\mathbb{Z}^n$ with $n \geq 2$. Then Λ is not a group of divisibility.*

Proof. We may suppose that Λ is complete. Suppose that Λ is a group of divisibility and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a triangular positive basis of Λ. Then there exist a field $\mathbb{K}$ and a semi-valuation $\nu : \mathbb{K}^* \to \Lambda$. If we define $v_i = \pi_i \circ \nu$, where π_i is the projection on the i-th coordinate, then, by Lemma 2.1, each v_i is a Krull valuation. We claim that they are independent: in fact, suppose that there exist $1 \leq i < j \leq n$ and $r \in \mathbb{Q}^*$ such that $\pi_i(\alpha) = r\pi_j(\alpha)$, for all $\alpha \in \Lambda$. In particular, $\pi_i(\alpha_m) = r\pi_j(\alpha_m)$, $m = 1, \ldots, n$. Hence, $x_{ii} = \pi_i(\alpha_i) = r\pi_j(\alpha_i) = r \cdot 0 = 0$ and thus $\det(\Lambda_{\mathcal{B}}) = 0$, a contradiction, proving the claim.

Consider the map

$$\Psi : \Lambda \to \pi_1(\Lambda) \times \cdots \times \pi_n(\Lambda),$$

given by

$$\Psi(v) := (\pi_1(v), \ldots, \pi_n(v)).$$

The Approximation Theorem (cf. [7]) tells us that given

$$w = (b_1, \ldots, b_n) \in \pi_1(H) \times \cdots \times \pi_n(H)$$

there exists $x \in \mathbb{K}^*$ such that $v_i(x) = b_i$, that is, $\Psi(\nu(x)) = w$ and so Ψ is an isomorphism. Obviously $\Psi(\Lambda_+) = (\pi_1(\Lambda) \times \cdots \times \pi_n(\Lambda))_+$ and hence Ψ is an order preserving isomorphism. Thus Λ is an l-group and a proper sublattice of $\mathbb{Z}^n$, contradicting Theorem 3.3.

Corollary 3.5. *Let Λ be a sublattice of $(\mathbb{Z}^n, +, \geq)$ with $n \geq 2$. Then Λ is a group of divisibility if and only if Λ is an l-group.*

Example 3.6. Let $n \geq 2$ be an integer and $p \in \mathbb{N}$ be a rational prime. Define

$$\hat{e} := \sum e_i \text{ and } \Lambda_{pn} := \{v \in \mathbb{Z}^n : \langle v \,|\, \hat{e} \rangle \equiv 0 \bmod p\}.$$

Then Λ_{pn} is a proper lattice and hence it is not a group of divisibility.

Proof. In fact, Λ_{pn} has a triangular positive basis $\mathcal{B}$ such that

$$\Lambda_{\mathcal{B}} = \begin{bmatrix} p & 0 & 0 & \cdots & 0 \\ p-1 & 1 & 0 & \cdots & 0 \\ p-1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p-1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

To prove this let $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ be given by the rows of $\Lambda_{\mathcal{B}}$ and let $v \in \Lambda_{pn}$, $v = \sum_{j=1}^{n} a_j e_j$. Then

$$v - \sum_{j=2}^{n} a_j \alpha_j = \left(a_1 - (p-1) \sum_{j=2}^{n} a_j \right) e_1.$$

Clearly $a_1 - (p-1)\sum_{j=2}^{n} a_j \equiv 0 \bmod p$ and so there exists $\lambda_1 \in \mathbb{Z}$ such that $v - \sum_{j=2}^{n} a_j \alpha_j = \lambda_1 \alpha_1$. This proves that $\mathcal{B}$ is a triangular positive basis for Λ_{pn}. Now use the criteria of Proposition 3.2 to conclude that Λ_{pn} is proper.

For $p = 2$ the above examples give us the well known checkerboard lattice D_n of $\mathbb{Z}^n$ (cf. [3]). Note also that we can consider Λ_{mn}, where m is not necessarily a rational prime. We obtain a generating matrix in the same way and hence the resulting lattice is still proper.

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