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APPLICATION OF NONCLASSICAL SYMMETRY ANALYSIS TO ORDINARY DIFFERENTIAL EQUATION

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Abstract

We demonstrate an application of a nonclassical symmetry analysis to ordinary differential equations (ODEs). Our method does not request solving the determining equations for a nonclassical symmetry and enables us to separate an $(n+1)$th-order ODE into a first-order ODE and an nth-order differential equation. Moreover, our method comprehends the variable separated ODE method and the factorization method. In this paper, the application of our method to nonlinear ODEs is elucidated with the aid of two examples from mathematical physics.

1. Introduction

The notion of local symmetry of differential equations presents a systematic framework for simplifying and solving them [4, 13]. For example, an invariance under symmetry admitted by differential equation enables us to reduce the order of one and constructs an exact solution called an *invariant solution*. In addition, for a first-order ODE, the method based on symmetry provides a systematic procedure to construct an integrating factor. The local symmetry is characterized by

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infinitesimal generators, whose coefficients are given by solutions of the determining equations.

Bluman and Cole proposed a generalization of classical symmetry which has since been called the *nonclassical symmetry* [1, 5]. In a nonclassical symmetry analysis, original equations are imposed on invariant conditions associated with infinitesimal generators. The determining equations are derived by invariant criteria to the original equations and the invariant conditions. It is well known that a class of nonclassical symmetries is larger than one of classical symmetries.

Recently, the variable separated ODE method and the factorization method are established to find particular solutions of ODEs. In the variable separated ODE method, a first-order ODE called the additional variable separated ODE is introduced. This method mainly depends on assuming the first-order equation. If we success, we can find an exact solution by calculating algebraic equations [16]. The factorization technique presented by Berkovich and it focuses on the factorable second-order ODE. The second-order ODE is regarded as the action of a second-order differential operator on a dependent variable and factorable means the second-order differential operator is factorized into two first-order differential operators [2, 3].

In this paper, we show applications of a nonclassical symmetry analysis to ODEs. Our method dose not request solving the determining equations for nonclassical symmetry and separates an $(n+1)$th-order ODE into a first and an nth differential equation. To illustrate our method, we show two examples from mathematical physics. Moreover, we show that the variable separated ODE method and the factorization method are understood as special cases in a framework based on nonclassical symmetry.

2. Nonclassical Symmetry of ODE

Consider the following nth order ODE:

$$u_{n+1} + f(t, u, u_1, \dots, u_n) = 0, \tag{1}$$

where u_n denotes the nth order derivative with respect to an independent variable t. This is a single $(n+1)$th order ODE involving the dependent variable u. We set an infinitesimal generator of nonclassical symmetry in the following form:

$$\mathbf{V} = \partial_t + \phi(t, u)\partial_u. \tag{2}$$

Then, the invariant condition can be expressed by

$$u_1 = \phi. \tag{3}$$

Lemma 1. *For second-order ODEs* $(i.e., n = 2)$, *an invariant solution associated with nonclassical symmetry* (2) *is a general solution of Eq.* (1).

Proof. The determining equation associated with the infinitesimal generator (2) is given by

$$D_2\phi + (D_1\phi)\frac{\partial f}{\partial u_1} + \phi\frac{\partial f}{\partial u} + \frac{\partial f}{\partial t} = 0, \tag{4}$$

where $D_i\phi$ is the ith total derivative of the function ϕ. The invariant solution that we seek satisfies Eqs. (1), (3) and (4). Eqs. (1) and (3) lead to a relation between the derivative of ϕ and f

$$D_1\phi + f = 0. \tag{5}$$

Since the determining equation (4) can be derived from Eq. (5), we only treat Eqs. (3) and (5). Then, its characteristic curves are

$$u_1 = \phi, \quad \phi_1 = f, \tag{6}$$

and they are equal to Eq. (1).

Consequently, finding an invariant solution to the nonclassical symmetry of a second-order ODE is the same as solving Eq. (1). □

Lemma 2. *An invariant solution of the nonclassical symmetry of Eq.* (1) *satisfies the following system of differential equations*:

$$u_1 = \phi, \quad D_n\phi + f = 0. \tag{7}$$

Proof. The nth total derivation of the invariant condition Eq. (3) leads to $u_{n+1} = D_n\phi$. According to Eq. (1),

$$D_n\phi + \Delta = 0. \tag{8}$$

By differentiating Eq. (8), we can derive the determining equation of the nonclassical symmetry of Eq. (1)

$$D_{n+1} + \sum_{j=0}^{n} (D_j \phi) \frac{\partial \Delta}{\partial u_j} + \frac{\partial \Delta}{\partial t} = 0. \tag{9}$$

□

3. The Application of Lemma 2 to the Reduction Order of ODEs

We provide two examples to illustrate the application of Lemma 2 to Eq. (1).

We begin with a fourth-order partial differential equation (PDE) describing the dynamics of discrete breathers on one-dimensional monotonic chains [9]

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4} \frac{\partial^4 u}{\partial^2 x \partial^2 t} + 4T(u) = 0, \tag{10}$$

where the function $T(u)$ is a cubic equation of u. In this example, we regard the function $T(u)$ as a function of u alone.

Among the solutions of Eq. (10), of particular interest is the traveling wave solution $u = u(y)$, where $y = 2x + ct$. By substituting the representation of a solution into Eq. (10), one finds

$$\frac{d^4 u}{dy^2} - \frac{d^2 u}{dy^2} - f(u) = 0, \quad \text{where } f = \frac{4T}{c^2}. \tag{11}$$

Let us seek nonclassical symmetry in which an infinitesimal generator is expressed by (2). Applying Lemma 2 yields the determining equation for the function ϕ

$$D_3 \phi - D\phi - f = 0. \tag{12}$$

Suppose $\phi = \phi(u)$ gives the integrable equation

$$2 \frac{d^3 \psi}{du^3} \psi + \frac{d^2 \psi}{du^2} \frac{d\psi}{du} - \frac{d\psi}{du} - f = 0, \tag{13}$$

where $\psi = \frac{\phi^2}{2}$.

By integrating Eq. (13) and from the invariant condition, we find the exact solution of Eq. (10) in an implicit form

$$\int \frac{du}{\sqrt{2\psi}} = \pm y, \tag{14}$$

where the function ψ satisfies

$$2\frac{d^2\psi}{du^2}\psi - \frac{1}{2}\left(\frac{d\psi}{dy}\right)^2 - \psi - \int f du = 0. \tag{15}$$

In the next example, we consider the class of nonlinear third-order PDEs [6, 10, 11, 14, 18]

$$\frac{\partial u}{\partial t} + a\frac{\partial^3 u}{\partial x^2 \partial t} + b\frac{\partial u}{\partial x} - u\frac{\partial^3 u}{\partial x^3} + c\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + du\frac{\partial u}{\partial x} = 0, \tag{16}$$

where a, b, c, d are constants. Some studies focus on special cases that correspond to physical situations. For example, Eq. (16) for (a, b, c, d) $= (-1, \text{arbitrary}, -2, 3), (-1, 1, -3, 1)$ and $(0, 0, -3, -1)$ are now referred to as the Camasa-Home equation, the Fornberg-Whitham equation, and the Rosenau-Hyman equation, respectively.

By using the traveling wave frame $y = x + t$, Eq. (16) is reduced to the ODE

$$(a - u)\frac{d^3 u}{dy^3} + c\frac{du}{dy}\frac{d^2 u}{dy^2} + (1 + b + du)\frac{du}{dy} = 0. \tag{17}$$

As a first example, we regard an infinitesimal generator (2) as having nonclassical symmetry whose coefficient ϕ satisfies

$$(a - u)D_2\phi + c\psi D\phi + (1 + b + du)\phi = 0. \tag{18}$$

We assume that $\phi = \phi(u)$; then Eq. (17) becomes

$$(a - u)\frac{d^2\psi}{du^2} + c\frac{d\psi}{du} + (1 + b + du) = 0, \tag{19}$$

where $\psi = \dfrac{1}{2}\phi^2$.

It is easy to integrate Eq. (19) twice. The analytical expression of ϕ is given as

$$\phi = \pm\sqrt{2\int \eta du}, \tag{20}$$

where

$$\eta = \frac{1}{(u-a)^c}\left\{\int (1+b+du)(u-a)^{c-1}du\right\}. \tag{21}$$

From an integration of the invariant condition, we find

$$\int \frac{du}{\sqrt{2\int \eta du}} = \pm y. \tag{22}$$

4. The Factorization Method, the Variable Separated ODE Method and Nonclassical Symmetry

In this section, we show Lemma 2 contains the factorization method and the variable separated ODE method.

The factorization method treats the following second-order differential equation:

$$\frac{d^2u}{dy^2} + f(u)\frac{du}{dy} + g(u). \tag{23}$$

Performing the factorization of Eq. (23) means that Eq. (23) is transformed into the following form:

$$(D - \phi_2(u))(D - \phi_1(u))u = 0, \tag{24}$$

where D denotes the total derivative with respect to y and the functions ϕ_1 and ϕ_2 satisfy

$$\phi_1 + \phi_2 + \frac{d\psi_1}{du}u = -f, \tag{25}$$

$$\phi_1\psi_2 = g. \tag{26}$$

An exact solution which we seek satisfies the following the first-order ODE

$$(D - \phi_1)u = 0. \tag{27}$$

The factorization technique applied to nonlinear ODEs to find particular solutions [7, 8, 15].

We consider the additional variable separated ODE of Eq. (23)

$$u_1 = \psi(u). \tag{28}$$

Here, in the original method, the function $\psi(u)$ gives by a combination of a suitable function such as sine, cosine, hyperbolic cosine, exponential, etc. Wazwaz has applied the variable separated ODE method to the Liuville equation and found some exact solutions of one [17].

Instead of assuming for the function $\psi(u)$, Murata [12] considered a determining equation for $\psi(u)$:

$$\psi \frac{d\psi}{dy} + f\psi + g = 0. \tag{29}$$

He proved the variable separated ODE method contains the factorization method by transformed Eq. (29) into Eqs. (25) and (26).

Furthermore, it is evident that the variable separated ODE method is a special case of Lemma 2. Namely, the additional variable separated ODE and its determining equation correspondent to the invariant condition and the relation between the invariant condition and the original ODE.

5. Summary

We focus on the nonclassical symmetry analysis of ODEs and its applications for solving them. Our method separates an $(n+1)$th-order ODE into a first-order differential equation and an nth differential equation. Especially when $n = 1$, our method is equivalent to integrating the original ODE. To illustrate the application of Lemma 2, we treat two examples. The first example is a fourth-order PDE governing the dynamics of discrete breathers on a one-dimensional chain, and the second example is a third-order PDE that includes the Camasa-Home equation, the Fornberg-Whitham equation, and the Rosenau-Hyman equation as special cases.

Furthermore, our method contains the variable separated ODE method and the factorization method. The first method needs introduction of the additional variable separated ODE and assuming an appropriate function $G(u)$. The later method demands factorability for the second ODEs. On the other hand, our method naturally introduces the variable separated ODE method and the factorization method.

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