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DYNAMICAL SYSTEMS, MEASUREMENTS, QUANTITATIVE LANGUAGES AND ZENO'S PARADOXES

SHIRO ISHIKAWA

Department of Mathematics
Faculty of Science and Technology
Keio University, 3-14-1 Hiyoshi, Kohoku-ku
Yokohama, 223-8522, Japan
e-mail: ishikawa@math.keio.ac.jp

Abstract

The main purpose of this paper is to assert that measurement theory is only a kind of quantitative language. This implies that dynamical system theory is also only a quantitative language, since dynamical system theory is characterized as one of fields of measurement theory. As a consequence of this assertion, Zeno's paradox is, from the spirit of the language game, clarified in terms of measurement theory.

1. Introduction

It is well known that dynamical system theory (=DST) in engineering is usually formulated as follows.

$$\text{“DST”} = \begin{cases} \dfrac{dx(t)}{dt} = f(x(t), u_2(t), t),\ x(0) = x_0 & \text{... (stochastic state equation)} \\ y(t) = g(x(t), u_1(t), t) & \text{... (measurement equation)} \end{cases} \tag{1}$$

where u_1 and u_2 are external forces (or noises).

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The DST (including probability theory, cf. [7]) is, of course, quite applicable to most fields in science. However it should be noted that DST is neither pure mathematics nor physics. Thus, the question "What is DST?" or "What kind of discipline does DST belong to?" should be answered. For example, it is usually said that Zeno's paradox (Achilles and the Tortoise) is elementarily justified in the framework of DST. However, strictly speaking, we consider that the dynamical system theoretical justification is nonsense if we have no answer to the question: "What kind of power does DST have?". Thus, we think that the question "What is DST?" or "What kind of power does DST have?" should be one of the most important problems in science. Although there may be several opinions (for example, the opinion $(\sharp_1)$ in Section 5 may be usual), in this paper we assert:

(A) DST is nothing more than a powerful quantitative language, in which the formula (1) should be regarded as the linguistic rule of the quantification.

Since the formula (1) can be interpreted as various meanings, it is not easy to find the assertion (A) directly in the formulation of DST(1). Thus, in order to show the above (A), we begin with the measurement theory (cf. [3, 4, 5] or author's papers in the references of the book [4]), which includes measurements in classical and quantum systems and is constructed in terms of operator algebras (cf. [10]). This measurement theory is characterized as the mathematical representation of "the mechanical world view", namely, an epistemology to understand and analyze every phenomenon in our usual life by an analogy of mechanics (cf. the $(\sharp_1)$ in Section 5 later) such that

$$\text{"measurement theory(= MT)"} = \underset{(\text{Axiom 1})}{[\text{measurement}]} + \underset{(\text{Axiom 2})}{[\text{causal relation}]}. \tag{2}$$

And we have the following classification:

$$\text{"MT"} = \begin{cases} \text{quantum measurement theory (= quantum mechanics [11])} \\ \text{classical measurement theory (= CMT),} \end{cases} \tag{3}$$

where the algebra is either non-commutative or commutative. We say, from the mathematical point of view, that $\text{DST} \subset \text{CMT}$ (cf. [3, 4]). Thus, in order to assert the above statement (A), it suffices to show that MT (or, CMT) is a quantitative language. Precisely speaking, we assert that the measurement theoretical language (=MTL) is created as follows.

$$\underset{\text{(quantitative language)}}{\text{“MTL”}} = \underset{\text{(qualitative language)}}{\text{“ordinary language”}} + \underset{\text{(rules made express statement)}}{\text{MT(= Axioms 1 and 2 in (2))}}, \quad (4)$$

or

$$\underset{\text{(qualitative language)}}{\text{ordinary language}} \xrightarrow[\text{by MT (Axioms 1 and 2)}]{\text{quantification}} \underset{\text{(quantitative language)}}{\text{MTL}}. \quad (5)$$

This will be asserted in Section 2. Also, in the sense of (4), we sometimes identify MT with MTL. Since $\text{DST} \subset \text{CMT}$, the (4) implies that the dynamical system theoretical language (=DSTL) is created as follows.

$$\underset{\text{(quantitative language)}}{\text{“DSTL”}} = \underset{\text{(qualitative language)}}{\text{“ordinary language”}} + \underset{\text{(expressed by (1))}}{\text{DST}}. \quad (6)$$

Similarly, we sometimes identify DST with DSTL. Here, we consider that Axiom 1 in (2) [resp. Axiom 2 in (2)] corresponds to the measurement equation in (1) [resp. the stochastic state equation (1)].

Remark 1. We consider the following classification:

$$\text{“language”} = \begin{cases} \text{natural language} \begin{cases} \text{ordinary language} \\ \text{scientific language} \end{cases} \\ \text{artificial language} \end{cases} \quad (7)$$

where the artificial language includes mathematics, pure logic, programming languages, etc. The language used in our usual life is, of course, an ordinary language. Also, for example, the electromagnetic language (i.e., the language used when electromagnetic phenomena are discussed in the framework of Maxwell's electromagnetism) is scientific. In general, the physical language (i.e., the language used when physical phenomena are discussed in physics) is scientific. In this paper, the term: “quantitative language” means a language in which quantities (i.e., calculation, logic, etc.) can be treated well. Thus, the physical language is quantitative. Of course, MTL and DSTL are also quantitative. The question: “Are MTL and DSTL ordinary or scientific in (7)?” may be significant. Although we believe that MTL and DSTL should be ordinary, in this paper we are not concerned with the question. Strictly speaking, a purely qualitative language may not exist. Thus, we consider that the term: "qualitative language" means "incomplete (or, poor, powerless) quantitative language". And therefore, we assume that our ordinary language is more or less qualitative. In Section 3, we will see that Zeno's paradox (Achilles and the Tortoise) is caused in the gap between a qualitative argument and a quantitative argument.

2. Measurement Theory as a Quantitative Language

In this paper, for simplicity we focus on CMT and not MT. Now we shall introduce CMT, which is formulated in a commutative algebra $L^{\infty}(\Omega, \mu)$. Let Ω be a locally compact Hausdorff topological space (called a *state space* later), and let $(\Omega, \mathcal{F}(\Omega), \mu)$ be a measure space such that $0 < \mu(U) \leq \infty$ for any open set $U(\subseteq \Omega)$, and $0 \leq \mu(\{\omega\}) < \infty$ $(\forall \omega \in \Omega)$. (For our direct purpose (in Section 4), it suffices to consider $\Omega = \mathbb{R}^2$, 2-dimensional real plane, and $\mu = m^2$, the usual (Lebesgue) measure on $\mathbb{R}^2$.) Define the Banach space $L^r(\Omega, \mu)$, $(r = 1, \infty)$, by the set of all complex valued measurable functions on Ω such that the norm $\| f \|_{L^r(\Omega, \mu)}$ is finite, where $\| f \|_{L^r(\Omega, \mu)} = \int_{\Omega} | f(\omega) | \mu(d\omega)$ (if $r = 1$), $= \text{ess.}\sup_{\omega \in \Omega} | f(\omega) |$ (if $r = \infty$). A function $f(\in L^{\infty}(\Omega, \mu))$ is said to be *essentially continuous at* $\omega_0(\in \Omega)$ if there exists a function $g(\in L^{\infty}(\Omega, \mu))$ such that g is continuous at ω_0 and $\mu(\{\omega \in \Omega \,|\, f(\omega) \neq g(\omega)\}) = 0$. And thus, $f(\omega_0)$ is defined by $g(\omega_0)$.

A triplet $(X, \mathcal{F}(X), F)$ is called an *observable* in $L^{\infty}(\Omega, \mu)$, if it satisfies:

1. X is a set with a σ-field $\mathcal{F}(X)$. That is, $(X, \mathcal{F}(X))$ is a measurable space.
2. F is a mapping from $\mathcal{F}(X)$ to $L^{\infty}(\Omega, \mu)$ satisfying: (i): for every $\Xi \in \mathcal{F}(X)$, $F(\Xi)$ is a non-negative function in $L^{\infty}(\Omega, \mu)$ such that $0 \leq [F(\Xi)](\omega) \leq 1$, $[F(\emptyset)](\omega) = 0$ and $[F(X)](\omega) = 1$ (a.e. $\omega \in \Omega$).

(ii): for any countable decomposition $\{\Xi_1, \Xi_2, ..., \Xi_n, ...\}$ of Ξ (i.e., $\Xi, \Xi_n \in \mathcal{F}(X)$ $(n = 1, 2, 3, ...)$, $\bigcup_{n=1}^{\infty} \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ $(i \neq j)$), it holds that

$$\int_{\Omega} [F(\Xi)](\omega) u(\omega) \mu(d\omega) = \lim_{N \to \infty} \sum_{n=1}^{N} \int_{\Omega} [F(\Xi_n)](\omega) u(\omega) \mu(d\omega)$$

for all $u \in L^1(\Omega, \mu)$.

For each $k = 1, 2$, consider an observable $\mathbf{O}_k := (X_k, \mathcal{F}(X_k), F_k)$ in $L^\infty(\Omega, \mu)$. Let $(X_1 \times X_2, \mathcal{F}(X_1 \times X_2))$ be the product measurable space of $(X_k, \mathcal{F}(X_k))$'s. An observable $\widetilde{\mathbf{O}} := (X_1 \times X_2, \mathcal{F}(X_1 \times X_2), \widetilde{F})$ in $L^\infty(\Omega, \mu)$ is called the *product observable of* $\{\mathbf{O}_1, \mathbf{O}_2\}$, if it satisfies $\widetilde{F}(\Xi_1 \times \Xi_2) = F_1(\Xi_1) \times F_2(\Xi_2)$ $(\forall \Xi_k \in \mathcal{F}(X_k), k = 1, 2)$. The product observable $\widetilde{\mathbf{O}}$ is also denoted by $\mathbf{O}_1 \times \mathbf{O}_2$ or $(X_1 \times X_2, \mathcal{F}(X_1 \times X_2), F_1 \times F_2)$. For the further argument, see [4].

With any *classical system* S, a commutative algebra $L^\infty(\Omega, \mu)$ can be associated in which the CMT of that system can be formulated. A *state* of the system S is represented by a point $\omega (\in \Omega)$ and an *observable* is represented by an observable $\mathbf{O} := (X, \mathcal{F}(X), F)$ in $L^\infty(\Omega, \mu)$. Also, the *measurement of the observable* $\mathbf{O}$ *for the system* S *with the state* ω is denoted by $\mathbf{M}_{L^\infty(\Omega, \mu)}(\mathbf{O}, S_{[\omega]})$ (or, $\mathbf{M}_{L^\infty(\Omega, \mu)}(\mathbf{O} := (X, \mathcal{F}(X), F), S_{[\omega]})$). We can obtain a measured value $x (\in X)$ by the measurement $\mathbf{M}_{L^\infty(\Omega, \mu)}(\mathbf{O}, S_{[\omega]})$.

The axiom presented below is motivated by Born's probabilistic interpretation of quantum mechanics.

Axiom$_\mathbf{c}$1 (the classical form of Axiom 1 in (2), cf. [5]). Only one measurement is permitted. And the probability that a measured value $x (\in X)$ obtained by the measurement $\mathbf{M}_{L^\infty(\Omega, \mu)}(\mathbf{O} := (X, \mathcal{F}(X), F), S_{[\omega_0]})$ belongs to a set $\Xi (\in \mathcal{F}(X))$ is given by $[F(\Xi)](\omega_0)$ if $F(\Xi)$ is essentially continuous at $\omega_0 (\in \Omega)$.

Next, we explain the classical form of Axiom 2 in (2). A continuous (or precisely, weak*-continuous) linear operator $\Phi_{1,2} : L^{\infty}(\Omega_2, \mu_2) \to L^{\infty}(\Omega_1, \mu_1)$ is called a *Markov operator*, if it satisfies that (i) $\Phi_{1,2}(f_2) \geq 0$ for any non-negative function f_2 in $L^{\infty}(\Omega_2, \mu_2)$, (ii) $\Phi_{1,2}(I_2) = I_1$, where $I_k(\omega_k) = 1$ for all $\omega_k \in \Omega_k$ $(k = 1, 2)$. Here note that, for any observable $(X, \mathcal{F}(X), F_2)$ in $L^{\infty}(\Omega_2, \mu_2)$, the $(X, \mathcal{F}(X), \Phi_{1,2}F_2)$ is also an observable in $L^{\infty}(\Omega_1, \mu_1)$, which is denoted by $\Phi_{1,2}\mathbf{O}_2$.

The following axiom should be regarded as the rule of the quantification of "causality".

Axiom$_{\mathbf{c}}$2 (the classical form of Axiom 2 in (2), cf. [5])**.** The causal relation between classical systems is represented by a Markov operator $\Phi_{1,2} : L^{\infty}(\Omega_2, \mu_2) \to L^{\infty}(\Omega_1, \mu_1)$. And, an observable $\mathbf{O_2} := (X, \mathcal{F}(X), F_2)$ in $L^{\infty}(\Omega_2, \mu_2)$ can be identified with the observable $\Phi_{1,2}\mathbf{O}_2 := (X, \mathcal{F}(X), \Phi_{1,2}F_2)$ in $L^{\infty}(\Omega_1, \mu_1)$ such as

$$\underset{\text{in } L^{\infty}(\Omega_1, \mu_1)}{\Phi_{1,2}\mathbf{O}_2 := (X, \mathcal{F}(X), \Phi_{1,2}F_2)} \xleftarrow[\textit{identification}]{\Phi_{1,2}} \underset{\text{in } L^{\infty}(\Omega_2, \mu_2)}{\mathbf{O}_2 := (X, \mathcal{F}(X), F_2)}. \tag{8}$$

The observable $\Phi_{1,2}\mathbf{O}_2$ is called the *Heisenberg picture representation of* $\mathbf{O}_2$.

Remark 2. Consider times $t_1, t_2 (\in \mathbb{R})$ such that $t_1 \leq t_2$. Then, the (8) usually implies that the observable $\Phi_{t_1,t_2}\mathbf{O}_{t_2}$ at time t_1 can be identified with the observable $\mathbf{O}_{t_2}$ at time t_2. For the more general description of Axiom$_c$2, see "Axiom 2" in [5]. Also, as seen in Section 4 later, we add the importance of time $\mathbb{R}$ and n-dimensional Euclidean space $\mathbb{R}^n$.

It should be noted that mathematics (or, logic) is a kind of language. In a similar sense, we consider that measurement theory is neither physics nor science but a kind of language, that is, Axiom$_c$1 and

$\text{Axiom}_c 2$ are not laws in nature (or, to more general, natural and social science) but linguistic rules. The two axioms teach us how to use the terms: "system", "state", "observable", "measurement", "probability", "measured value", "causal relation" and "Heisenberg picture" in measurement theory (or more precisely, measurement theoretical language). That is, measurement theory is a method to improve a qualitative ordinary language to a powerful qualitative language. Also, in order to use $\text{Axiom}_c 1$ and $\text{Axiom}_c 2$ well, it may be convenient to regard "observable" [resp. $\text{Axiom}_c 1$; $\text{Axiom}_c 2$] as the quantification of division (or, classification)" [resp. "belief" ; "causality (cf. [1, 6])"]. In this sense, the term: "probability" in $\text{Axiom}_c 1$ might have had to be called "belief degree".

Recall, as stated in [4], that measurement theory covers DST, probability theory, Fisher's statistics, Bayesian statistics, control theory, practical logic etc. Thus, we expect that the measurement theoretical language (=MTL in (4), (5)) has a great power to describe most phenomena in our usual life (i.e., economics, psychology, engineering, biology and so on). If we are allowed to use Wittgenstein's term [12], we can say:

(A') Measurement theory (or precisely, measurement theoretical language (=MTL)) is the language game under the linguistic rules (i.e., $\text{Axiom}_c 1$ and $\text{Axiom}_c 2$). That is, we see:

$$\underset{\text{(quantitative language)}}{\text{“MTL”}} = \underset{\text{(qualitative language)}}{\text{“ordinary language”}} + \underset{\text{(rules made express statement)}}{\text{MT(= Axioms 1 and 2 in (2))}}. \tag{9}$$

Here, $\text{Axiom}_c 1$ [resp. $\text{Axiom}_c 2$] is, roughly speaking, considered as the rule of the quantification of "belief" [resp. "causality"].

According to the spirit of "the linguistic turn (cf. [9, 12])", we agree to the view that *language constitutes reality*, though it is contrary to the common sense of physics. Thus, in the language game, the question "What is probability?" should be replaced by the question "How do we use the term: probability?" That is because *it is a situation that finally decides the meaning of the word*, or, *the meaning of a word is its use in the* *language game*. For completeness, again note that measurement theory does not insist on something (like the truth) but it is only a language.

The following example will promote a better understanding of the above axioms. Also, this is the preparation for the argument of Section 4.

Example 1. Put $T_n = \{0, 1, 2, ..., n-1, n\}$. For each $t \in T_n$, consider an observable $\mathbf{O}_t (:= (X_t, \mathcal{F}(X_t), F_t))$ in $L^\infty(\Omega_t, \mu_t)$. And, for each $t \in T_n \setminus \{0\}$, consider a Markov operator $\Phi_{t-1,t} : L^\infty(\Omega_t, \mu_t) \to L^\infty(\Omega_{t-1}, \mu_{t-1})$. Put $\widetilde{\mathbf{O}}_n^{T_n} = \mathbf{O}_n$. Axiom$_c$2 says that $\widetilde{\mathbf{O}}_n^{T_n} (:= \mathbf{O}_n)$ in $L^\infty(\Omega_n, \mu_n)$ can be identified with $\Phi_{n-1,n}\widetilde{\mathbf{O}}_n^{T_n}$ in $L^\infty(\Omega_{n-1}, \mu_{n-1})$. Thus, we get the product observable $\mathbf{O}_{n-1} \times \Phi_{n-1,n}\widetilde{\mathbf{O}}_n^{T_n} (:= (X_{n-1} \times X_n, \mathcal{F}(X_{n-1} \times X_n), F_{n-1} \times \Phi_{n-1,n}F_n))$ in $L^\infty(\Omega_{n-1}, \mu_{n-1})$. Similarly, putting $\widetilde{\mathbf{O}}_{n-1}^{T_n} = \mathbf{O}_{n-1} \times \Phi_{n-1,n}\widetilde{\mathbf{O}}_n^{T_n}$, we get $\widetilde{\mathbf{O}}_{n-2}^{T_n} := \mathbf{O}_{n-2} \times \Phi_{n-2,n-1}\widetilde{\mathbf{O}}_{n-1}^{T_n}$ in $L^\infty(\Omega_{n-2}, \mu_{n-2})$. And, finally, we have $\widetilde{\mathbf{O}}_{\mathbf{0}}^{T_n} := \mathbf{O}_0 \times \Phi_{0,1}\widetilde{\mathbf{O}}_1^{T_n}$ in $L^\infty(\Omega_0, \mu_0)$. Thus, for any initial state $\omega_0 (\in \Omega_0)$, we have a measurement $\mathbf{M}_{L^\infty(\Omega_0, \mu_0)}(\widetilde{\mathbf{O}}_{\mathbf{0}}^{T_n}, S_{[\omega_0]})$, which is called "*a measurement in time series* T_n" if T_n is interpreted as time series. Put $\widetilde{\mathbf{O}}_{\mathbf{0}}^{T_n} = (\times_{t \in T_n} X_t, \mathcal{F}(\times_{t \in T_n} X_t), \widetilde{F}_0^{T_n})$. Then, Axiom$_c$1 says that the probability that a measured value $(x_t)_{t \in T_n} (\in \times_{t \in T_n} X_t)$ obtained by the measurement $\mathbf{M}_{L^\infty(\Omega_0, \mu_0)}(\widetilde{\mathbf{O}}_0^{T_n}, S_{[\omega_0]})$ belongs to $\times_{t \in T_n} \Xi_t$ is given by $[\widetilde{F}_0^{T_n}(\times_{t \in T_n} \Xi_t)](\omega_0)$. Also, we add that the pair $[\omega_0; \{\Phi_{t-1,t}\}_{t \in T_n \setminus \{0\}}]$ is called a *general state*.

3. Dynamical System Theoretical Explanation of Zeno's Paradox

We believe that the problem concerning quantitative language takes its origin from Zeno's paradoxes (cf. [8]). Thus we study Zeno's paradox (Achilles and the Tortoise) in what follows. The following problem was first stated 2500 years ago by Zeno of Elea.

(P_Z) In a race, can the quickest runner catch up with the slowest?

His interesting answer is as follows:

(A_Z) It is impossible. That is because the pursuer must first reach the point whence the pursued started, so that the slowest must always hold a lead.

Of course, we (as well as Zeno) know that this (A_Z) is not true. The paradox (A_Z) is clearly due to the incomplete quantitativeness of our ordinary language.

Someone may consider that Zeno's paradox was already clarified completely, since the sum of infinite series was completely understood by several great mathematicians (e.g., Cauchy, Dedekind, Cantor, etc.). The following elementary explanation (A_D) is usually considered to be standard. (In the next Section 4 we will assert that the (A_D) is not final, cf. Remark 3(a) later.)

(A_D) For example, assume that the velocity v_q [resp. v_s] of the quickest [resp. slowest] runner is equal to $v(>0)$ [resp. γv $(0<\gamma<1)$]. And further, assume that the position of the quickest [resp. slowest] runner at time $t=0$ is equal to 0 [resp. $a(>0)$]. Thus, we can assume that the position $\xi(t)$ [resp. $\eta(t)$] of the quickest [resp. slowest] runner at time $t(\geq 0)$ is represented by

$$\frac{d\xi}{dt}=v, \quad \xi(0)=0 \quad \left[\text{resp. } \frac{d\eta}{dt}=\gamma v, \quad \eta(0)=a\right].$$

Thus, $\xi(t)=vt$ [resp. $\eta(t)=\gamma vt+a$]. Put $t_k=\dfrac{(1-\gamma^k)a}{(1-\gamma)v}$ $(k=0,1,\dots)$. Then we see that

$$(\xi(t_k), \eta(t_k))=\left(\frac{(1-\gamma^k)a}{1-\gamma}, \frac{(1-\gamma^{k+1})a}{1-\gamma}\right)\to\left(\frac{a}{1-\gamma}, \frac{a}{1-\gamma}\right)$$

as $k\to\infty$. Here note that $\eta(t_k)=\xi(t_{k+1})<\eta(t_{k+1})=\xi(t_{k+2})$ $(k=0,1,2,\dots)$. Also, the quickest runner catches up with the slowest at time $s_0=\dfrac{a}{(1-\gamma)v}$ (since $\xi(s_0)=\eta(s_0)$ must hold).

Although this explanation (A_D) is, from the practical point of view, satisfactory, we assume, from the pure theoretical point of view, that it is not sufficient. The above argument should not be based on Newton mechanics since Zeno's paradox (Achilles and the Tortoise) may be the economic competition among countries. Thus, now we have the question: "What kind of quantification is the above explanation (A_D) based on?". We consider that the above argument should be based on DST. However, if it is so, the above dynamical system theoretical explanation (A_D) is nonsense without the answer to the question: "What is DST?". That is, we must answer the question: "Does DST not have a certain assertion or have it?" or, "Is DST more than a language or not?". This is important. That is because, if DST asserts a certain principle, we must say that the explanation (A_D) is true "under the principle". For example, if we start from the $(\sharp_1)$ in Section 5, we must say that the explanation (A_D) is true "under the mechanical world view". However, we believe, as mentioned in the (A) in Section 1, that DST is nothing more than a language. This will be discussed in Section 4 after presenting the translation of the above (A_D) to measurement theory.

4. Measurement Theoretical Explanation of Zeno's Paradox

Since MT (or, CMT) is a language (as stated in Section 2), and moreover, $\text{DST} \subset \text{CMT}$, it is natural to consider that DST is also a language, and therefore, the dynamical system theoretical explanation (A_D) in Section 3 can be translated to CMT. This will be done in what follows. Thus the following argument should be read in comparison with the (A_D) in Section 3.

Put $\Omega = \mathbb{R}^2_{\xi\eta} = \{(\xi, \eta) : \xi, \eta \in \mathbb{R}\}$, which is regarded as the state space in Zeno's problem (P_Z). And consider the 2-dimensional Lebesgue measure space $(\mathbb{R}^2_{\xi\eta}, \mathcal{F}(\mathbb{R}^2_{\xi\eta}), m^2)$. A state $(\xi(t), \eta(t))\ (\in \mathbb{R}^2_{\xi\eta})$ means that $\xi(t)$ [resp. $\eta(t)$] is the position of the quickest [resp. slowest] runner at time $t (\in [0, \infty))$. Recalling the situation in (P_Z) and (A_D), we define the

Markov operator $\Phi_{t,s} : L^\infty(\mathbb{R}^2_{\xi\eta}, m^2) \to L^\infty(\mathbb{R}^2_{\xi\eta}, m^2)$, $(0 \le \forall t \le \forall s < \infty)$, such that:

$$(\Phi_{t,s} f)(\xi, \eta) = f(\xi + v_q(s-t), \eta + v_s(s-t)) \quad (v_q = v, v_s = \gamma v \text{ in } (A_D))$$

$$(\forall(\xi, \eta) \in \mathbb{R}^2_{\xi\eta}, \forall f \in L^\infty(\mathbb{R}^2_{\xi\eta}, m^2), \forall t, s \in [0, \infty) \text{ such that } t \le s). \quad (10)$$

Put $T_n = \{t_0(=0), t_1, t_2, \ldots, t_{n-1}\} \cup \{s_0\}$ where $t_k = \dfrac{(1-\gamma^k)a}{(1-\gamma)v}$, $(k = 0, 1, \ldots, n-1)$ and $s_0 = \dfrac{a}{(1-\gamma)v}$. And, for each $t (\in T_n)$, consider the *position observable* $\mathbf{O} := (\mathbb{R}^2_{\xi\eta}, \mathcal{F}(\mathbb{R}^2_{\xi\eta}), F)$ in $L^\infty(\mathbb{R}^2_{\xi\eta}, m^2)$ such that:

$$F(\Xi)(\xi, \eta) = 1 \text{ (if } (\xi, \eta) \in \Xi \in \mathcal{F}(\mathbb{R}^2_{\xi\eta})), \ = 0 \text{ (if } (\xi, \eta) \notin \Xi \in \mathcal{F}(\mathbb{R}^2_{\xi\eta})). \quad (11)$$

Then, by the same arguments in Example 1, we can get the observable $\widetilde{\mathbf{O}}^{T_n}_{t_0} := ((\mathbb{R}^2_{\xi\eta})^{T_n}, \mathcal{F}((\mathbb{R}^2_{\xi\eta})^{T_n}), \widetilde{F}^{T_n}_{t_0})$ where $(\mathbb{R}^2_{\xi\eta})^{T_n} = (\mathbb{R}^2_{\xi\eta})^{n+1}$. Since it holds that $\Phi_{t,s}(f \times g) = \Phi_{t,s}(f) \times \Phi_{t,s}(g)$, we see, for any $\times_{t \in T_n} \Xi_t$ $(\forall \Xi_t \in \mathcal{F}(\mathbb{R}^2_{\xi\eta}), \forall t \in T_n)$,

$$\widetilde{F}^{T_n}_{t_0}(\mathop{\times}_{t \in T_n} \Xi_t) = F(\Xi_{t_0}) \times \Phi_{t_0, t_1} \widetilde{F}^{T_n}_{t_1}(\mathop{\times}_{t \in T_n \setminus \{t_0\}} \Xi_t)$$

$$= F(\Xi_{t_0}) \times \Phi_{t_0, t_1} \widetilde{F}(\Xi_{t_1}) \times \Phi_{t_0, t_2} \widetilde{F}^{T_n}_{t_2}(\mathop{\times}_{t \in T_n \setminus \{t_0, t_1\}} \Xi_t) = \cdots = \mathop{\times}_{t \in T_n} \Phi_{t_0, t} F(\Xi_t),$$

which and (10) imply that

$$[\widetilde{F}^{T_n}_{t_0}(\mathop{\times}_{t \in T_n} \Xi_t)](\xi, \eta) = \mathop{\times}_{t \in T_n}([F(\Xi_t)](\xi + vt, \eta + \gamma vt)). \quad (12)$$

Next, put $T = \{t_0(=0), t_1, t_2, \ldots\} \cup \{s_0\}$. Define the observable $\widetilde{\mathbf{O}}^T_{t_0} := ((\mathbb{R}^2_{\xi\eta})^T, \mathcal{F}((\mathbb{R}^2_{\xi\eta})^T), \widetilde{F}^T_{t_0})$, (symbolically, $\widetilde{\mathbf{O}}^T_{t_0} = \lim_{n \to \infty} \widetilde{\mathbf{O}}^{T_n}_{t_0}$), such that

$$\widetilde{F}^T_{t_0}((\mathop{\times}_{t \in T_n} \Xi_t) \times (\mathop{\times}_{t \in T \setminus T_n} \mathbb{R}^2_{\xi\eta})) = \widetilde{F}^{T_n}_{t_0}((\mathop{\times}_{t \in T_n} \Xi_t) \ ((\forall \Xi_t \in \mathcal{F}(\mathbb{R}^2_{\xi\eta}), \forall n = 1, 2, \ldots).$$

The existence of $\widetilde{\mathbf{O}}_{t_0}^T$ is shown by W^*-algebraic Kolmogorov's extension theorem (cf. [4]), or directly by the following formula (obtained by the hint of (12) and (11)):

$$[\widetilde{F}_{t_0}^T(\times_{t \in T_n} \Xi_t)](\xi, \eta) = 1 \text{ (if } (\xi + vt, \eta + \gamma vt) \in \Xi_t (\forall t \in T)), = 0 \text{ (otherwise)}. \quad (13)$$

And therefore, we get the measurement $\mathbf{M}_{L^\infty(\mathbb{R}^2_{\xi\eta}, m^2)}(\widetilde{\mathbf{O}}_{t_0}^T, S_{[(0,a)]})$ *in time series T*. (It is surprising that infinite measurements are possible!) For any positive ε, put

$$\Xi_t^\varepsilon = \{(\xi, \eta) \in \mathbb{R}^2_{\xi\eta} : |\xi - vt| < \varepsilon, \ |\eta - (a + \gamma vt)| < \varepsilon\} \quad (\forall t \in T). \quad (14)$$

Then, Axiom$_c$1 says, (by (13), (14)),

- the probability that a measured value $((x_t, y_t))_{t \in T}$ obtained by $\mathbf{M}_{L^\infty(\mathbb{R}^2_{\xi\eta}, m^2)}(\widetilde{\mathbf{O}}_{t_0}^T, S_{[(0,a)]})$ belongs to the set $\times_{t \in T} \Xi_t^\varepsilon$ is given by $[\widetilde{F}_0^T(\times_{t \in T} \Xi_t^\varepsilon)](0, a) = 1$.

Since ε is arbitrary positive, we see that $((x_{t_k}, y_{t_k}))_{k=0}^\infty = ((vt_k, a + \gamma vt_k))_{k=0}^\infty$ $= \left(\left(\dfrac{(1-\gamma^k)a}{1-\gamma}, \dfrac{(1-\gamma^{k+1})a}{1-\gamma}\right)\right)_{k=0}^\infty$, and $(x_{s_0}, y_{s_0}) = \left(\dfrac{a}{1-\gamma}, \dfrac{a}{1-\gamma}\right)$. Thus, the quickest runner catches up with the slowest at time $s_0 = \dfrac{a}{(1-\gamma)v}$.

This is the measurement theoretical answer to Zeno's problem (P_Z). From the spirit of "the language game (cf. [9, 12])", we can conclude that the above explanation can be trusted under the hypothesis that we have the ability to use MTL well. Also, note that Zeno's paradox (A_Z) suggests that the logical power of a qualitative ordinary language is not sufficient.

In general, a language has two aspects, i.e., "expression" and "logic". For example, note that a picture (or, photograph) does not have the aspect of "logic" but "expression". In fact, the phrase "negative sentence" is meaningful, but the phrase "negative picture" is not so. Thus, we consider that the logical aspect of a language is surprising. The question: "Why does a language have the logical aspect?" is important. As seen in the proof of "syllogism" (cf. [3]), we believe that the logical property is essentially due to the quantitative property of a language (particularly, Axiom 1).

Since $\mathrm{DST} \subset \mathrm{CMT}$, we consider that DST is also a quantitative language. Thus, we can trust the dynamical system theoretical explanation (A_D) in Section 3 as well as the measurement theoretical explanation in this section. That is because the two are essentially paraphrases, i.e., the translation from DST to CMT.

Remark 3. (a) The main idea in the above explanation is a slight improvement of the idea used in Chap. 10 (Newtonian mechanics in measurement theory) in [4]. In measurement theory, only one measurement is permitted (as stated in $\mathrm{Axiom}_c 1$). Thus, the essential part in the above explanation is to regard "infinite measurements" (i.e., $\{\mathbf{M}_{L^\infty(\mathbb{R}^2_{\xi\eta}, m^2)}(\Phi_{t_0, t}\mathbf{O}, S_{[(0, a)]})\}_{t \in T}$) as one measurement (i.e., $\mathbf{M}_{L^\infty(\mathbb{R}^2_{\xi\eta}, m^2)}(\widetilde{\mathbf{O}}_{t_0}^T, S_{[(0, a)]})$). We admit that the measurement theoretical explanation is not completely corresponding to our usual sense of "motion". In spite of this disagreement, we believe that the measurement theoretical explanation is the final conclusion concerning Zeno's paradoxes in the long history of 2500 years. That is because any formal explanation should be always represented after answering "What kind of quantification is used?" (i.e., after declaring the (9) (or, the (A')) in Section 2). Also, note that the disagreement between theory and sense is rather ordinary in modern physics (particularly, the theory of relativity, quantum theory).

(b) In general, Zeno's paradoxes (cf. [8]) are composed of four paradoxes, i.e., "Achilles and the tortoise", "dichotomy", "arrow" and "stadium". The three formers can be easily formulated in measurement theory since these are essentially the same. For the stadium paradox, it suffices to change continuous time $[0, \infty)$ to discrete time $\{0, 1, 2, \ldots\}$. Zeno's paradoxes were studied from various view points, e.g., the consideration in the theory of relativity (cf. Part 4 in [8]), the infinite divisibility of time and space (which may contradict Heisenberg's uncertainty relation, cf. [2, 4, 11]), and so on. However, we consider, from the viewpoint of the language game, these are not natural.

Remark 4. Although we do not intend to propose measurement theory as philosophy, they might have to relate. In fact, Socrates' original motivation was to clarify the definition of words (or, concepts, observables). If it is allowed to assume the correspondences: "observable $\leftrightarrow$ idea(Plato)", "state $\leftrightarrow$ eidos(Aristotle)", "system (i.e., measuring object) $\leftrightarrow$ hule + eidos", we can understand Greek philosophy (due to Plato and Aristotle) in terms of measurement theory. In fact, we consider that what Axiom_c 1 says is essentially the same as the following concrete statement:

- $\underset{\text{(measure)}}{\text{Examing}}$ whether the $\underset{\text{(system)}}{\text{water}}$ of $\underset{(\text{state};\, eidos)}{5^\circ\text{C}}$ is $\underset{(\text{observable};\, idea)}{\text{cold or hot}}$, $\underset{\text{(observer)}}{\text{I}}$ $\underset{\text{(probability)}}{\text{surely}}$ $\underset{\text{(obtain)}}{\text{feel}}$ it $\underset{\text{(measured value)}}{\text{cold.}}$

which seems "tautology (or, definition, directions for use)" such as "My daughter is a woman". Also, we like the following metaphor:

$$\underset{\text{(in the real world)}}{\overset{\text{(ordinary language)}}{\text{qualitative language}}} \;\underset{\text{forgetting}}{\overset{\text{anamnesis}}{\rightleftarrows}}\; \underset{\text{(in the idea world)}}{\overset{\text{(MTL)}}{\text{qualitative language}}} \tag{15}$$

Further, note that "perception ($\approx$ measurement)" and "causal relation" are central subjects in modern philosophy (i.e., from Descartes to Kant). We consider that "Copernican revolution" and "linguistic turn" are similar in some sense, though the former [resp. the latter] is somewhat related to cognitive science [resp. logos] (cf. [6, 9, 12]). Also recall that I. Kant emphasized the importance of time and space $\mathbb{R} \times \mathbb{R}^n$ (cf. Remark 2), though the completeness of $\mathbb{R}$ is due to Dedekind.

5. Conclusions

In [4] we asserted:

($\sharp_1$) Measurement theory (=MT) is the mathematical representation of the epistemology called "the mechanical world view". And thus, dynamical system theory (=DST) is also so, since $\mathrm{DST} \subset \mathrm{MT}$.

Although this is an opinion, in this paper we focus on the linguistic aspect of measurement theory. In [9], which made the impressive advertisement "the linguistic turn" known, R. Rorty said that *philosophical problems are problems which may be solved* (*or dissolved*) *either by reforming language*, *or by understanding more about the language we presently use*. We agree with him. In fact, in this paper we propose measurement theory as a method to improve a qualitative language to a powerful quantitative language. That is, instead of the above $(\sharp_1)$, we assert:

$(\sharp_2)$ Measurement theoretical language (=MTL) is a powerful quantitative language with the linguistic rules (i.e., $\text{Axiom}_c 1$ and $\text{Axiom}_c 2$). And thus, dynamical system theory is also so (i.e., the assertion (A) in Section 1 holds)), since $\mathrm{DST} \subset \mathrm{MT}$,

though there may be an opinion that this $(\sharp_2)$ is the paraphrase of the above $(\sharp_1)$. Here we believe that this $(\sharp_2)$ is superior to the above $(\sharp_1)$. That is because the meaning of the philosophical phrase: "the mechanical world view" in the $(\sharp_1)$ is somewhat ambiguous. On the other hand, under the hypothesis $(\sharp_2)$, measurement theory can acquire a "neutral" position, and thus, we can discuss measurement theory in the spirit of the language game.

Under the assumption $(\sharp_2)$, MT as well as DST can be considered as a kind of language respectively. Thus, it is natural to show that the explanation in Section 3 (i.e., the dynamical system theoretical explanation about Zeno's paradox) is paraphrased (or, translated) to the measurement theoretical explanation in Section 4. Then, we can conclude, from the spirit of "the language game (cf. [9, 12])", that the two explanations can be trusted under the hypothesis that we have the ability to use MTL well.

If we want to describe phenomena precisely, we must have the language with a great power of expression. Thus, it is a matter of course that a lot of people have been interested in the problem: how to create a powerful quantitative language. In fact, Zeno's paradoxes and Alistotle's syllogism (cf. [3]) may be the oldest linguistic problems concerning quantativeness ($\approx$ logic). Also, recall the enthusiastic fashion of fuzzy sets theory [13] about twenty years ago, which is clearly related to quantification. We believe that measurement theory has a great power of expression of our usual phenomena, and therefore, we hope that measurement theory (or, measurement theoretical language) will be generally accepted as the standard quantitative ordinary language.

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