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SOME WEIGHTED ZETA FUNCTIONS OF DIGRAPHS

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Abstract

We consider a weighted zeta function and a weighted L-function of a digraph D, and give determinant expressions of them. Furthermore, we show that a weighted zeta function and a weighted L-function of D are equal to that of the line digraph $\vec{L}(D)$ of D, respectively. As corollaries, we show that a zeta function and an L-function of D are equal to that of the line graph $\vec{L}(D)$ of D, respectively. Moreover, we express characteristic polynomials of a weighted matrix and the adjacency matrix of $\vec{L}(D)$ by those of D.

1. Introduction

Graphs and digraphs treated here are finite and simple. Let $G = (V(G), E(G))$ be a connected graph with vertex $V(G)$ and edge set $E(G)$, and D be the symmetric digraph corresponding to G. Furthermore, set $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. Note that $D(G) = A(D)$. For $e = (u, v) \in D(G)$, let $o(e) = u$ and $t(e) = v$. The inverse arc of e is denoted by e^{-1}. A *path P of length n* in G is a sequence $P = (e_1, \ldots, e_n)$ of n arcs such that

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$e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ $(1 \le i \le n-1)$. If $e_i = (v_{i-1}, v_i)$, $1 \le i \le n$, then we also denote $P = (v_0, v_1, \dots, v_n)$. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$-*path*. We say that a path $P = (e_1, \dots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some $i\,(1 \le i \le n-1)$. A (v, w)-path is called a v-*cycle* (or v-*closed path*) if $v = w$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \dots, e_m)$ and $C_2 = (f_1, \dots, f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if C has no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle.

The (*Ihara*) *zeta function* of a graph G is defined to be a formal power series of a variable u, by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $|C|$ is the length of C.

Zeta functions of graphs started from zeta functions of regular graphs by Ihara [7]. In [7], he showed that their reciprocals are explicit polynomials. A zeta function of a regular graph G associated to a unitary representation of the fundamental group of G was developed by Sunada [17, 18]. Hashimoto [6] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is a polynomial:

$$\mathbf{Z}(G, u)^{-1} = (1 - u^2)^{r-1} \det(\mathbf{I} - u\mathbf{A}(G) + u^2(\mathbf{D} - \mathbf{I})),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i (V(G) = \{v_1, \dots, v_n\})$.

Various proofs of Bass' Theorem were given by Stark and Terras [16], Foata and Zeilberger [4] and Kotani and Sunada [10].

Cycles, reduced cycles and prime cycles in a simple digraph which is not symmetric are defined similarly to the case of a symmetric digraph. Let D be a connected digraph. Then, the *zeta function* of D is defined to be a formal power series of a variable u, by

$$\mathbf{Z}(D, u) = \mathbf{Z}_D(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D.

Let D have n vertices $v_1, \ldots, v_n$. The *adjacency matrix* $\mathbf{A} = \mathbf{A}(D) = (a_{ij})$ of D is the square matrix of order n such that $a_{ij} = 1$ if there exists an arc starting at the vertex v_i and terminating at the vertex v_j, and $a_{ij} = 0$ otherwise.

Bowen and Lanford [3] gave a determinant expression of the zeta function of a connected digraph D (c.f., [10, 11]).

Theorem 1 (Bowen and Lanford)**.** $\mathbf{Z}(D, u)^{-1} = \det(\mathbf{I} - \mathbf{A}(D)u)$.

Kotani and Sunada [10] stated a connection between zeta functions of graphs and that of oriented line graphs. Let G be a connected non-circuit graph. Then the *oriented line graph* $\vec{L}(G) = (V_L, A_L)$ of G is defined as follows:

$$V_L = D(G);\ A_L = \{(e_1, e_2) \in D(G) \times D(G) \mid e_1^{-1} \neq e_2,\ t(e_1) = o(e_2)\}.$$

There exist no reduced cycles in the oriented line graph. Thus, there is a one-to-one correspondence between prime cycles in $\vec{L}(G)$ and prime, reduced cycles in G, and so

$$\mathbf{Z}(G, u) = \mathbf{Z}(\vec{L}(G), u).$$

Foata and Zeilberger [4] gave a new proof of Bass' Theorem by using the algebra of Lyndon words. Let X be a finite nonempty set, $<$ be a total order in X, and X^* be the free monoid generated by X. Then the total order $<$ on X derive the lexicographic order $<$ on X^*. A *Lyndon word* in X is defined to a nonempty word in X^* which is prime, i.e., not the power l^r of any other word l and any $r \geq 2$, and which is also minimal in the class of its cyclic rearrangements under $<$ (see [9]). Let L denote the set of all Lyndon words in X.

Let $\mathbf{B}$ be a square matrix whose entries $b(x, x')$ $(x, x' \in X)$ form a set of commuting variables. If $w = x_1 x_2 \cdots x_m$ is a word in X^*, then define

$$\beta(w) = b(x_1, x_2) b(x_2, x_3) \cdots b(x_{m-1}, x_m) b(x_m, x_1).$$

Furthermore, let

$$\beta(L) = \prod_{l \in L} (1 - \beta(l)).$$

The following theorem played a central role in [4].

Theorem 2 (Foata and Zeilberger)**.** $\beta(L) = \det(\mathbf{I} - \mathbf{B})$.

Foata and Zeilberger [4] gave a short proof of Amitsur's identity [1] by using Theorem 2.

Theorem 3 (Amitsur)**.** *For square matrices* $\mathbf{A}_1, \ldots, \mathbf{A}_k$,

$$\det(\mathbf{I} - (\mathbf{A}_1 + \cdots + \mathbf{A}_k)) = \prod_{l \in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where the product runs over all Lyndon words in $\{1, \ldots, k\}$, *and* $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ *for* $l = i_1 \cdots i_p$.

In Section 2, we consider a weighted zeta function of a digraph D, and give a determinant expression of it. In Section 3, we show that the weighted zeta function of the line digraph $\vec{L}(D)$ of D is equal to that of D. As corollaries, we express characteristic polynomials of a weighted matrix and the adjacency matrix of $\vec{L}(D)$ by those of D. In Section 4, we define a weighted L-function of D, and present its determinant expression. Furthermore, we show that the weighted L-function of $\vec{L}(D)$ of D is equal to that of D.

For a general theory of the representation of groups, the reader is referred to [15].

2. Weighted Zeta Functions of Digraphs

Let D be a connected digraph and $V(D) = \{v_1, \dots, v_n\}$. Then we consider an $n \times n$ matrix $\mathbf{W} = \mathbf{W}(D) = (w_{ij})_{1 \le i, j \le n}$ with ij entry the nonzero variable w_{ij} if $(v_i, v_j) \in A(D)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W}(D)$ is called the *weighted matrix* of D. For each path $P = (v_{i_1}, \dots, v_{i_r})$ of D, the *norm* $w(P)$ of P is defined as follows: $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(D)$ and $w(e) = w_{ij}$, $e = (v_i, v_j) \in A(D)$. The *weighted zeta function* of D is defined by

$$\mathbf{Z}(D, w, u) = \prod_{[C]} (1 - w(C) u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D.

Theorem 4. *Let D be a connected digraph. Then the reciprocal of the weighted zeta function of D is given by*

$$\mathbf{Z}(D, w, u)^{-1} = \det(\mathbf{I} - u\mathbf{W}(D)).$$

Proof. Let $V(D) = \{v_1, \dots, v_n\}$ and $v_1 < v_2 < \cdots < v_n$ be a total order of $V(D)$. We consider the free monoid $V(D)^*$ generated by $V(D)$, and the lexicographic order on $V(D)^*$ derived from $<$. If a cycle C is prime, then there exists a unique cycle in $[C]$ which is a Lyndon word in $V(D)$.

For $z \in V(D)^*$, let

$$\beta(z) = \begin{cases} w(z)u & \text{if } z \text{ is a prime cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\beta(L) = \prod_{l \in L} (1 - \beta(l)) = \prod_{[C]} (1 - w(C) u^{|C|}),$$

where $[C]$ runs over all equivalence classes of prime cycles of D. Furthermore, we define variables $b(x, x')(x, x' \in V(D))$ as follows:

$$b(x, x') = \begin{cases} w(x, x')u & \text{if } (x, x') \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2 implies that

$$\prod_{[C]} (1 - w(C)u^{|C|}) = \det(\mathbf{I} - \mathbf{B}) = \det(\mathbf{I} - u\mathbf{W}(D)).$$

The formula $\mathbf{Z}(D, w, u)^{-1} = \det(\mathbf{I} - u\mathbf{W}(D))$ is also a specialization of Theorem 3. Furthermore, Theorem 4 is obtained from [12, Theorem 4].

3. Weighted Zeta Functions of Line Graphs of Digraphs

Let D be a connected digraph and $\mathbf{W}(D)$ be a weighted matrix of D. Then the *line digraph* $\vec{L}(D) = (V_L, A_L)$ of D is defined as follows:

$$V_L = A(D); \quad A_L = \{(e_1, e_2) \in A(D) \times A(D) \mid t(e_1) = o(e_2)\}.$$

Furthermore, we define the *weighted matrix* $\vec{\mathbf{W}} = \mathbf{W}(\vec{L}(D)) = (\vec{w}_L(e, f))$ of $\vec{L}(D)$ *derived from* $\mathbf{W}(D)$ as follows:

$$\vec{w}_L(e, f) := \begin{cases} w(e) & \text{if } (e, f) \in A(\vec{L}(D)), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5. *Let D be a connected digraph and $\mathbf{W}(D)$ be a weighted matrix of D. Then*

$$\mathbf{Z}(\vec{L}(D), \vec{w}_L, u) = \mathbf{Z}(D, w, u).$$

Proof. At first, there is a one-to-one correspondence between prime cycles in $\vec{L}(D)$ and prime cycles in D. Let $\vec{C}$ be the prime cycle of $\vec{L}(D)$ corresponding to a prime cycle C in D. Then we have

$$w(C) = \vec{w}_L(\vec{C}),$$

and so

$$\mathbf{Z}(\vec{L}(D), \vec{w}_L, u) = \mathbf{Z}(D, w, u).$$

Next, we give another proof of Theorem 5 by using Theorem 3.

Another proof of Theorem 5. Let $A(D) = \{e_1, \dots, e_m\}$. For each arc $e_r \in A(D)$, let $\mathbf{X}_{e_r}$ be the $m \times m$ matrix whose r row is the r row of $\vec{\mathbf{W}}(\vec{L}(D))$, and whose other rows are $\mathbf{0}$. Set $\mathbf{M} = \mathbf{I} - u\sum_{e \in A(D)} \mathbf{X}_e$. Then, for any sequence of arcs π,

$$\det(\mathbf{I}_m - u\mathbf{X}_\pi) = \begin{cases} 1 - \vec{w}_L(\pi)u^{|\pi|} & \text{if } \pi \text{ is a cycle,} \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathbf{X}_\pi = \mathbf{X}_{e_1} \cdots \mathbf{X}_{e_r}$ for $\pi = (e_1 \cdots e_r)$. By Theorems 3 and 4, we have

$$\mathbf{Z}(D, w, u)^{-1} = \det \mathbf{M} = \mathbf{Z}(\vec{L}(D), \vec{w}_L, u)^{-1}.$$

Corollary 1. *Let D be a connected digraph with n vertices and l unoriented edges, and let $\mathbf{W}(D)$ be a weighted matrix of D. Then we have*

$$\det(\mathbf{I}_l - u\mathbf{W}(\vec{L}(D))) = \det(\mathbf{I}_n - u\mathbf{W}(D)).$$

The *characteristic polynomial* of a square matrix $\mathbf{B}$ is defined by $\Phi(\mathbf{B}; \lambda) = \det(\lambda\mathbf{I} - \mathbf{B})$.

Corollary 2. *Let D be a connected digraph with n vertices and l unoriented edges, and let $\mathbf{W}(D)$ be a weighted matrix of D. Then we have*

$$\Phi(\mathbf{W}(\vec{L}(D)); \lambda) = \lambda^{l-n}\Phi(\mathbf{W}(D); \lambda).$$

Proof. By Corollary 1, we have

$$\det(\mathbf{I}_l - u\mathbf{W}(\vec{L}(D))) = \det(\mathbf{I}_n - u\mathbf{W}(D)),$$

and so

$$u^l \det\left(\frac{1}{u}\mathbf{I}_l - \mathbf{W}(\vec{L}(D))\right) = u^n \det\left(\frac{1}{u}\mathbf{I}_n - \mathbf{W}(D)\right).$$

Therefore the result follows.

For a connected digraph D, let $\vec{L}^n(D) = \vec{L}(\vec{L}^{n-1}(D))$, $n \geq 1$. We define the *weighted matrix* $\mathbf{W}(\vec{L}^n(D)) = (\vec{w}_{L^n}(e, f))$ of $\vec{L}^n(D)$ derived from $\mathbf{W}(\vec{L}^{n-1}(D))$ as follows:

$$\vec{w}_{L^n}(e, f) := \begin{cases} \vec{w}_{L^{n-1}}(e) & \text{if } (e, f) \in A(\vec{L}^n(D)), \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3. *Let D be a connected digraph and* $\mathbf{W}(D)$ *be a weighted matrix of D. Then we have*

$$\mathbf{Z}(\vec{L}^n(D), \vec{w}_{L^n}, u) = \mathbf{Z}(D, w, u),$$

i.e.,

$$\det(\mathbf{I} - u\mathbf{W}(\vec{L}^n(D))) = \det(\mathbf{I} - u\mathbf{W}(D)).$$

Corollary 4. *Let D be a connected digraph with n vertices, and let* $\mathbf{W}(D)$ *be a weighted matrix of D. Then we have*

$$\Phi(\mathbf{W}(\vec{L}^n(D)); \lambda) = \lambda^{|A(\vec{L}^n(D))|-n}\Phi(\mathbf{W}(D); \lambda).$$

In the case that $w(e) = 1$ for any arc e of a digraph D, the weighted matrix $\mathbf{W}(D)$ is the adjacency matrix $\mathbf{A}(D)$ of D.

By Theorem 5 and Corollary 1, we obtain the following result.

Corollary 5. *Let D be a connected digraph with n vertices and l unoriented edges. Then we have*

$$\mathbf{Z}(\vec{L}(D), u) = \mathbf{Z}(D, u),$$

i.e.,

$$\det(\mathbf{I}_l - u\mathbf{A}(\vec{L}(D))) = \det(\mathbf{I}_n - u\mathbf{A}(D)).$$

Corollary 6. *Let D be a connected digraph. Then, for any positive integer k, we have*

$$\mathbf{Z}(\vec{L}^k(D), u) = \mathbf{Z}(D, u),$$

i.e.,

$$\det(\mathbf{I}_l - u\mathbf{A}(\vec{L}^k(D))) = \det(\mathbf{I}_n - u\mathbf{A}(D)).$$

Kotani and Sunada [10] showed that

$$\mathbf{Z}(D, u) = \exp\left(\sum_{k \geq 1} \frac{N_k}{k} u^k\right),$$

where N_k is the number of cycles with length k in D for each $k \geq 1$.

Corollary 7. *Let D be a connected digraph and k be any positive integer. Furthermore, let $N_k^{(s)}$ be the number of cycles with length k in $\vec{L}^s(D)$ $(s \geq 1)$, where $N_k = N_k^{(1)}$. Then we have*

$$N_k = N_k^{(s)} \text{ for any } s, k \geq 1.$$

Let D be a digraph and $\mathbf{A}(D)$ be its adjacency matrix. Then the *characteristic polynomial* $\Phi(D; \lambda)$ of D is defined by $\Phi(D; \lambda) = \det(\lambda\mathbf{I} - \mathbf{A}(D))$.

By Corollary 2, we obtain the following results (see [8, 13, 14]).

Corollary 8 (Lin, Ning and Zhang; Rosenfeld)**.** *Let D be a connected digraph with n vertices and l unoriented edges. Then we have*

$$\Phi(\vec{L}(D); \lambda) = \lambda^{l-n}\Phi(D; \lambda).$$

Corollary 9 (Pakoński, Tanner and Życzkowski)**.** *Let G be a connected graph with n vertices and m unoriented edges, and let D_G be the symmetric digraph corresponding to G. Then we have*

$$\Phi(\vec{L}(D_G); \lambda) = \lambda^{2m-n}\Phi(G; \lambda).$$

4. Weighted L-functions of Digraphs

Let D be a connected digraph, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. We define the *net voltage* $\alpha(P)$ of each path $P = (v_1, \dots, v_l)$ of D by $\alpha(P) = \alpha(v_1, v_2)\cdots\alpha(v_{l-1}, v_l)$. Furthermore, let ρ be a representation of Γ and d be its degree. The *weighted L-function* of D associated to ρ and α is defined by

$$\mathbf{Z}_D(w, u, \rho, \alpha) = \prod_{[C]} \det(\mathbf{I}_d - \rho(\alpha(C))w(C)u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of D.

For $g \in \Gamma$, let the matrix $\mathbf{W}_g = (w_{uv}^{(g)})$ be defined by

$$w_{uv}^{(g)} := \begin{cases} w(u, v) & \text{if } \alpha(u, v) = g \text{ and } (u, v) \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

Let $1 \le i, j \le n$. Then, the (i, j)-*block* $\mathbf{B}_{i,j}$ of a $dn \times dn$ matrix $\mathbf{B}$ is the submatrix of $\mathbf{B}$ consisting of $d(i-1)+1, \dots, di$ rows and $d(j-1)+1, \dots, dj$ columns.

Theorem 6. *Let D be a connected digraph with l oriented edges, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ, and d be the degree of ρ.*

For $m \ge 1$, let $\mathcal{C}_m$ be the set of all cycles of length m in D. Set

$$N_m(D, w) = N_m = \sum_{C \in \mathcal{C}_m} \mathrm{tr}(\rho(\alpha(C))w(C)).$$

Then the reciprocal of the weighted L-function of D associated to ρ and α is

$$\mathbf{Z}_D(w, u, \rho, \alpha)^{-1} = \det\left(\mathbf{I} - u\sum_{h \in \Gamma} \rho(h) \otimes \mathbf{W}_h\right) = \exp\left(\sum_{m \ge 1} \frac{N_m}{m} u^m\right).$$

Proof. At first, by the Jacobi formula $\det \exp \mathbf{A} = \exp \mathrm{tr}\mathbf{A}$, we have

$$\mathbf{Z}_D(w, u, \rho, \alpha)^{-1} = \prod_{[C]} \det \exp\{-\log(\mathbf{I} - \rho(\alpha(C))w(C)u^{|C|})\}$$

$$= \prod_{[C]} \exp \operatorname{tr}\left(\sum_{m\geq 1} \frac{1}{m} \rho(\alpha(C^m)) w(C)^m u^{m|C|}\right)$$

$$= \exp\left(\sum_{[C]} \sum_{m\geq 1} \frac{1}{m} \operatorname{tr}(\rho(\alpha(C^m))) w(C)^m u^{m|C|}\right)$$

$$= \exp\left(\sum_{m\geq 1} \sum_{C} \frac{1}{m|C|} \operatorname{tr}(\rho(\alpha(C^m))) w(C)^m u^{m|C|}\right)$$

$$= \exp\left(\sum_{m\geq 1} \frac{1}{m} N_m u^m\right).$$

Next, let $V(D) = \{v_1, \ldots, v_n\}$ and consider the lexicographic order on $V(D) \times V(D)$ derived from a total order of $V(D)$: $v_1 < v_2 < \cdots < v_n$. If (v_i, v_j) is the m-th pair under the above order, then we define the $nd \times nd$ matrix $\mathbf{A}_m = ((\mathbf{A}_m)_{p,q})_{1\leq p,q\leq n}$ as follows:

$$(\mathbf{A}_m)_{p,q} = \begin{cases} \rho(\alpha(v_p, v_q)) w(v_p, v_q) u & \text{if } p = i,\ q = j \text{ and } (v_i, v_j) \in A(D), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathbf{B} = \mathbf{A}_1 + \cdots + \mathbf{A}_k$, $k = n^2$. Then we have

$$\mathbf{B} = u \sum_{h} \mathbf{W}_h \otimes \rho(h).$$

Let L be the set of all Lyndon words in $V(D) \times V(D)$. Then we can also consider L as the set of all Lyndon words in $\{1, \ldots, k\}$: $(v_{i_1}, v_{j_1}) \cdots (v_{i_q}, v_{j_q})$ corresponds to $m_1 m_2 \cdots m_q$, where (v_{i_r}, v_{j_r}) $(1 \leq r \leq q)$ is the m_r-th pair. Theorem 3 implies that

$$\det(\mathbf{I}_{nd} - \mathbf{B}) = \prod_{l\in L} \det(\mathbf{I} - \mathbf{A}_l),$$

where $\mathbf{A}_l = \mathbf{A}_{i_1} \cdots \mathbf{A}_{i_p}$ for $l = i_1 \cdots i_p$. Note that $\det(\mathbf{I} - \mathbf{A}_l)$ is the alternating sum of the diagonal minors of A_l. Thus, we have

$$\det(\mathbf{I} - \mathbf{A}_l) = \begin{cases} \det(\mathbf{I} - \rho(\alpha(C)) w(C) u^{|C|}) & \text{if } l \text{ is a prime cycle } C, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$\mathbf{Z}_D(w, u, \rho, \alpha)^{-1} = \det\left(\mathbf{I}_{nd} - u\sum_{h\in\Gamma}\mathbf{W}_h \otimes \rho(h)\right)$$

$$= \det\left(\mathbf{I}_{nd} - u\sum_{h\in\Gamma}\rho(h) \otimes \mathbf{W}_h\right).$$

Hence the result is obtained.

Let D be a connected digraph and $\mathbf{W}(D)$ be a weighted matrix of D. Then we define the function $\alpha_{\vec{L}} : A(\vec{L}(D)) \to \Gamma$ as follows: $\alpha_{\vec{L}}(e, f) = \alpha(e)$, $(e, f) \in A(\vec{L}(D))$. For each path $P = (e_1, \dots, e_r)$ of $\vec{L}(D)$, let $\alpha_{\vec{L}}(P) = \alpha(e_1)\cdots\alpha(e_r)$.

Now, we consider the weighted L-function $\mathbf{Z}_{\vec{L}(D)}(\vec{w}_L, u, \rho, \alpha_{\vec{L}})$ of the line digraph $\vec{L}(D)$ of D associated to ρ and $\alpha_{\vec{L}}$.

For $g \in \Gamma$, let the matrix $\vec{\mathbf{W}}_g = (\vec{w}_{ef}^{(g)})$ be defined by

$$\vec{w}_{ef}^{(g)} := \begin{cases} w(e) & \text{if } \alpha(e) = g \text{ and } (e, f) \in A(\vec{L}(D)), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7. *Let D be a connected digraph with l oriented edges, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ, and d be the degree of ρ. Then the reciprocal of the weighted L-function of D associated to ρ and α is*

$$\mathbf{Z}_D(w, u, \rho, \alpha)^{-1} = \mathbf{Z}_{\vec{L}(D)}(\vec{w}, u, \rho, \alpha_{\vec{L}})^{-1} = \det\left(\mathbf{I} - u\sum_{h\in\Gamma}\rho(h) \otimes \vec{\mathbf{W}}_h\right).$$

Proof. Let $\vec{C}$ be the prime cycle of $\vec{L}(D)$ corresponding to a prime cycle C in D. Then we have

$$\alpha(C) = \alpha_{\vec{L}}(\vec{C}), \quad \text{and} \quad w(C) = \vec{w}_L(\vec{C}),$$

and so

$$\mathbf{Z}_D(w, u, \rho, \alpha)^{-1} = \mathbf{Z}_{\vec{L}(D)}(\vec{w}, u, \rho, \alpha_{\vec{L}})^{-1}.$$

Furthermore, by Theorem 6, we have

$$\mathbf{Z}_{\vec{L}(D)}(\vec{w}, u, \rho, \alpha_{\vec{L}})^{-1} = \det\left(\mathbf{I} - u\sum_{h\in\Gamma}\rho(h)\otimes\vec{\mathbf{W}}_h\right).$$

Note that Theorem 7 is also proved by using Theorem 3.

By Theorems 6 and 7, the following result holds.

Corollary 10. *Let D be a connected digraph with n vertices and l oriented edges, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ, and d be the degree of ρ. Then*

$$\det\left(\mathbf{I}_{ld} - u\sum_{h\in\Gamma}\rho(h)\otimes\vec{\mathbf{W}}_h\right) = \det\left(\mathbf{I}_{nd} - u\sum_{h\in\Gamma}\rho(h)\otimes\mathbf{W}_h\right).$$

By Theorems 6 and 7, the following result holds.

Corollary 11. *Let D be a connected digraph, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. Furthermore, let ρ be a representation* of Γ, *and d be the degree of ρ. Then*

$$N_k(\vec{L}(D), \vec{w}_L) = N_k(D, w) \text{ for any } k \geq 1.$$

By Corollary 10, we obtain a generalization of Corollary 2.

Corollary 12. *Let D be a connected digraph with n vertices and l unoriented edges, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. Furthermore, let ρ be a representation of Γ, and d be the degree of ρ. Then we have*

$$\Phi\left(\sum_{h\in\Gamma}\rho(h)\otimes\vec{\mathbf{W}}_h;\ \lambda\right) = \lambda^{(l-n)d}\Phi\left(\sum_{h\in\Gamma}\rho(h)\otimes\mathbf{W}_h;\ \lambda\right).$$

5. Remark

We state a background for weighted L-functions of digraphs.

We can generalize the notion of a Γ-covering of a graph to a simple digraph. Let D be a connected digraph and Γ be a finite group. Then a mapping $\alpha : A(D) \to \Gamma$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in A(D)$ such that $(v, u) \in A(D)$. The pair (D, α) is called an *ordinary voltage digraph*. The *derived digraph* D^{α} of the ordinary voltage digraph (D, α) is defined as follows: $V(D^{\alpha}) = V(D) \times \Gamma$ and $((u, h), (v, k)) \in A(D^{\alpha})$ if and only if $(u, v) \in A(D)$ and $k = h\alpha(u, v)$. The digraph D^{α} is called a Γ-*covering of* D. Note that a Γ-covering of the symmetric digraph corresponding to a graph G is a Γ-covering of G.

Let D be a connected digraph, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. In the Γ-covering D^{α}, set $v_g = (v, g)$ and $e_g = (e, g)$, where $v \in V(D)$, $e \in A(D)$, $g \in \Gamma$. For $e = (u, v) \in A(D)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1} = (e^{-1})_{g\alpha(e)}$.

Furthermore, we define the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(D^{\alpha}) = (\tilde{w}(u_g, v_h))$ of D^{α} *derived from* $\mathbf{W}(D)$ as follows:

$$\tilde{w}(u_g, v_h) := \begin{cases} w(u, v) & \text{if } (u, v) \in A(D) \text{ and } h = g\alpha(u, v), \\ 0 & \text{otherwise.} \end{cases}$$

Then the following result holds.

Corollary 13. *Let D be a connected digraph, $\mathbf{W}(D)$ be a weighted matrix of D, Γ be a finite group and $\alpha : A(D) \to \Gamma$ be an ordinary voltage assignment. Then we have*

$$\mathbf{Z}(D^{\alpha}, \tilde{w}, u) = \prod_{\rho} \mathbf{Z}_D(w, u, \rho, \alpha)^{\deg \rho} = \mathbf{Z}(\vec{L}(D^{\alpha}), \tilde{\vec{w}}, u)$$

$$= \prod_{\rho} \mathbf{Z}_{\vec{L}(D)}(\vec{w}_L, u, \rho, \alpha_{\vec{L}})^{\deg \rho},$$

where ρ *runs over all irreducible representations of* Γ.

Proof. By a similar proof to that of Theorem 5 in [11] and Theorem 7. Furthermore, the result is obtained from [12, Corollary 1].

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