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A NOTE ON THE POSITIVE SOLUTIONS OF THE DIFFERENCE EQUATION SYSTEM $x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$

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Abstract

The main purpose of this study aims to give general formulas for the positive solutions of the difference equation system

$$x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},\ n = 0, 1, 2, \ldots,$$

where $x_{-1} = k$, $x_0 = h$ and $y_0 = b$, $y_{-1} = a$ are positive real numbers. Thus, we improve the paper in reference [1].

1. Introduction

Cinar [1] investigated the solutions of the difference equation system

$$x_{n+1} = \frac{1}{y_n},\ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},\ n = 0, 1, 2, \ldots, \tag{1.1}$$

where $x_{-1} = k$, $x_0 = h$ and $y_0 = b$, $y_{-1} = a$ are positive real numbers

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and in that paper, he gave only the following two theorems:

Theorem 1 [1]. *Let* $\{x_n, y_n\}$ *be a solution of difference equation system* (1.1). *Then all solutions of equating system* (1.1) *are periodic with period four*.

Theorem 2 [1]. *Let* $\{x_n, y_n\}$ *be a solution of difference equation system*. *Then for* $n = 0, 1, 2, \ldots,$ *all solutions of equation system* (1.1) *are*

$$x_{4n+1} = \frac{1}{b}, \qquad y_{4n+1} = \frac{b}{ka},$$

$$x_{4n+2} = \frac{ak}{b}, \qquad y_{4n+2} = \frac{1}{akh},$$

$$x_{4n+3} = akh, \qquad y_{4n+3} = \frac{1}{h},$$

$$x_{4n+4} = h, \qquad y_{4n+4} = b.$$

From Theorem 2, it is seen that in [1], the solutions $\{x_n, y_n\}_{n=0}^{\infty}$ of equating system (1.1) have been presented in eight forms as x_{4n+1}, x_{4n+2}, x_{4n+3}, x_{4n+4} and y_{4n+1}, y_{4n+2}, y_{4n+3}, y_{4n+4} for $n = 0, 1, 2, \ldots$ In this paper, we have simplified this situation reducing from eight forms to two forms. Also, using periodicity of the sine and cosine functions, we say immediately that the solutions $\{x_n, y_n\}_{n=0}^{\infty}$ of equating system (1.1) are periodic with period four.

2. Main Theorem

Theorem 3. *Let* $\{x_n, y_n\}$ *be a solution of difference equation system*. *Then all solutions of equation system* (1.1) *are*

$$y_n = b^{\frac{1}{2}\left(1+\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right)} k^{\frac{1}{2}\left(-1+\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right)} a^{\sin\left(\frac{n\pi}{2}\right)}, \; n=1, 2, \ldots,$$

$$x_{n+1} = b^{-\frac{1}{2}\left(1+\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right)} k^{-\frac{1}{2}\left(-1+\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right)} a^{-\sin\left(\frac{n\pi}{2}\right)}, \; n=0, 1, 2, \ldots$$

and all solutions of equating system (1.1) *are periodic with period four*.

Proof. By substituting

$$x_{n+1} = \frac{1}{y_n} \tag{2.1}$$

into the equation

$$y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}},$$

we obtain

$$y_{n+1} = \frac{y_{n-2}y_n}{y_{n-1}}$$

or

$$y_{n-1}y_{n+1} = y_{n-2}y_n. \tag{2.2}$$

Letting $z_n = y_{n-1}y_{n+1}$, $n = 0, 1, 2, \ldots$ in equation (2.2) creates

$$z_n - z_{n-1} = 0,\ n = 0, 1, 2, \ldots. \tag{2.3}$$

The characteristic equation of the difference equation (2.3) is $\lambda - 1 = 0$, and thus we have the general solution of (2.3) as

$$z_n = y_{-2}y_0,\ n = 0, 1, 2, \ldots$$

or

$$y_{n-1}y_{n+1} = y_{-2}y_0,\ n = 0, 1, 2, \ldots. \tag{2.4}$$

Since $y_n > 0$ for $n = 0, 1, 2, \ldots,$ using the transformation

$$w_n = \ln y_n,\ n = 0, 1, 2, \ldots \tag{2.5}$$

in equation (2.4), we obtain

$$\exp(w_{n-1} + w_{n+1}) = \exp(w_{-2} + w_0),\ n = 0, 1, 2, \ldots.$$

From where we get

$$w_{n-1} + w_{n+1} = w_{-2} + w_0,\ n = 0, 1, 2, \ldots. \tag{2.6}$$

Since the characteristic equation roots of the homogeneous equation are $\mu_1 = i$ and $\mu_2 = -i$, we obtain the general solution of equation (2.6) as

$$w_n = c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right) + \frac{1}{2}(w_{-2} + w_0), \; n = 0, 1, 2, \dots . \tag{2.7}$$

From equations (2.5) and (2.7), we have

$$y_n = \exp\left(c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right)\right)\sqrt{y_{-2}y_0}, \; n = 0, 1, 2, \dots .$$

To find the constants c_1 and c_2, we use the initial data

$$y_0 = e^{c_1}\sqrt{y_{-2}y_0},$$

$$y_1 = e^{c_2}\sqrt{y_{-2}y_0}.$$

Finally, after solving the above system of equations, we obtain $c_1 = \frac{1}{2}\ln(y_0x_{-1}) = \frac{1}{2}\ln bk$ and $c_2 = \ln\frac{\sqrt{y_0}}{y_{-1}\sqrt{x_{-1}}} = \ln\frac{\sqrt{b}}{a\sqrt{k}}$. Hence all the solutions $\{x_n, y_n\}_{n=0}^{\infty}$ of the equation system are given by

$$y_n = \exp\left[\frac{1}{2}\cos\left(\frac{n\pi}{2}\right)\ln bk + \frac{1}{2}\sin\left(\frac{n\pi}{2}\right)\ln\frac{b}{ka^2}\right]\sqrt{\frac{b}{k}}$$

$$= (bk)^{1/2\cos\left(\frac{n\pi}{2}\right)}\left(\frac{b}{ka^2}\right)^{1/2\sin\left(\frac{n\pi}{2}\right)}\sqrt{\frac{b}{k}}$$

$$= b^{\frac{1}{2}\left(1+\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right)}k^{\frac{1}{2}\left(-1+\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right)}a^{\sin\left(\frac{n\pi}{2}\right)}, \; n=1, 2, \dots$$

and

$$x_{n+1} = \frac{1}{y_n}$$

$$= b^{-\frac{1}{2}\left(1+\cos\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)\right)}k^{-\frac{1}{2}\left(-1+\cos\left(\frac{n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right)}a^{-\sin\left(\frac{n\pi}{2}\right)}, \; n=0, 1, 2, \dots .$$

Since the functions $f(x) = \sin x$ and $g(x) = \cos x$ have 2π period, solutions of equating system (1.1) are periodic with period four. Therefore, the proof is completed.

References

[1] C. Cinar, On the positive solutions of the difference equation system $x_{n+1} = \frac{1}{y_n}$, $y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$, Appl. Math. Comp. 158 (2004), 303-305.

[2] S. N. Elaydi, An Introduction to Difference Equations, Springer-Verlag, New York, 1996.

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