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A MULTIPERSON PURSUIT PROBLEM ON A CLOSED CONVEX SET IN HILBERT SPACE

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Abstract

A differential game of pursuit of an evader by a finite number of pursuers on a closed convex set in l_2-space is studied. The game is described by simple differential equations and players' controls obeyed the integral constraints. The game is deemed to be completed if exact contact of a pursuer with the evader is occured. It is shown that even if the resources for controls of an individual pursuer is less than that of the evader, the completion of game is still possible.

1. Introduction

Numerous researches deal with differential games and fundamental results are given by Isaacs [4], Pontryagin [7], Krasovskii and Subbotin [5]. Differential pursuit games involving several players in $\mathcal{R}^n$ are discussed by authors [1, 2, 3, 8]. Recently, Leong and Gafurjan [6] investigated pursuit game in the space of l_2.

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In this paper, we consider differential games with integral constraints on controls of players; namely a pursuit of one player by finite number of dynamical players in a closed convex subset of the space of l_2.

Consider the l_2-space of elements $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots)$, $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$ with inner product $(\alpha, \beta) = \sum_{k=1}^{\infty} \alpha_k \beta_k$ and norm $\| \alpha \| = \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2}$. Let S be a closed convex subset of l_2-space. Define $H(x_0, r) = \{x \in l_2 : \| x - x_0 \| \leq r\}$ as a ball in the l_2-space with center x_0 and radius r.

The motions of the finite number of pursuers P_i and the evader E are described by the equations:

$$\begin{aligned} &P_i : \dot{x}_i = u_i, \ x_i(0) = x_{i0}, \ i = \overline{1, m}, \\ &E : \dot{y} = v, \ y(0) = y_0, \end{aligned} \tag{1}$$

where $x_i, x_0, y, y_0, u_i, v \in l_2$, $u_i = (u_{i1}, u_{i2}, \ldots, u_{ik}, \ldots)$ is the control parameter of the ith pursuer P_i, and $v = (v_1, v_2, \ldots, v_k, \ldots)$ is the control parameter of the evader E.

It is necessary to give the following definitions:

Definition 1. A function $u_i(\cdot)$; $u_i : [0, \theta] \to l_2$ with components u_{ik} are Borel measurable functions for all k, such that

$$\int_0^{\infty} \| u_i(t) \|^2 dt \leq \rho_i^2, \tag{2}$$

where ρ_i and θ are some given positive numbers, is called an *admissible control* of the ith pursuer, P_i.

Definition 2. A function $v(\cdot)$; $v : [0, \theta] \to l_2$ with components v_k are Borel measurable functions for all k, such that

$$\int_0^{\infty} \| v(t) \|^2 dt \leq \sigma^2, \tag{3}$$

where σ and θ are some given positive numbers, is called an *admissible control* of the evader, E.

When the players' admissible controls $u_i(\cdot)$ and $v(\cdot)$ are chosen, the corresponding motions $x_i(\cdot)$ and $y(\cdot)$ of the players can be easily obtained as

$$x_i(t) = (x_{i1}(t), x_{i2}(t), ..., x_{ik}(t), ...), x_{ik}(t) = x_{ik}(0) + \int_0^t u_{ik}(s)ds \qquad (4)$$

and

$$y = (y_1(t), y_2(t), ..., y_k(t), ...), y_k(t) = y_k(0) + \int_0^t v_k(s)ds. \qquad (5)$$

It can be verified that for a positive number θ, $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$, where $C(0, \theta; l_2)$ is the space of functions

$$f(t) = (f_1(t), f_2(t), ..., f_k(t), ...) \in l_2, t \geq 0,$$

such that

1. $f_k(t)$, $0 \leq t \leq \theta$ are absolutely continuous functions;

2. $f(t)$, $0 \leq t \leq \theta$ is a continuous function in the norm of l_2.

Definition 3. A function $U_i(x_i, y, v), U_i : l_2 \times l_2 \times l_2 \to l_2$ such that the system of

$$\dot{x}_i = U_i(x_i, y, v), x_i(0) = x_{i0},$$
$$\dot{y} = v, y(0) = y_0,$$

has a unique solution $(x_i(\cdot), y(\cdot))$, where $x_i(\cdot), y(\cdot) \in C(0, \theta; l_2)$ for any admissible controls $u_i = u_i(t)$, and $v = v(t)$, $0 \leq t \leq \theta$ of the pursuers P_i and the evader E is called a *strategy* of the pursuer P_i. A strategy U_i is admissible if each control that is used to form this strategy is admissible.

Definition 4. It is said that the differential game (1) with the initial position $\{y_0, x_{10}, x_{20}, ..., x_{m0}\}, x_{i0} \neq y_0, i = \overline{1, m}$ can be completed, if there exists strategies $U_i, i = \overline{1, m}$ of pursuers such that for any admissible control $v(\cdot)$ of evader E, $x_k(t) = y(t)$ for some $t \geq 0$ and an index k.

2. A Pursuit Problem in l_2

Let us consider the game (1) with admissible controls of the pursuers and evader as (2) and (3), respectively such that

$$\rho_i^2 < \sigma^2, \text{ for all } i = \overline{1, m}. \tag{6}$$

In other words, we are considering a game, where the evader E has advantage in control resources over each pursuer of the game. However, we prove that the completion of the game is still possible.

Theorem 1. *Consider the game* (1) *with admissible controls of the pursuers and evader as* (2) *and* (3), *respectively such that* (6) *holds. If*

$$\sum_{i=1}^{m} \rho_i^2 > \sigma^2, \tag{7}$$

then for any initial position $\{y_0, x_{10}, x_{20}, \ldots, x_{m0}\}$, *the game* (1) *is completable.*

Proof. Without loss of generality, we assume that $y_0 \neq x_{i0}$, for all $i = \overline{1, m}$. Let $\rho^2 = \sum_{i=1}^{m} \rho_i^2$ and $\sigma_i = \frac{\sigma}{\rho} \rho_i$, for $i = \overline{1, m}$. Clearly, $\sigma_i^2 < \rho_i^2$ and $\sum_{i=1}^{m} \sigma_i^2 = \sigma^2$.

Let T be a finite and sufficiently large constant denotes the prefix terminal time of the game.

We shall construct the strategies as follows:

First, we divide the duration of the game into m periods.

At the first period, we define the pursuers' strategy as follows:

$$u_1(t) = \frac{y_0 - x_{10}}{\theta_1} + v(t),\ 0 \leq t \leq \theta_1; \text{ and } u_i(t) = 0,\ \forall i = \overline{2, m}, \tag{8}$$

where $\theta_1 = \left(\frac{\| y_0 - x_{10} \|}{\rho_1 - \sigma_1}\right)^2$. If $\int_0^{\theta_1} \| v(t) \|^2 dt \le \sigma_1^2$, then the strategy defined is admissible, since

$$\left(\int_0^{\theta_1} \| u_1(t) \|^2 dt\right)^{1/2} = \left(\int_0^{\theta_1} \left\| \frac{y_0 - x_{10}}{\theta_1} + v(t) \right\|^2 dt\right)^{1/2}$$

$$\le \left(\int_0^{\theta_1} \left\| \frac{y_0 - x_{10}}{\theta_1} \right\|^2 dt\right)^{1/2} + \left(\int_0^{\theta_1} \| v(t) \|^2 dt\right)^{1/2}$$

$$\le \frac{\| y_0 - x_{10} \|}{\theta_1^{1/2}} + \sigma_1$$

$$= \rho_1 - \sigma_1 + \sigma_1 = \rho_1.$$

Consequently,

$$x_1(\theta_1) = x_{10} + \int_0^{\theta_1} u(t) dt$$

$$= x_{10} + \int_0^{\theta_1} \left(\frac{y_0 - x_{10}}{\theta_1} + v(t)\right) dt$$

$$= x_{10} + y_0 - x_{10} + \int_0^{\theta_1} v(t) dt = y(\theta_1)$$

and hence completes the proof. Therefore, if $x_1(\tau_1) \ne y(\tau_1)$ for all $\tau_1 \in [0, \theta_1]$, then we will have either $\int_0^{\theta_1} \| v(t) \|^2 dt > \rho_1^2$ or $> \sigma_1^2$. Since $\rho_1^2 > \sigma_1^2$, the inequality $\int_0^{\theta_1} \| v(t) \|^2 dt > \sigma_1^2$ holds if there is no contact between the first pursuer and the evader during the time interval $[0, \theta_1]$. In such case, the game will continue into next period. In a general period after the first period, says at ith period, for any $i = \overline{2, m}$, we construct the pursuers' strategy as follows:

Denote $\overline{\Theta}_i = \sum_{k=1}^{k=i} \theta_k$, where $\overline{\Theta}_0 = 0$,

$$u_i(t) = \frac{y(\overline{\Theta}_{i+1}) - x_{i0}}{\theta_i} + v(t),\ \overline{\Theta}_{i-1} \le t \le \overline{\Theta}_i;$$

$$u_k(t) = 0,\ \forall k = \overline{1,\ m} \setminus \{i\}, \tag{9}$$

where $\theta_i = \left(\dfrac{\| y(\overline{\Theta}_{i+1}) - x_{i0} \|}{\rho_i - \sigma_i} \right)^2$. Again, if $\int_{\overline{\Theta}_{i-1}}^{\overline{\Theta}_i} \| v(t) \|^2 dt \le \sigma_i^2$, then the considered strategy is admissible and it guarantees the completion of the game. Thus, if the contact between the i-pursuer and the evader is not occured, then

$$\int_{\overline{\Theta}_{i-1}}^{\overline{\Theta}_i} \| v(t) \|^2 dt > \sigma_i^2.$$

Now, assume that in contrary the theorem is false, in which there is no contact occured between the evader with all the pursuers until the terminal time. Since $x_i(\tau) \ne y(\tau)$ for all $i = \overline{1, m}$ and $\tau \in [0, \overline{\Theta}_m]$, we must have

$$\int_0^{\overline{\Theta}_1} \| v(t) \|^2 dt > \sigma_1^2$$

$$\int_{\overline{\Theta}_1}^{\overline{\Theta}_2} \| v(t) \|^2 dt > \sigma_2^2$$

$$\vdots$$

$$\int_{\overline{\Theta}_{m-1}}^{\overline{\Theta}_m} \| v(t) \|^2 dt > \sigma_m^2.$$

It follows that

$$\int_0^{\infty} \| v(t) \|^2 dt \ge \int_0^{\overline{\Theta}_1} \| v(t) \|^2 dt + \int_{\overline{\Theta}_1}^{\overline{\Theta}_2} \| v(t) \|^2 dt + \cdots + \int_{\overline{\Theta}_{m-1}}^{\overline{\Theta}_m} \| v(t) \|^2 dt$$

$$> \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2 = \sigma^2,$$

which is a contradiction with (3). Obviously, there must exist an i such that $\int_{\overline{\Theta}_{i-1}}^{\overline{\Theta}_i} \| v(t) \|^2 dt \le \sigma_i^2$. At that particular time interval, contact is ensured and thus, complete the proof.

3. A Pursuit Problem in a Closed Convex Set of l_2

In this section, we still consider the game (1) with admissible controls of the pursuers and evader as (2) and (3), respectively such that (6) holds. However, unlike in the preceding section, the considered game is restricted within a closed convex subset $S \subset l_2$.

Theorem 2. *Consider the game* (1) *in which* $x_i, x_0, y, y_0 \in S \subset l_2$ *with admissible controls of the pursuers and evader as* (2) *and* (3), *respectively such that* (6) *holds. If*

$$\sum_{i=1}^{m} \rho_i^2 > \sigma^2, \tag{10}$$

then for any initial position $\{y_0, x_{10}, x_{20}, \ldots, x_{m0}\}$, *the completion of the game* (1) *is possible.*

Proof. First, we introduce m fictitious pursuers $P_i', i = \overline{1, m}$ by means of their motions can be described as

$$P_i' : \dot{z} = u_i', \; i = \overline{1, m}; \; z_i(0) \tag{11}$$

with their controls satisfy

$$\int_0^\infty \| u_i'(t) \|^2 dt \le \rho_i^2. \tag{12}$$

Here, the fictitious players can go beyond the boundaries of S. Let us define a continuous function $F : l_2 \to S$,

$$F(z_i) = x_i, \text{ if } z_i \in l_2 \setminus S,$$

$$F(z_i) = z_i = x_i, \text{ if } z_i \in S. \tag{13}$$

Define the strategy of the fictitious pursuers with the same notations as in Theorem 1.

$$u_i'(t) = \frac{y(\overline{\Theta}_{i-1}) - z_{i0}}{\theta_i} + v(t),\ \overline{\Theta}_{i-1} \le t \le \overline{\Theta}_i;$$

$$u_k'(t) = 0,\ \forall k = \overline{1,\ m} \setminus \{i\}. \tag{14}$$

Now, we construct the strategy for the actual pursuers as follows:

$$u_i(t) = u_i'(t),\ \text{if}\ z_i(t) \in S;$$

$$u_i(t) = \dot{F}(z_i(t)),\ \text{if}\ z_i(t) \notin S. \tag{15}$$

We need to prove that the contact between the actual pursuer and the evader occurs and the above strategy is admissible.

From Theorem 1, we can conclude that at some moment $\tau \in [0, \overline{\Theta}_m]$, the equality

$$z_i(\tau) = y(\tau) \tag{16}$$

holds for an index i. Hence taking into account of the condition $y(t) \in S$ by (13), we have $z_i(\tau) = x_i(\tau) = y(\tau)$. Now, we shall show the admissibility of the pursuers' strategy:

1. If $z_i(t) \in S$ for some moment $t \in [0, \overline{\Theta}_m]$, then

$$\int_0^{\overline{\Theta}_m} \| u_i(t) \|^2 dt = \int_0^{\overline{\Theta}_m} \| u_i'(t) \|^2 dt \le \rho_i^2. \tag{17}$$

2. If $z_i(t) \notin S$ for some moment $t \in [0, \overline{\Theta}_m]$, then

$$\begin{aligned}\int_0^{\overline{\Theta}_m} \| u_i(t) \|^2 dt &= \int_0^{\overline{\Theta}_m} \| \dot{F}(z_i(t)) \|^2 dt \\ &= \int_0^{\overline{\Theta}_m} \left\| \lim_{h \to 0} \frac{F(z_i(t+h)) - F(z_i(t))}{h} \right\|^2 dt \\ &= \int_0^{\overline{\Theta}_m} \lim_{h \to 0} \frac{\| F(z_i(t+h)) - F(z_i(t)) \|^2}{h^2} dt. \end{aligned} \tag{18}$$

Since $F(z_i(\cdot)) = x_i(\cdot) \in S$ and S is closed convex,

$$\| F(z_i(t+h)) - F(z_i(t)) \| \le \| z_i(t+h) - z_i(t) \|. \tag{19}$$

Therefore,

$$\begin{aligned}\int_0^{\overline{\Theta}_m} \| u_i(t) \|^2 dt &= \int_0^{\overline{\Theta}_m} \lim_{h\to 0} \frac{\| F(z_i(t+h)) - F(z_i(t)) \|^2}{h^2} dt \\ &\le \int_0^{\overline{\Theta}_m} \left\| \lim_{h\to 0} \frac{z_i(t+h) - z_i(t)}{h} \right\|^2 dt \\ &= \int_0^{\overline{\Theta}_m} \| \dot{z}_i(t) \|^2 dt \\ &= \int_0^{\overline{\Theta}_m} \| u_i'(t) \|^2 dt \le \rho_i^2. \end{aligned} \tag{20}$$

4. Conclusion

We show that under the integral constraints, an evading player cannot avoid an exact contact with any finite number of pursuing players whose individual resources are less than of this evading player. In the case, where the pursuers are of countably many, an anologue proof should be possible.

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