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WEYL GROUPS OF THE NONREDUCED 4-EXTENDED AFFINE ROOT SYSTEMS

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Abstract

We describe the Weyl group associated to the nonreduced 4-extended affine root systems which satisfy an assumption, the quotient affine root system $R/(\mathbb{R}d \oplus \mathbb{R}c \oplus \mathbb{R}a)$ is reduced in terms of the 4-extended affine diagram. The nonreduced 4-extended affine root systems have been classified by the author [12].

1. Introduction

In 1985, Saito [4] introduced the notion of an extended affine root system and considered the classification of 2-extended affine root systems which are the root systems belong to a positive semi-definite quadratic form whose radical has rank two. Since 2-extended affine root systems are associated to the elliptic singularities, they are also called *elliptic root systems*. Saito achieved a complete classification of elliptic root systems when the quotient of the root system modulo a certain one-dimensional subspace called a *marking* is a reduced affine root system. In 1997,

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Allison et al. [1] also introduced the extended affine root systems associated to the extended affine Lie algebras and gave a complete description of them by using the concept of a semilattice. The generators and their relations of elliptic Weyl groups associated to the elliptic root systems were described from the viewpoint of a generalization of Coxeter groups by Saito and the author [5]. In the cases of n-extended affine root systems, Azam and Shahsanaei [2, 3] have given a presentation of the corresponding Weyl groups. After that in the case of 3-extended affine root systems, the author described them in terms of the 3-extended affine diagrams [7-9]. In the previous papers [10, 11] in the case of reduced 4-extended affine root systems, the author described their Weyl groups in the terms of the 4-extended affine diagrams. In this paper, we describe the Weyl group associated to the nonreduced 4-extended affine root systems which satisfy an assumption, the quotient affine root system $R/(\mathbb{R}d \oplus \mathbb{R}c \oplus \mathbb{R}a)$ is reduced in terms of the 4-extended affine diagram. The nonreduced 4-extended affine root systems have been classified by the author [12].

2. The Nonreduced 4-Extended Affine Root Systems

We classify the nonreduced 4-extended affine root systems which satisfy an assumption, the quotient affine root system $R/(\mathbb{R}d \oplus \mathbb{R}c \oplus \mathbb{R}a)$ is reduced.

Type $BC_l^{(2,1,1,1)}(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l)(n, m, k, r \in \mathbb{Z}),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l)(n, m, k, r \in \mathbb{Z}),$

$\pm 2\varepsilon_i + (2n+1)b + ma + kc + rd\ (1 \leq i \leq l)(n, m, k, r \in \mathbb{Z}).$

Type $BC_l^{(2,1,1,1)*}(1)(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l)(n, m, k, r \in \mathbb{Z}),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l)(n, m, k, r \in \mathbb{Z}),$

$\pm 2\varepsilon_i + (2n+1)b + ma + 2kc + 2rd\ (1 \le i \le l)(n, m, k, r \in \mathbb{Z}),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + 2rd\ (1 \le i \le l)(n, m, k, r \in \mathbb{Z}),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd\ (1 \le i \le l)(n, m, k, r \in \mathbb{Z}).$

Type $BC_l^{(2,1,1,1)*}(2)(l \ge 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd\ (1 \le i \le l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \le i < j \le l),$

$\pm 2\varepsilon_i + (2n+1)b + ma + kc + 2rd\ (1 \le i \le l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd\ (1 \le i \le l).$

Type $BC_l^{(2,1,1,1)*}(3)(l \ge 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd\ (1 \le i \le l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \le i < j \le l),$

$\pm 2\varepsilon_i + (2n+1)b + ma + kc + rd\ (1 \le i \le l), (m \cdot k \equiv 0 \bmod 2).$

Type $BC_l^{(2,1,1,1)*}(4)(l \ge 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd\ (1 \le i \le l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \le i < j \le l),$

$\pm 2\varepsilon_i + (2n+1)b + ma + kc + rd\ (1 \le i \le l), (m \cdot k \cdot r \equiv 0 \bmod 2).$

Type $BC_l^{(2,2,1,1)}(1)(l \ge 2)$

$R : \pm\varepsilon_i + nb + ma + kc + rd\ (1 \le i \le l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \le i < j \le l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + rd\ (1 \le i \le l).$

Type $BC_l^{(2,2,1,1)}(2)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,1,1)*}(1)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + rd\ (1 \leq i \leq l),\ (k \cdot r \equiv 0 \bmod 2).$

Type $BC_l^{(2,2,1,1)*}(2)\,(l \geq 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + (2k+1)c + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,1,1)*}(3)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,1,1)*}(4)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + rd\ (1 \leq i \leq l),\ (k \cdot r \equiv 0 \bmod 2).$

Type $BC_l^{(2,2,1,1)*}(5)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + kc + rd\ (1 \leq i \leq l),\ (k \cdot r \equiv 0 \bmod 2),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + kc + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)}(1)\,(l \geq 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)}(2)\,(l \geq 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + (2k+1)c + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)}(3)\,(l \geq 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd\ (1 \leq i < j \leq l),\ (m \cdot k \equiv 0 \bmod 2),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)}(4)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)}(5)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,1)}(6)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$, $(m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,1)*}(1)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,1)*}(2)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \leq i \leq l)$.

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$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l).$

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$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)*}(5)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,1)*}(6)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,1)*}(7)\,(l \ge 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l),\ (m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,1)*}(8)\,(l \ge 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l),\ (m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,2)}(1)\,(l \ge 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,2)}(2)\,(l \ge 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + 2rd \ (1 \le i < j \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)}(3)\,(l \geq 2)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + ma + kc + rd \ (1 \leq i < j \leq l)$, $(m \cdot k \cdot r \equiv 0 \bmod 2)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)}(4)\,(l \geq 2)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)}(5)\,(l \geq 2)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$, $(m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)}(6)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)}(7)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)}(8)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)}(9)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(1)\,(l \geq 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(2)\,(l \geq 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),\ (m \cdot k \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(3)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 4rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 4rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(4)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(5)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(6)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(7)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 4rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 4rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(8)\,(l \geq 1)$

R : $\pm\varepsilon_i + nb + ma + kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 4rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 4rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(9)\,(l \geq 1)$

$R:\ \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),\ (k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 4rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 4rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(10)\,(l \geq 1)$

$R:\ \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),\ (m \cdot k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 4rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 4rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(11)\,(l \geq 1)$

$R:\ \pm\varepsilon_i + nb + ma + 2kc + 2rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + kc + 2rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd\ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(12)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(13)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(14)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,2,2,2)*}(15)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(16)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + 2ma + kc + rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(17)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(18)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \cdot r \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l)$.

Type $BC_l^{(2,2,2,2)*}(19)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \leq i \leq l)$,

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \le i \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,2)*}(20)\,(l \ge 1)$

$R : \pm\varepsilon_i + nb + ma + kc + 2rd \ (1 \le i \le l)$,

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \le i \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,2)*}(21)\,(l \ge 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l), \ (m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \le i \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,2,2,2)*}(22)\,(l \ge 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l), \ (m \cdot k \cdot r \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 2kc + 4rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 2ma + 4kc + 2rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,1)}(1)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,1)}(2)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,1)*}(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(1)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(2)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),\ (m \cdot k \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(3)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(4)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + 2kc + 2rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + kc + 2rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(5)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + 2rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd\ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd\ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd\ (1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(6)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \;(1 \leq i \leq l), \;(k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \;(1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \;(1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(7)\,(l \geq 1)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \;(1 \leq i \leq l), \;(m \cdot k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \;(1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \;(1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(8)\,(l \geq 2)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \;(1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd \;(1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 2kc + 2rd \;(1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(9)\,(l \geq 2)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \;(1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd \;(1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + (4m+2)a + 2kc + 2rd \;(1 \leq i \leq l).$

Type $BC_l^{(2,4,2,2)}(10)\,(l \geq 2)$

$R : \pm\varepsilon_i + nb + ma + kc + rd \;(1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + kc + rd \;(1 \leq i < j \leq l), \;(k \cdot r \equiv 0 \bmod 2),$

$\pm 2\varepsilon_i + (2n+1)b + (4m+2)a + 2kc + 2rd \;(1 \leq i \leq l).$

Type $BC_l^{(2,4,4,1)}(1)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,4,1)}(2)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,4,2)}(1)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,4,2)}(2)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,4,2)}(3)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,4,4,2)}(4)\,(l \ge 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l), \ (m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,4,4,2)}(5)\,(l \ge 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l), \ (m \cdot k \cdot r \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,4,4,2)}(6)\,(l \ge 2)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l), \ (m \cdot k \equiv 0 \bmod 2)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 2rd \ (1 \le i \le l)$.

Type $BC_l^{(2,4,4,4)}(1)\,(l \ge 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \le i \le l)$,

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \le i < j \le l)$,

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 4rd \ (1 \le i \le l)$.

Type $BC_l^{(2,4,4,4)}(2)\,(l \ge 1)$

$R: \pm\varepsilon_i + nb + ma + 2kc + 2rd \ (1 \le i \le l)$,

$\pm\varepsilon_i + nb + 2ma + kc + 2rd \ (1 \le i \le l)$,

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 4rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,4,4)}(3)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + 2rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i + nb + 2ma + 2kc + rd \ (1 \leq i \leq l),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 4rd \ (1 \leq i \leq l).$

Type $BC_l^{(2,4,4,4)}(4)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 4rd \ (1 \leq i \leq l).$

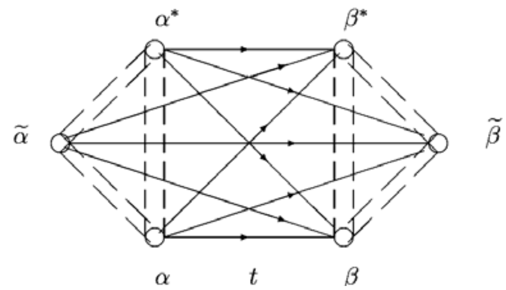
Type $BC_l^{(2,4,4,4)}(5)\,(l \geq 1)$

$R: \pm\varepsilon_i + nb + ma + kc + rd \ (1 \leq i \leq l), \ (m \cdot k \cdot r \equiv 0 \bmod 2),$

$\pm\varepsilon_i \pm \varepsilon_j + nb + 2ma + 2kc + 2rd \ (1 \leq i < j \leq l),$

$\pm 2\varepsilon_i + (2n+1)b + 4ma + 4kc + 4rd \ (1 \leq i \leq l).$

The 3-extended affine diagram is defined to be consisting of all vertices $\alpha_i, \alpha_i^*, \widetilde{\alpha}_i \ (0 \leq i \leq l)$, so that it has the following subdiagram for all α, β s.t. $\langle \alpha, \beta^\vee \rangle = -t, \langle \alpha^\vee, \beta \rangle = -1.$



Further, we define the 4-extended affine diagram $\Gamma(R)$ by adding the vertices $\underline{\alpha}_i$ $(0 \leq i \leq l)$ to the 3-extended affine diagram.

3. The Weyl Group of the 4-Extended Affine Root System

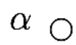
The *Weyl group* of the 4-extended affine root system is defined as follows [1, 4]. Let V be an $(l+4)$-dimensional real vector space equipped with a positive semi-definite bilinear form. Let V^0 be the 4-dimensional radical of the form $\langle\, ,\rangle$ and $(V^0)^*$ be the dual space of V^0. Set $V = \dot{V} \oplus V^0$ and $\tilde{V} = \dot{V} \oplus V^0 \oplus (V^0)^*$. Let $\{\varepsilon_1, \ldots, \varepsilon_l\}$ be the standard basis of $\dot{V}$ satisfying $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for all $i, j = 1, \ldots, l$. Define the bilinear form $\langle\, ,\rangle$ on $\tilde{V}$ so that $\langle\, ,\rangle$ extends the form on V and $\langle\, ,\rangle$ is nondegenerate on $\tilde{V}$. For $\alpha \in R$, we define the reflection $w_\alpha \in GL(\tilde{V})$ by $w_\alpha(u) = u - \langle u, \alpha^\vee \rangle \alpha \; (u \in \tilde{V})$ with $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. Set $\tilde{W}_R = \langle w_\alpha \,|\, \alpha \in R \rangle \subseteq GL(\tilde{V})$. Then $\tilde{W}_R$ is the Weyl group of the 4-extended affine root system R. Our main result is the following:

Theorem 3.1. *The Weyl groups of the nonreduced* 4-*extended affine root systems are described as follows*:

Generators: *for each* $\alpha \in \Gamma(R)$, *we attach a generator* $a_\alpha := w_\alpha$. *For simplicity*, *we shall write* $a, a^*, \tilde{a}, \underline{a}, b, b^*, \tilde{b}, \underline{b}, \ldots$ *instead of* $a_\alpha, a_{\alpha^*}, a_{\tilde{\alpha}}, a_{\underline{\alpha}}, a_\beta, a_{\beta^*}, a_{\tilde{\beta}}, a_{\underline{\beta}}, \cdots$.

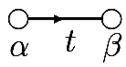
Relations:

0



$\Rightarrow \quad a^2 = 1$

I.t

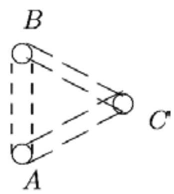


$t = 0 \quad \Rightarrow \quad (ab)^2 = 1$

$t = 1 \quad \Rightarrow \quad (ab)^3 = 1$

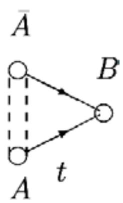
$t = 2 \quad \Rightarrow \quad (ab)^4 = 1$

II.0



$\Rightarrow \quad (ABC)^2 = (BCA)^2 = (CAB)^2$

II.t

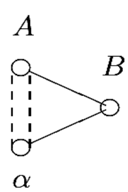


$t = 1 \quad \Rightarrow \quad (AB\overline{A}B)^3 = 1$

$t = 2^{\pm 1} \quad \Rightarrow \quad (AB\overline{A}B)^2 = 1$

$t = 4^{\pm 1} \quad \Rightarrow \quad (A\overline{A}B)^2 = (\overline{A}BA)^2 = (BA\overline{A})^2$

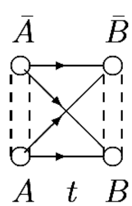
where $A \neq \overline{A} \in \{\alpha, \alpha^*, \tilde{\alpha}, \underline{\alpha}\}$



$\Rightarrow \quad (A\alpha AB)^3 = 1,$ *where* $A = \tilde{\alpha},\ B = \beta^*, \underline{\beta},$

$A = \alpha^*,\ B = \tilde{\beta}, \underline{\beta}$ or $A = \underline{\alpha},\ B = \beta^*, \tilde{\beta}$

III.t



$t = 1,$

(i) $A\overline{B}A = B\overline{A}B,$ *where*

$A \neq \overline{A} \in \{\alpha, \alpha^*, \tilde{\alpha}, \underline{\alpha}\},\ B \neq \overline{B} \in \{\beta, \beta^*, \tilde{\beta}, \underline{\beta}\},$

A, B and $\overline{A}$, $\overline{B}$ are chosen in the same tunts, respectively.

(ii) $(AB\overline{B})^2\overline{A} = \overline{A}(AB\overline{B})^2,\ (A\overline{A}B)^2\overline{B} = \overline{B}(A\overline{A}B)^2,$

where $A \neq \overline{A}$ *and* $B \neq \overline{B}$ *are arbitrary chosen.*

In the sequel, we set $\begin{cases} A = \alpha + pp_1, \; B = \beta + qp_1 \\ \overline{A} = \alpha + rp_2, \; \overline{B} = \beta + sp_2. \end{cases}$

$t = 2,$

(i) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \Rightarrow \overline{A}B\overline{A} = A\overline{B}A$

(ii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \Rightarrow \overline{B}A\overline{B} = B\overline{A}B$

(iii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \Rightarrow \begin{cases} bAb = BaB \\ a\overline{B}a = \overline{A}b\overline{A} \end{cases}$

$t = 4,$

(i) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \overline{A}B\overline{A} = A\overline{B}A$

(ii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} aAbB = AbBa = bBaA = BaAb \\ \overline{A}b\overline{A} = a\overline{B}a \end{cases}$

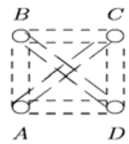
(iii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow A\overline{A}B\overline{B} = \overline{A}B\overline{B}A = B\overline{B}A\overline{A} = \overline{B}A\overline{A}B$

(iv) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{cases} bAb = BaB \\ a\overline{A}b\overline{B} = \overline{A}b\overline{B}a = b\overline{B}a\overline{A} = \overline{B}a\overline{A}b \end{cases}$

(v) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} bAb = BaB \\ a\overline{B}a = \overline{A}b\overline{A} \end{cases}$

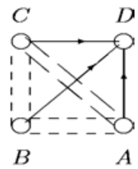
(vi) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \Rightarrow \overline{B}A\overline{B} = B\overline{A}B$

III.0



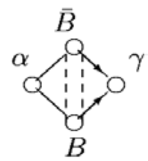
$$\Rightarrow \quad (ABC)^2 D = D(ABC)^2$$

III.∞



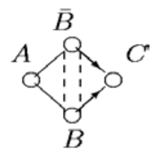
$$t = 1,\ 2^{\pm 1},\ 4^{\pm 1} \quad \Rightarrow \quad (ABC)^2 D = D(ABC)^2$$

IV(i).t



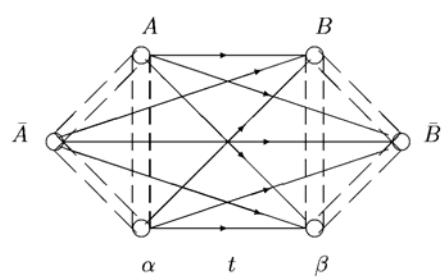
$$t = 1, 2^{\pm 1} \Rightarrow (aBa\overline{B}c\overline{B})^2 = 1,\ (a\overline{B}aBcB)^2 = 1,$$
$$where\ B \neq \overline{B} \in \{\beta,\ \beta^*,\ \widetilde{\beta},\ \underline{\beta}\}$$

IV(ii).t



$$t = 1,\ 2^{\pm 1} \quad \Rightarrow \quad (BAB\overline{B}C\overline{B})^2 = 1$$

V.t



$t = 2,$

$$\text{(i)} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} \overline{B}aB\overline{A} = B\overline{A}bA,\ \overline{B}A\overline{A}B = A\overline{A}Bb \\ \overline{B}B\overline{A}a = A\overline{B}B\overline{A} = B\overline{A}Ab \end{cases}$$

(ii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 2 \\ 2 & 2 \end{pmatrix} \Rightarrow \begin{cases} \overline{B}B\overline{A}a = A\overline{B}B\overline{A},\ A\overline{A}B\overline{B} = bA\overline{A}B \\ \overline{B}A\overline{A}B = A\overline{A}Bb \end{cases}$

$t = 4$,

(i) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} A(a\overline{A})^2 b\overline{B} = (a\overline{A})^2 b\overline{B}A \\ B(a\overline{A})^2 b\overline{B} = (a\overline{A})^2 b\overline{B}B \end{cases}$

(ii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} aAbB\overline{A} = \overline{A}aAbB,\ aAbB\overline{B} = \overline{B}aAbB \\ (a\overline{A})^2 b\overline{B}A = A(a\overline{A})^2 b\overline{B},\ (a\overline{A})^2 b\overline{B}B = B(a\overline{A})^2 b\overline{B} \\ A\overline{A}Bb = \overline{B}A\overline{A}B \end{cases}$$

(iii) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$

$$\Rightarrow aA(bB)^2\overline{A} = \overline{A}aA(bB)^2,\ aA(bB)^2\overline{B} = \overline{B}aA(bB)^2$$

(iv) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$

$$\Rightarrow \begin{cases} aA(bB)^2\overline{A} = \overline{A}aA(bB)^2,\ aA(bB)^2\overline{B} = \overline{B}aA(bB)^2 \\ (a\overline{A})^2 b\overline{B}A = A(a\overline{A})^2 b\overline{B},\ (a\overline{A})^2 b\overline{B}B = B(a\overline{A})^2 b\overline{B} \\ A\overline{A}B\overline{B}a = \overline{A}Bb\overline{A}\overline{B},\ \overline{A}B\overline{A}a\overline{B}A = a\overline{A}B\overline{A}a\overline{B} \\ Ba\overline{A}B\overline{B} = bBa\overline{A}B,\ bBa\overline{A}Bb = \overline{B}bBa\overline{A}B \end{cases}$$

(v) $\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow \begin{cases} a\overline{A}b\overline{B}A = Aa\overline{A}b\overline{B},\ a\overline{A}b\overline{B}B = Ba\overline{A}b\overline{B} \\ A\overline{A}B\overline{B} = \overline{A}B\overline{B}a,\ a\overline{A}B\overline{B} = \overline{A}B\overline{B}A. \end{cases}$

Proof. It is proved similarly to the 3-extended affine cases [7, 9].

Let us denote $\dot{w}_\alpha$ be the reflection in $GL(V)$ such that $w_\alpha|_V = \dot{w}_\alpha$, and set $W_R = \langle \dot{w}_\alpha \mid \alpha \in R \rangle$. Then we see the following:

Proposition 3.2. (i) *The central elements* γ_1, γ_2 *and* γ_3 *in* $\widetilde{W}(\Gamma(R))$ *are given as follows*:

$$\gamma_1 = \left(\prod_{\alpha \in \Gamma(R/(\mathbb{R}c \oplus \mathbb{R}d),\, \mathbb{R}a) \setminus (\Gamma_{\max} \cup \Gamma^*_{\max})} w_\alpha \prod_{\alpha \in \Gamma_{\max}} w_\alpha w^*_\alpha \right)^m,$$

where $\Gamma(R/(\mathbb{R}c \oplus \mathbb{R}d), \mathbb{R}a)$ *is the elliptic Dynkin diagram*, $\Gamma_{\max}$, $\Gamma^*_{\max}$ *are the subdiagrams of* $\Gamma(R/(\mathbb{R}c \oplus \mathbb{R}d), \mathbb{R}a)$ (*see* [4]), $m : m(R/(\mathbb{R}c \oplus \mathbb{R}d), \mathbb{R}a)$ *is the order of the Coxeter element in the elliptic Weyl group*.

Similarly,

$$\gamma_2 = \left(\prod_{\alpha \in \Gamma(R/(\mathbb{R}a \oplus \mathbb{R}d),\, \mathbb{R}c) \setminus (\Gamma_{\max} \cup \Gamma^*_{\max})} w_\alpha \prod_{\alpha \in \Gamma_{\max}} w_\alpha \widetilde{w}_\alpha \right)^m,$$

$$\gamma_3 = \left(\prod_{\alpha \in \Gamma(R/(\mathbb{R}a \oplus \mathbb{R}c),\, \mathbb{R}d) \setminus (\Gamma_{\max} \cup \Gamma^*_{\max})} w_\alpha \prod_{\alpha \in \Gamma_{\max}} w_\alpha \underline{w}_\alpha \right)^m.$$

Further, we set $t_i := w^*_i w_i$, $s_i := \widetilde{w}_i w_i$, $q_i := \underline{w}_i w_i$, $[A, B] := A^{-1}B^{-1}AB$, *then* γ_4, γ_5, γ_6 *are given as follows*:

$BC_l^{(2,1,1,1)}$, $BC_l^{(2,1,1,1)*}(i)\,(1 \le i \le 4)$

$\gamma_4 = [s_0, t_0] = T^a_c$, $\gamma_5 = [q_0, t_0] = T^a_d$, $\gamma_6 = [q_0, s_0] = T^c_d$

$BC_l^{(2,2,1,1)}(1)$, $BC_l^{(2,2,1,1)*}(i)\,(i = 1, 2)$, $BC_l^{(2,2,2,1)}(2)$

$$\gamma_4 = \begin{cases} [s_0, t_0] = T^{2a}_c \ (l = 1, 2) \\ [t_1, s_2] = T^a_c \ (l \ge 3), \end{cases} \quad \gamma_5 = \begin{cases} [q_0, t_0] = T^{2a}_d \ (l = 1, 2) \\ [t_1, q_2] = T^a_d \ (l \ge 3), \end{cases}$$

$\gamma_6 = [q_0, s_0] = T^c_d$

$BC_l^{(2,2,1,1)}(2)$, $BC_l^{(2,2,1,1)*}(i)\,(3 \leq i \leq 5)$

$$\gamma_4 = [s_0, t_0] = T_c^{2a},\ \gamma_5 = [q_0, t_0] = T_d^{2a},\ \gamma_6 = [q_0, s_0] = T_d^{c}$$

$BC_l^{(2,2,2,1)}(1)$, $BC_l^{(2,2,2,1)}(3)$

$$\gamma_4 = \begin{cases} [t_1, s_2] = T_c^{2a}\ (l = 2) \\ [t_1, s_2] = T_c^{a}\ (l \geq 3), \end{cases} \quad \gamma_5 = \begin{cases} [q_0, t_0] = T_d^{2a}\ (l = 2) \\ [t_1, q_2] = T_d^{a}\ (l \geq 3), \end{cases}$$

$$\gamma_6 = \begin{cases} [q_0, s_0] = T_d^{2c}\ (l = 2) \\ [s_1, q_2] = T_d^{c}\ (l \geq 3), \end{cases}$$

$BC_l^{(2,2,2,1)}(4)$

$$\gamma_4 = \begin{cases} [s_0, t_0] = T_c^{4a}\ (l = 1) \\ [t_2, s_1] = T_c^{2a}\ (l \geq 2), \end{cases} \quad \gamma_5 = [s_0, t_0] = T_d^{2a},$$

$$\gamma_6 = \begin{cases} [q_0, s_0] = T_d^{2c}\ (l = 1, 2) \\ [s_1, q_2] = T_d^{c}\ (l \geq 3) \end{cases}$$

$BC_l^{(2,2,2,1)}(i)\,(i = 5, 6)$, $BC_l^{(2,2,2,1)*}(i)\,(1 \leq i \leq 8)$

$$\gamma_4 = [s_0, t_0] = T_c^{4a},\ \gamma_5 = [q_0, t_0] = T_d^{2a},\ \gamma_6 = [q_0, s_0] = T_d^{2c}$$

$BC_l^{(2,2,2,2)}(i)\,(1 \leq i \leq 3)$

$$\gamma_4 = \begin{cases} [s_0, t_0] = T_c^{4a}\ (l = 1) \\ [t_1, s_2] = T_c^{2a}\ (l = 2) \\ [t_1, s_2] = T_c^{a}\ (l \geq 3), \end{cases} \quad \gamma_5 = \begin{cases} [q_0, t_0] = T_d^{4a}\ (l = 1) \\ [t_1, q_2] = T_d^{2a}\ (l = 2) \\ [t_1, q_2] = T_d^{a}\ (l \geq 3), \end{cases}$$

$$\gamma_6 = \begin{cases} [q_0, s_0] = T_d^{4c}\ (l = 1) \\ [s_1, q_2] = T_d^{2c}\ (l = 2) \\ [s_1, q_2] = T_d^{c}\ (l \geq 3) \end{cases}$$

$BC_l^{(2,2,2,2)}(i)\,(4 \le i \le 9)$, $BC_l^{(2,2,2,2)*}(i)\,(i = 1, 2)$

$$\gamma_4 = [s_0, t_0] = T_c^{4a}, \quad \gamma_5 = \begin{cases} [q_0, t_0] = T_d^{4a} \ (l = 1) \\ [t_2, q_1] = T_d^{2a} \ (l \ge 2), \end{cases}$$

$$\gamma_6 = \begin{cases} [q_0, s_0] = T_d^{4c} \ (l = 1) \\ [s_2, q_1] = T_d^{2c} \ (l \ge 2) \end{cases}$$

$BC_l^{(2,2,2,2)*}(i)\,(3 \le i \le 22)$

$$\gamma_4 = [s_0, t_0] = T_c^{4a}, \ \gamma_5 = [q_0, t_0] = T_d^{4a}, \ \gamma_6 = [q_0, s_0] = T_d^{4c}$$

$BC_l^{(2,4,2,1)}(1)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \ \gamma_5 = [q_l, t_l] = T_d^{4a}, \ \gamma_6 = \begin{cases} [q_0, s_0] = T_d^{2c} \ (l = 1, 2) \\ [s_1, q_2] = T_d^{c} \ (l \ge 3) \end{cases}$$

$BC_l^{(2,4,2,1)}(2)$, $BC_l^{(2,4,2,1)*}$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \ \gamma_5 = [t_1, q_0] = T_d^{2a}, \ \gamma_6 = [q_0, s_0] = T_d^{2c}$$

$BC_l^{(2,4,4,1)}(i)\,(i = 1, 2)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \quad \gamma_5 = \begin{cases} [q_l, t_l] = T_d^{4a} \ (l = 1, 2) \\ [t_1, q_2] = T_d^{2a} \ (l \ge 3), \end{cases}$$

$$\gamma_6 = \begin{cases} [q_l, s_l] = T_d^{4c} \ (l = 1, 2) \\ [s_1, q_2] = T_d^{2c} \ (l \ge 3) \end{cases}$$

$BC_l^{(2,4,4,2)}(i)\,(1 \le i \le 5)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \ \gamma_5 = [q_l, t_l] = T_d^{4a}, \ \gamma_6 = [q_l, s_l] = T_d^{4c}$$

$BC_l^{(2,4,4,2)}(6)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \quad \gamma_5 = \begin{cases} [q_1, t_1] = T_d^{4a} \ (l = 1) \\ [t_2, q_1] = T_d^{2a} \ (l \geq 2), \end{cases}$$

$$\gamma_6 = \begin{cases} [q_1, s_1] = T_d^{4c} \ (l = 1) \\ [s_2, q_1] = T_d^{2c} \ (l \geq 2) \end{cases}$$

$BC_l^{(2,4,2,2)}(i)\,(i = 1, 2)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \quad \gamma_5 = \begin{cases} [q_1, t_1] = T_d^{4a} \ (l = 1) \\ [t_2, q_1] = T_d^{2a} \ (l \geq 2), \end{cases}$$

$$\gamma_6 = \begin{cases} [q_0, s_0] = T_d^{4c} \ (l = 1) \\ [s_2, q_1] = T_d^{2c} \ (l \geq 2) \end{cases}$$

$BC_l^{(2,4,2,2)}(i)\,(3 \leq i \leq 10)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \ \gamma_5 = [q_l, t_l] = T_d^{4a}, \ \gamma_6 = [q_l, s_l] = T_d^{4c}$$

$BC_l^{(2,4,4,4)}(i)\,(1 \leq i \leq 5)$

$$\gamma_4 = [s_l, t_l] = T_c^{4a}, \ \gamma_5 = [q_l, t_l] = T_d^{4a}, \ \gamma_6 = [q_l, s_l] = T_d^{4c}$$

(ii) *We have an isomorphism* $\widetilde{W}(\Gamma(R))/\langle \gamma_i \ (1 \leq i \leq 6)\rangle \cong W_R$.

Proof. (i) is directly checked and (ii) is trivial.

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