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WEAK SEPARATION AXIOMS VIA $\widetilde{g}$-OPEN SETS

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Abstract

In this paper, $\widetilde{g}$-open sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms T_i, $(i = 0, 1/2, 1, 2)$ are investigated.

1. Introduction

In 1970, Levine [5] initiated the study of the so-called g-closed sets. The notion has been studied extensively in recent years by many topologists. In the same paper Levine also introduced the notion of $T_{1/2}$-spaces which properly lie between T_1 spaces and T_0 spaces. In a recent

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year, a generalization of closed sets, $\widetilde{g}$-closed sets were introduced and studied by Jafari et al. [3]. This notion was further studied by Rajesh and Ekici [7-10]. In this paper, we continue the study of related spaces with $\widetilde{g}$-open sets (i.e., complements of $\widetilde{g}$-closed sets). We introduce and characterize four new separation axioms called $\widetilde{g} - T_i$, ($i = 0, 1/2, 1, 2$). We show that $\widetilde{g} - T_i$, $i = 0, 1/2, 1, 2$ is weaker than T_i, $i = 0, 1/2, 1, 2$, respectively.

Throughout this paper, a space stands for a topological space and a function $f : X \to Y$ denotes a function from a space X into a space Y. For a subset A of a space X, the closure and the interior of A in X are denoted by $cl(A)$ and $int(A)$, respectively.

2. Preliminaries

Before entering our work we recall the following definitions and results which are used in this paper.

Definition 2.1. A subset A of a space X is said to be *semi-open* [6] if $A \subset cl(int(A))$. The complement of a semi-open set is called *semi-closed*. The intersection of all semi-closed subsets of X that contains A, or equivalently, the smallest semi-closed subset of X that contains A, is called the *semi-closure* of A [2] and is denoted by $scl(A)$.

Definition 2.2. Let A be a subset of a space X. Then

(i) A is *generalized closed* (briefly *g-closed* [5]) if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X.

(ii) A is $\hat{g}$*-closed* [13] if $cl(A) \subset U$ whenever $A \subset U$ and U is semi-open in X. The complement of a $\hat{g}$-closed set is called $\hat{g}$*-open*.

(iii) A is ${}^*g$*-closed* [12] if $cl(A) \subset U$ whenever $A \subset U$ and U is $\hat{g}$-open in X. The complement of a ${}^*g$-closed set is called ${}^*g$*-open*.

(iv) A is ${}^\sharp g$*-semi-closed* [14] if $scl(A) \subset U$ whenever $A \subset U$ and U is ${}^*g$-open. The complement of a ${}^\sharp g$-semi-closed set is called ${}^\sharp g$-*semi-open*.

(v) A is $\widetilde{g}$ *-closed* [3] if $cl(A) \subset U$ whenever $A \subset U$ and U is $^{\sharp}g$ - semi-open. The complement of a $\widetilde{g}$ -closed set is called $\widetilde{g}$ *-open*. The class of all $\widetilde{g}$ -open (resp. $\widetilde{g}$ -closed) subsets of X is denoted by $\widetilde{g}(X)$ (resp. $\widetilde{g}C(X)$).

Definition 2.3. The intersection of all $\widetilde{g}$ -closed (resp. $\widetilde{g}$ -open) sets containing A is called the $\widetilde{g}$ *-closure* (resp. $\widetilde{g}$ *-kernel*) of A [10] and is denoted by $\widetilde{g} - cl(A)$ (resp. $\widetilde{g} - ker(A)$).

Definition 2.4. A space X is called a $T_{1/2}$ *-space* [5] if every g-closed subset of X is closed in X, or equivalently, if every singleton subset of X is open or closed.

Theorem 2.5 [3]**.** *In any space X, the following hold*:

(i) *An arbitrary intersection of* $\widetilde{g}$ *-closed sets is* $\widetilde{g}$ *-closed.*

(ii) *The finite union of* $\widetilde{g}$ *-closed sets is* $\widetilde{g}$ *-closed.*

Remark 2.6. A subset is $\widetilde{g}$ -closed if and only if it coincides with its $\widetilde{g}$ -closure.

Definition 2.7 [4]**.** A subset U_x of a space X is said to be a $\widetilde{g}$ *-neighborhood* of a point $x \in X$ if there exists a $\widetilde{g}$ -open set G in X such that $x \in G \subset U_x$.

Lemma 2.8 [4]**.** *A subset A of a space X is* $\widetilde{g}$ *-open in X if and only if it is a* $\widetilde{g}$ *-neighborhood of each of its points.*

Definition 2.9 [8]**.** A function $f : X \to Y$ is said to be $\widetilde{g}$ *-continuous* if the inverse image of every open set in Y is $\widetilde{g}$ -open in X.

Definition 2.10 [7]**.** A function $f : X \to Y$ is said to be $\widetilde{g}$ *-irresolute* if the inverse image of every $\widetilde{g}$ -open set in Y is $\widetilde{g}$ -open in X.

Definition 2.11 [1]**.** A function $f : X \to Y$ is said to be $\widetilde{g}^*$ *-closed* (resp. $\widetilde{g}$ *-closed*) if the image of every $\widetilde{g}$ -closed (resp. closed) set in X is $\widetilde{g}$ -closed in Y.

Definition 2.12 [1]**.** A function $f : X \to Y$ is said to be $\widetilde{g}^*$*-open* (resp. $\widetilde{g}$ *-open*) if the image of every $\widetilde{g}$ -open (resp. open) set in X is $\widetilde{g}$ -open in Y.

Definition 2.13 [9]**.** A space X is said to be $\widetilde{g}$ *-regular* if for each closed subset F of X and each point $x \in F^c$, there exist disjoint $\widetilde{g}$ -open sets U and V such that $F \subset U$ and $x \in V$.

Theorem 2.14. *A function* $f : X \to Y$ *is* $\widetilde{g}$ *-irresolute if and only if for each* $\widetilde{g}$ *-open subset W of Y and for each* $x \in X$ *such that* $f(x) \in W$, *then there exists a* $\widetilde{g}$ *-open subset* U *of* X *such that* $x \in U$ *and* $f(U) \subset W$.

3. $\widetilde{g} - T_0$ Spaces

Definition 3.1. A space X is said to be $\widetilde{g} - T_0$ if to each pair of distinct points x, y of X there exists a $\widetilde{g}$ -open set A containing x but not y or a $\widetilde{g}$ -open set B containing y but not x.

Theorem 3.2. *For a space X, the following are equivalent*:

(i) *X is* $\widetilde{g} - T_0$.

(ii) *For each* $x \in X$, $\{x\} = \bigcap \{F \in \widetilde{g}(X) \cup \widetilde{g}C(X) : x \in F\} = \widetilde{g} - cl(\{x\}) \cap \widetilde{g} - ker(\{x\})$.

Proof. The proof follows from the definitions. □

Theorem 3.3. *A space X is* $\widetilde{g} - T_0$ *if and only if for each pair of distinct points x, y of X,* $\widetilde{g} - cl(\{x\}) \neq \widetilde{g} - cl(\{y\})$.

Proof. Necessity. Let X be a $\widetilde{g} - T_0$ space and x, y be any two distinct points of X. There exists a $\widetilde{g}$ -open set G containing x but not y or containing y but not x, say, x but not y. Thus $X - G$ is a $\widetilde{g}$ -closed set which does not contain x but contains y. Since $\widetilde{g} - cl(\{y\})$ is the smallest $\widetilde{g}$ -closed set containing y, $\widetilde{g} - cl(\{y\}) \subset X - G$, and so $x \notin \widetilde{g} - cl(\{y\})$. Consequently, $\widetilde{g} - cl(\{x\}) \neq \widetilde{g} - cl(\{y\})$.

Sufficiency. Let $x, y \in X, x \neq y.$ Then by assumption, $\widetilde{g} - cl(\{x\}) \neq \widetilde{g} - cl(\{y\})$. Thus there exists a point $z \in X$ such that z belongs to $\widetilde{g} - cl(\{x\})$ but not to $\widetilde{g} - cl(\{y\})$ or z belongs to $\widetilde{g} - cl(\{y\})$ but not to $\widetilde{g} - cl(\{x\})$, say, $\widetilde{g} - cl(\{x\})$ but not to $\widetilde{g} - cl(\{y\})$. If we suppose that $x \in \widetilde{g} - cl(\{y\})$, then $z \in \widetilde{g} - cl(\{x\}) \subset \widetilde{g} - cl(\{y\})$, which is a contradiction. Thus $x \in X - (\widetilde{g} - cl(\{y\}))$, but $X - (\widetilde{g} - cl(\{y\}))$ is $\widetilde{g}$ -open and does not contain y, hence X is $\widetilde{g} - T_0$. □

Definition 3.4. A function $f : X \to Y$ is said to be *point* $\widetilde{g}$ *-closure one-to-one* if for each $x, y \in X$ such that $\widetilde{g} - cl(\{x\}) \neq \widetilde{g} - cl(\{y\})$, then $\widetilde{g} - cl(\{f(x)\}) \neq \widetilde{g} - cl(\{f(y)\})$.

Theorem 3.5. *If* $f : X \to Y$ *is a point* $\widetilde{g}$ *-closure one-to-one function and X is* $\widetilde{g} - T_0$ *space, then f is one-to-one.*

Proof. Let $x, y \in X$ with $x \neq y$. Since X is $\widetilde{g} - T_0$, by Theorem 3.3, $\widetilde{g} - cl(\{x\}) \neq \widetilde{g} - cl(\{y\})$. But f is point $\widetilde{g}$ -closure one-to-one, so $\widetilde{g} - cl(\{f(x)\}) \neq \widetilde{g} - cl(\{f(y)\})$. Hence $f(x) \neq f(y)$. Thus, f is one-to-one. □

Theorem 3.6. *Let* $f : X \to Y$ *be a function from a* $\widetilde{g} - T_0$ *space X into a* $\widetilde{g} - T_0$ *space Y. Then f is point* $\widetilde{g}$ *-closure one-to-one if and only if f is one-to-one.*

Proof. Follows from Theorem 3.5 and from the definitions. □

Theorem 3.7. *Let* $f : X \to Y$ *be an injective* $\widetilde{g}$ *-irresolute function. If Y is* $\widetilde{g} - T_0$, *then X is* $\widetilde{g} - T_0$.

Proof. Let $x, y \in X$ with $x \neq y$. Since f is injective, $f(x) \neq f(y)$, but Y is $\widetilde{g} - T_0$, so there exists a $\widetilde{g}$ -open set V_x in Y such that $f(x) \in V_x$ and $f(y) \notin V_x$ or there exists a $\widetilde{g}$ -open set V_y in Y such that $f(y) \in V_y$ and $f(x) \notin V_y$. By $\widetilde{g}$ -irresoluteness of f, $f^{-1}(V_x)$ is $\widetilde{g}$ -open in X such that $x \in f^{-1}(V_x)$ and $y \notin f^{-1}(V_x)$ or $f^{-1}(V_y)$ is

$\widetilde{g}$-open in X such that $y \in f^{-1}(V_y)$ and $x \notin f^{-1}(V_y)$. This shows that X is $\widetilde{g} - T_0$. □

Theorem 3.8. *Let* $f : X \to Y$ *be an injective* $\widetilde{g}$*-continuous function. If Y is* T_0, *then X is* $\widetilde{g} - T_0$.

Proof. The proof is similar to that of Theorem 3.7. □

4. $\widetilde{g} - T_1$ Spaces

Definition 4.1. A space X is said to be $\widetilde{g} - T_1$ if to each pair of distinct points x, y of X, there exist two $\widetilde{g}$-open sets, one containing x but not y and the other containing y but not x.

It is evident that every T_1 space is $\widetilde{g} - T_1$. However, the next question asks about the converse.

Question 1. Is there an example of a $\widetilde{g} - T_1$ space that is not T_1?

Theorem 4.2. *For a space X, the following statements are equivalent*:

(i) *X is* $\widetilde{g} - T_1$.

(ii) *Each singleton subset of X is* $\widetilde{g}$*-closed in X.*

(iii) *For every subset A of X,* $A = \widetilde{g} - ker(A)$, *or equivalently, every subset of X is the intersection of* $\widetilde{g}$*-open sets.*

(iv) *For each* $x \in X$, $\{x\} = \widetilde{g} - ker(\{x\})$, *or equivalently, every singleton subset of X is the intersection of* $\widetilde{g}$*-open sets.*

Proof. (i) $\Rightarrow$ (ii) : Let $x \in X$. Then by (i), for any $y \in X$, $y \neq x$, there exists a $\widetilde{g}$-open set V_y containing y but not x. Hence $y \in V_y \subset \{x\}^c$. Now varying y over $\{x\}^c$ we get $\{x\}^c = \bigcup\{V_y : y \in \{x\}^c\}$. So $\{x\}^c$ is the union of $\widetilde{g}$-open sets. Since an arbitrary union of $\widetilde{g}$-open sets is $\widetilde{g}$-open, $\{x\}$ is $\widetilde{g}$-closed.

(ii) $\Rightarrow$ (iii): If $A \subset X$, then for each point $y \notin A$, $\{y\}^c$ is $\widetilde{g}$ -open by (ii). Hence $A = \bigcap \{\{y\}^c : y \in A^c\}$ is the intersection of $\widetilde{g}$ -open sets.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (i): Let $x, y \in X$ and $x \neq y$. Then by (iv), there exists a $\widetilde{g}$ -open set U_x such that $x \in U_x$ and $y \notin U_x$. Similarly, there exists a $\widetilde{g}$ -open set U_y such that $y \in U_y$ and $x \notin U_y$. Hence X is $\widetilde{g} - T_1$. □

Theorem 4.3. *Let X be a T_1 space and $f : X \to Y$ be a $\widetilde{g}$ -closed surjective function. Then Y is $\widetilde{g} - T_1$.*

Proof. Suppose $y \in Y$. Since f is surjective, there exists a point $x \in X$ such that $y = f(x)$. Since X is T_1, $\{x\}$ is closed in X. Since f is $\widetilde{g}$ -closed, $f(\{x\}) = \{y\}$ is $\widetilde{g}$ -closed in Y. Hence by Theorem 4.2, Y is $\widetilde{g} - T_1$. □

Theorem 4.4. *Let X be a $\widetilde{g} - T_1$ space and f be a $\widetilde{g}^*$ -closed function from X onto a space Y. Then Y is $\widetilde{g} - T_1$.*

Proof. Similar to that of Theorem 4.3. □

Definition 4.5. Let A be a subset of a space X and $x \in X$. Then x is said to be a $\widetilde{g}$ *-limit point* of A if for each $U \in \widetilde{g}(X)$, $x \in U$, then $U \cap (A \setminus \{x\}) \neq \emptyset$ and the set of all $\widetilde{g}$ -limit points of A is called the $\widetilde{g}$ *-derived set* of A and is denoted by $\widetilde{g}d(A)$.

Theorem 4.6. *If X is $\widetilde{g} - T_1$ and $x \in \widetilde{g}d(A)$ for some $A \subset X$, then every $\widetilde{g}$ -neighborhood of x contains infinitely many points of A.*

Proof. Suppose U is a $\widetilde{g}$ -neighborhood of x such that $U \cap A$ is finite. Let $U \cap A = \{x_1, x_2, ..., x_n\} = B$. Clearly B is a $\widetilde{g}$ -closed set. Hence $V = U - (B - \{x\})$ is a $\widetilde{g}$ -neighborhood of x and $V \cap (A - \{x\}) = \emptyset$, which implies that $x \notin \widetilde{g}d(A)$, a contradiction. □

The proof of the following theorem is straightforward and thus omitted.

Theorem 4.7. *If A is a subset of a $\widetilde{g}-T_1$ space X, then $\widetilde{g}d(A)$ is $\widetilde{g}$-closed.*

Theorem 4.8. *Let $f: X \to Y$ be an injective $\widetilde{g}$-irresolute function. If Y is $\widetilde{g}-T_1$, then X is $\widetilde{g}-T_1$.*

Proof. Similar to the proof of Theorem 3.7 □

Definition 4.9. A space X is said to be $\widetilde{g}-R_0$ [4] if every $\widetilde{g}$-open subset of X contains the $\widetilde{g}$-closure of each of its singletons.

Theorem 4.10. *A space X is $\widetilde{g}-T_1$ if and only if it is $\widetilde{g}-T_0$ and $\widetilde{g}-R_0$.*

Proof. Let X be a $\widetilde{g}-T_1$ space. Then by definitions, X is $\widetilde{g}-T_0$. It follows also by Theorem 4.2 that X is $\widetilde{g}-R_0$.

Conversely, suppose that X is both $\widetilde{g}-T_0$ and $\widetilde{g}-R_0$. We want to show that X is $\widetilde{g}-T_1$. Let x, y be any distinct points of X. Since X is $\widetilde{g}-T_0$, there exists a $\widetilde{g}$-open set G such that $x \in G$ and $y \notin G$ or there exists a $\widetilde{g}$-open set H such that $y \in H$ and $x \notin H$. Without loss of generality, we may assume that there exists a $\widetilde{g}$-open set G such that $x \in G$ and $y \notin G$. Since X is $\widetilde{g}-R_0$, $\widetilde{g}cl(\{x\}) \subset G$. As $y \notin G$, $y \notin \widetilde{g}cl(\{x\})$. Hence $y \in H = X - \widetilde{g}cl(\{x\})$ and it is clear that $x \notin H$. Thus it follows that there exist $\widetilde{g}$-open sets G and H containing x and y, respectively, such that $y \notin G$ and $x \notin H$. Hence X is $\widetilde{g}-T_1$. □

Definition 4.11. A subset A of a space X is called $\overline{g}$-*closed* if $\widetilde{g}-cl(A) \subset U$ whenever $A \subset U$ and U is $\widetilde{g}$-open in X, or equivalently, if $\widetilde{g}-cl(A) \subset \widetilde{g}-ker(A)$.

It is clear from the above definition that every $\widetilde{g}$-closed set is $\overline{g}$-closed.

Definition 4.12. A space X is said to be $\tilde{g} - T_{1/2}$ if every $\bar{g}$ -closed subset of X is $\tilde{g}$ -closed.

The following two theorems are immediate consequences of the definitions.

Theorem 4.13. *For a space X, the following statements are equivalent*:

(i) *X is $\tilde{g} - T_{1/2}$.*

(ii) *Every singleton subset of X is $\tilde{g}$ -open or $\tilde{g}$ -closed.*

Theorem 4.14. *For a space X, the following statements are equivalent*:

(i) *X is $\tilde{g} - T_{1/2}$.*

(ii) *For each subset A of X, $A = \bigcap \{F \in \tilde{g}(X) \cup \tilde{g}C(X) : A \subset F\} = \tilde{g} - cl(A) \cap \tilde{g} - ker(A)$.*

Clearly, every $\tilde{g} - T_1$ space is $\tilde{g} - T_{1/2}$, every $\tilde{g} - T_{1/2}$ space is $\tilde{g} - T_0$ and every $T_{1/2}$ space is $\tilde{g} - T_{1/2}$. However, the converses are not true as shown by the following examples.

Example 4.15. Let $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$. Then the space X is $\tilde{g} - T_0$ but not $\tilde{g} - T_{1/2}$. Observe that the $\tilde{g}$ -open subsets of X are the open sets.

Example 4.16. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then the space X is $\tilde{g} - T_{1/2}$ but not $T_{1/2}$ as every singleton subset of X is $\tilde{g}$ -open or $\tilde{g}$ -closed. Observe that every semi-open subset of X is open and thus the $\tilde{g}$ -closed sets are the closed sets together with $\{b, d\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}$. Also X is $\tilde{g} - T_{1/2}$ but not $\tilde{g} - T_1$. It is also an example of a $\tilde{g} - T_0$ space but not T_0.

Definition 4.17. A space X is called *weak* $\widetilde{g}-R_0$ if for each $x \in X$ such that $\{x\} = \widetilde{g}-cl(\{x\}) \cap \widetilde{g}-\ker(\{x\})$, then $\{x\} = \widetilde{g}-\ker(\{x\})$.

It is easy to see that every $\widetilde{g}-R_0$ space is weak $\widetilde{g}-R_0$. However, the converse is not true as shown by the following example.

Example 4.18. Let $X = \{a, b, c\}$ with the topology $\tau = \{\varnothing, \{a\}, X\}$. Then the space X is weak $\widetilde{g}-R_0$ but not $\widetilde{g}-R_0$. Observe that the $\widetilde{g}$-open subsets of X are the open sets.

It is easy to verify now the following improvement of Theorem 4.10.

Theorem 4.19. *For a space* X, *the following are equivalent*:

(i) X *is* $\widetilde{g}-T_1$.

(ii) X *is* $\widetilde{g}-T_0$ *and* $\widetilde{g}-R_0$.

(iii) X *is* $\widetilde{g}-T_0$ *and weak* $\widetilde{g}-R_0$.

Definition 4.20. Let f be a function from a space X into a space Y. Then the graph $G(f) = \{(x, f(x)) : x \in X\}$ of f is said to be *strongly* $\widetilde{g}$*-closed* if for each $(x, y) \in (X \times Y) - G(f)$, there exist a $\widetilde{g}$-open subset U of X and an open subset V of Y containing x and y, respectively, such that $(U \times V) \cap G(f) = \varnothing$.

Lemma 4.21. *Let f be a function from a space X into a space Y. Then its graph $G(f)$ is strongly $\widetilde{g}$-closed if and only if for each point $(x, y) \in (X \times Y) - G(f)$, there exist a $\widetilde{g}$-open subset U of X and an open subset V of Y containing x and y, respectively, such that $f(U) \cap V = \varnothing$.*

Proof. Follows immediately from the above definition. □

Theorem 4.22. *If $f : X \to Y$ is an injective function with a strongly $\widetilde{g}$-closed graph, then X is $\widetilde{g}-T_1$.*

Proof. Suppose that x and y are distinct points of X. Since f is injective, $f(x) \neq f(y)$. Thus $(x, f(y)) \in (X \times Y) - G(f)$, but $G(f)$ is strongly $\widetilde{g}$-closed, so there exist a $\widetilde{g}$-open set U and an open set V containing x and $f(y)$, respectively, such that $f(U) \cap V = \emptyset$. Hence $y \notin U$. Similarly there exist a $\widetilde{g}$-open set M and an open set N containing y and $f(x)$, respectively, such that $f(M) \cap N = \emptyset$. Hence $x \notin M$. Thus it follows that X is $\widetilde{g} - T_1$. □

Theorem 4.23. *If $f : X \to Y$ is a surjective function with a strongly $\widetilde{g}$-closed graph, then Y is T_1.*

Proof. Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists $x \in X$ such that $f(x) = y_2$. Hence $(x, y_1) \notin G(f)$ and thus by Lemma 4.21 there exist a $\widetilde{g}$-open set U and an open set V containing x and y_1, respectively, such that $f(U) \cap V = \emptyset$. Hence $y_2 \notin V$. Similarly there exists $x_0 \in X$ such that $f(x_0) = y_1$. Hence $(x_0, y_2) \notin G(f)$ and thus there exist a $\widetilde{g}$-open set M and an open set N containing x_0 and y_2, respectively, such that $f(M) \cap N = \emptyset$. Hence $y_1 \notin N$. Thus it follows that Y is T_1. □

Remark 4.24. In Definition 4.20, if we consider U and V both are $\widetilde{g}$-open, then Theorem 4.23 yields that Y is $\widetilde{g} - T_1$.

5. $\widetilde{g} - T_2$ Spaces

Definition 5.1. A space X is said to be $\widetilde{g} - T_2$ if to each pair of distinct points x, y of X, there exist two disjoint $\widetilde{g}$-open sets, one containing x and the other containing y.

It is clear that every T_2-space is $\widetilde{g} - T_2$. However, the next question asks about the converse.

Question 2. Is there an example of a $\widetilde{g} - T_2$ space that is not T_2?

Remark 5.2. We observe that every $\widetilde{g} - T_2$ space is $\widetilde{g} - T_1$. However, the converse is not true as shown by the following example.

Remark 5.3. An infinite set X with the finite complement topology is $\widetilde{g}-T_1$. It is, however, not $\widetilde{g}-T_2$ since any two non-empty open subsets of X and hence any two non-empty $\widetilde{g}$-open subsets of X intersect. Observe that a $\widetilde{g}$-open subset of X is open.

Theorem 5.4. *For a space X, the following statements are equivalent*:

(i) *X is $\widetilde{g}-T_2$.*

(ii) *For each $x \in X$, $\bigcap\{\widetilde{g}-cl(U_x) : U_x$ is a $\widetilde{g}$-neighborhood of $x\}$ $= \{x\}$ or equivalently, every singleton subset of X is the intersection of $\widetilde{g}$-closed neighborhoods of x.*

Proof. (i) $\Rightarrow$ (ii): Let X be a $\widetilde{g}-T_2$ space and $x \in X$. Then to each $y \in X$, $y \neq x$, there exist $\widetilde{g}$-open sets G and H such that $x \in G$, $y \in H$ and $G \cap H = \varnothing$. Since $x \in G \subset X-H$, $X-H$ is a $\widetilde{g}$-closed $\widetilde{g}$-neighborhood of x to which y does not belong. Consequently, the intersection of all $\widetilde{g}$-closed $\widetilde{g}$-neighborhoods of x is reduced to $\{x\}$.

(ii) $\Rightarrow$ (i): Suppose that $x, y \in X$ and $x \neq y$. Then by hypothesis there exists a $\widetilde{g}$-closed $\widetilde{g}$-neighborhood U of x such that $y \notin U$. Now there is a $\widetilde{g}$-open set G such that $x \in G \subset U$. Thus G and $X-U$ are disjoint $\widetilde{g}$-open sets containing x and y, respectively. Hence X is $\widetilde{g}-T_2$. □

The proof of the following theorem is straightforward and thus omitted.

Theorem 5.5. *A space X is $\widetilde{g}-T_2$ if and only if for each $x, y \in X$ such that $x \neq y$, there exist $\widetilde{g}$-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.*

Recall that a subset A of a space X is called *sg-closed* if whenever $A \subset U$, where U is semi-open in X, then $scl(A) \subset U$.

Remark 5.6. The product of two $\widetilde{g}$-open sets need not be $\widetilde{g}$-open as the following example tells.

Example 5.7. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a, b\}, X\}$. Then $A = \{b, c\}$ is $\widetilde{g}$ -closed. Now $A \times X$ is not *sg*-closed because if $U = (X \times X) - \{(a, c)\}$, then U is semi-open in $X \times X$ and $A \times X \subset U$. However, $X \times X = scl(A \times X) \not\subseteq U$. Since every $\widetilde{g}$ -closed set is *sg*-closed, it follows that $A \times X$ is not $\widetilde{g}$ -closed. From this we conclude that the product of two $\widetilde{g}$ -closed sets need not be $\widetilde{g}$ -closed. Since the union of $\widetilde{g}$ -open sets is $\widetilde{g}$ -open, it follows that the product of two $\widetilde{g}$ -open sets need not be $\widetilde{g}$ -open.

Theorem 5.8. *Every* $\widetilde{g}$ *-regular* T_0 *space is* $\widetilde{g} - T_2$.

Proof. Let X be a $\widetilde{g}$ -regular T_0 space and let $x, y \in X$ be such that $x \neq y$. Since X is T_0, there exists an open set V containing x but not y or y but not x, say x but not y. Then $y \in X - V$, $X - V$ is closed and $x \notin X - V$. By $\widetilde{g}$ -regularity of X, there exist $\widetilde{g}$ -open sets G and H such that $x \in G$, $y \in X - V \subset H$ and $G \cap H = \varnothing$. Hence X is $\widetilde{g} - T_2$. □

Theorem 5.9. *If* $f : X \to Y$ *is an injective* $\widetilde{g}$ *-irresolute* (*resp.* $\widetilde{g}$ *-continuous*) *function and* Y *is* $\widetilde{g} - T_2$ (*resp.* T_2), *then* X *is* $\widetilde{g} - T_2$.

Proof. We show the first case, the other case is similar. Suppose that $x, y \in X$, $x \neq y$. Since f is injective, $f(x) \neq f(y)$, but Y is $\widetilde{g} - T_2$, so there exist $\widetilde{g}$ -open sets G, H in Y such that $f(x) \in G$, $f(y) \in H$ and $G \cap H = \varnothing$. Let $U = f^{-1}(G)$ and $V = f^{-1}(H)$. Then by hypothesis, U and V are $\widetilde{g}$ -open sets in X. Also $x \in f^{-1}(G) = U$, $y \in f^{-1}(H) = V$ and $U \cap V = \varnothing$. Hence X is $\widetilde{g} - T_2$. □

The following three theorems have easy proofs and thus omitted:

Theorem 5.10. *If* $f : X \to Y$ *is a bijective* $\widetilde{g}$ *-open* (*resp.* $\widetilde{g}^*$ *-open*) *function and* X *is* T_2 (*resp.* $\widetilde{g} - T_2$), *then* Y *is* $\widetilde{g} - T_2$.

Theorem 5.11. *If f is a* $\widetilde{g}$*-open function from a space X onto a space Y and the set* $\{(x_1, x_2) : f(x_1) = f(x_2)\}$ *is closed in* $X \times X$, *then Y is* $\widetilde{g} - T_2$.

Theorem 5.12. *If f is a* $\widetilde{g}^*$*-open function from a space X onto a space Y and f has a strongly* $\widetilde{g}$*-closed graph, then Y is* $\widetilde{g} - T_2$.

Remark 5.13. The above theorem is still true if we consider in the definition of a strongly $\widetilde{g}$-closed graph *U* and *V* to be both $\widetilde{g}$-open.

Definition 5.14. A space *X* is said to be $\widetilde{g} - R_1$ [4] if for each $x, y \in X$ with $\widetilde{g} - cl(\{x\}) \neq \widetilde{g} - cl(\{y\})$, there exist disjoint $\widetilde{g}$-open sets *U* and *V* such that $\widetilde{g} - cl(\{x\}) \subset U$ and $\widetilde{g} - cl(\{y\}) \subset V$.

Theorem 5.15. *A space X is* $\widetilde{g} - T_2$ *if and only if it is* $\widetilde{g} - R_1$ *and* $\widetilde{g} - T_0$.

Proof. Similar to that of Theorem 4.10. □

Remark 5.16. In the following diagram we denote by arrows the implications between the separation axioms which we have introduced and discussed in this paper. However, none of these implications is reversible.

$$\begin{array}{ccc} T_2 & \Rightarrow & \widetilde{g} - T_2 \\ \Downarrow & & \Downarrow \\ T_1 & \Rightarrow & \widetilde{g} - T_1 \\ \Downarrow & & \Downarrow \\ T_{1/2} & \Rightarrow & \widetilde{g} - T_{1/2} \\ \Downarrow & & \Downarrow \\ T_0 & \Rightarrow & \widetilde{g} - T_0 \end{array}$$

Remark 5.17. It is not difficult to see that every $\widetilde{g} - R_1$ space is $\widetilde{g} - R_0$. However, it follows from Theorem 4.19 and Theorem 5.15 that any space which is $\widetilde{g} - T_1$ but not $\widetilde{g} - T_2$ is an example of a $\widetilde{g} - R_0$ space that is not $\widetilde{g} - R_1$.

References

[1] M. Caldas, S. Jafari, N. Rajesh and M. L. Thivagar, $\widetilde{g}$-homeomorphisms in topological spaces, submitted.

[2] S. G. Crossley and S. K. Hildebrand, Semi-closure, Texas J. Sci. 22 (1971), 99-112.

[3] S. Jafari, T. Noiri, N. Rajesh and M. L. Thivagar, Another generalized closed sets, Kochi J. Math. 3 (2008), to appear.

[4] S. Jafari and N. Rajesh, Characterizations of $\widetilde{g} - R_0$ and $\widetilde{g} - R_1$, preprint.

[5] N. Levine, Generalized closed sets in topology, Rend. Circ. Math. Palermo 19(2) (1970), 89-96.

[6] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly 70 (1963), 36-41.

[7] N. Rajesh and E. Ekici, On a new form of irresoluteness and weak forms of strong continuity, submitted.

[8] N. Rajesh and E. Ekici, On $\widetilde{g}$-continuous mappings in topological spaces, Proc. Inst. Math. (to appear).

[9] N. Rajesh and E. Ekici, On $\widetilde{g}$-regular and $\widetilde{g}$-normal spaces, Kochi J. Math. 2 (2007), 71-78.

[10] N. Rajesh and E. Ekici, $\widetilde{g}$-locally closed sets in topological space, Kochi J. Math. 2 (2007), 1-9.

[11] M. K. Singal and A. K. Singal, Almost continuous mappings, Yokohama Math. J. 16 (1968), 63-73.

[12] M. K. R. S. Veerakumar, Between g^*-closed sets and g-closed sets, Antartica J. Math. 3(1) (2006), 43-65.

[13] M. K. R. S. Veerakumar, $\hat{g}$-closed sets in topological spaces, Allahabad Math. Soc. 18 (2003), 99-112.

[14] M. K. R. S. Veerakumar, $^{\sharp}g$-semi-closed sets in topological spaces, Antartica J. Math. 2(2) (2005), 201-222.