Far East J. Math. Sci. (FJMS) **20**(3) (2006), 341-346

DUO MODULES AND DUO RINGS

N. AGAYEV, T. KOŞAN, A. LEGHWEL

and

A. HARMANCI

(Received December 1, 2005)

Submitted by K. K. Azad

Abstract

A module M is called duo module if every submodule of M is fully invariant. M has the SIP (or the SSP) if the intersection (or the sum) of two direct summands of M is direct summand. In this note we prove that every duo module has the SSP and the SIP.

1. Introduction

Throughout R will be a ring with identity, and modules are unital right R-modules. Let M be a module. Then we use $N \leq M$ to mean that N is a submodule of M and $N \subseteq^{\oplus} M$ to indicate that N is a direct summand of M. Let $S = End_R(M)$. A submodule N of M is said to be *fully invariant* if $f(N) \leq N$ for each $f \in S$. A module M is called *duo module* if every submodule of M is fully invariant. The ring R is called *right duo ring* if every right ideal in R is a left ideal. In fact, R is a duo ring if every one-sided ideal is two-sided.

A right artinian right self-injective ring is called *quasi-Frobenius ring* (or *QF-ring* for short). When we were discussing and reporting papers on

2000 Mathematics Subject Classification: 16D60, 16D99, 16S90.

Keywords and phrases: duo module, modules with the SSP, modules with the SIP.

the question of when a right perfect right self-injective ring is QF, we encountered the following results in the context:

(I) In [3, Corollary 2.5] it is proved that: Let M be a quasi-p-injective and duo module. If A and B are direct summands of M, then so are $A \cap B$ and $A + B$.

(II) In [6, Proposition 3.3] it is proved that: Let M be a duo and PQ-injective module. Then M has the SIP, and in [6, Proposition 3.4] it is shown that: Let M be a duo, principal and PQ-injective. Then M has both the SIP and the SSP.

(III) In [7, Theorem 3.1] it has been shown that: Let R be a right P-injective right duo ring. If A and B are right ideals of R with $A \subseteq^{\oplus} R_R$ and $B \subseteq^{\oplus} R_R$, then $(A \cap B) \subseteq^{\oplus} R_R$ and $(A \oplus B) \subseteq^{\oplus} R_R$.

(IV) In [8, Lemma 2.5] it is proved that: Let M be a quasi-principally injective module and A and B be its submodules.

(1) If A is a direct summand of M and $B \cong A$, then B is a direct summand of M.

(2) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

(V) In [9, Proposition 4.4] it is shown that: Every duo and semi-projective module has the SIP, and in [9, Proposition 4.6] it is also proved that: Every duo and semi-projective module with the property (C_3) has the SSP.

(VI) In [11, Proposition 3.6] it has been shown that every duo and semi-injective module has the SIP and the SSP.

We generalize the aforementioned results (I)-(VI) and we prove that every duo module has the SIP and the SSP. For the unexplained terminology, the reader is referred to [2], [5] or [10].

2. Duo Modules

We start by proving the following results.

Lemma 1. *R is a right duo ring if and only if the right R-module R_R is a duo module.*

Proof. Assume that R is a right duo ring. Then every right ideal is a left ideal. Let I be any right ideal and $f \in End(R_R)$. Let $x \in I$. Then $f(x) = f(1)x \in I$ since f is also a left ideal. Hence $f(I) \leq I$.

Conversely, suppose that R_R is a duo module, and let I be any right ideal in R. For $x \in R$ define $R \xrightarrow{f_x} R$ by $f_x(r) = xr$, where $r \in R$. f_x is a well-defined right R-homomorphism of R. By hypothesis $f_x(I) \leq I$. Hence $xI \leq I$ for all $x \in R$. It follows that R is a left ideal of R.

Lemma 2. *Let a module $M = M' \oplus M''$ be the direct sum of modules M' and M''. If M is a duo module, then M' and M'' are duo modules, $Hom(M', M'') = 0$ and $Hom(M'', M') = 0$.*

Proof. Let $N \leq M'$ and $f \in Hom(M', M'')$. Then f can be extended to a $g \in End(M_R)$, so that $g(M'') = 0$. By assumption, $f(N) = g(N) \leq N$. The rest is clear.

Lemma 3. *Let M be any duo left R-module. Then the natural projections of M are central idempotents of the endomorphism ring $S = End(M)$ of M.*

Proof. Let $M = M' \oplus M''$ and π denote the projection of M with kernel M''. Let $f \in S$ and $m = m' + m''$, where $m' \in M', m'' \in M''$. By Lemma 2, $f = f' + f''$, where $f' \in End(M'), f'' \in End(M'')$. Now $\pi(m) = m'$, $f(m) = f'(m') + f''(m''), f(m') = f'(m'), f(m'') = f''(m'')$. Then $f(\pi(m)) = f(m') = f'(m') = \pi(f(m))$ for all $m \in M$. Hence $f\pi = \pi f$ for all $f \in S$.

Example 4. Let Z denote the ring of integers. Then for any prime integer p and positive integer n, Z/Zp^n as Z-modules are duo but the rational numbers Q as a Z-module is not duo.

Proof. Let $f \in End_Z(Z)$ and $I = nZ$ be an ideal of Z. Since f is determined by $f(1)$, for all $t \in Z$, $f(nt) = f(1)nt \in I$. Hence $f(I) \leq I$ and so Z is duo module. Same holds for Z/Zp^n since any $f \in End_Z(Z/Zp^n)$ is determined by $f(\overline{1})$, where $\overline{1} = 1 + Zp^n \in Z/Zp^n$. Let $f \in End_Z(Q)$ be given by $f(a/b) = a/2b$, where $a/b \in Q$ and let $I = (1/2)Z$ be the Z-submodule of Q. Then $f(1/2) = 1/4$ and $1/4 \notin I$. It follows that I is not invariant under f and so Q is not a duo Z-module.

We now come to our main result and prove the following theorem which unifies and generalizes several known results in [3, 6-9, 11].

Theorem 5. *Let M be a duo module. Then M has the SIP and the SSP.*

Proof. Suppose that M is a duo module. Let N and K be two direct summands of M. Notice that for the natural projections α and β as $M = N \oplus N' \xrightarrow{\alpha} N$ and $M = K \oplus K' \xrightarrow{\beta} K$, respectively, $N = \alpha(M)$ and $K = \beta(M)$. Hence $M = \alpha(M) \oplus N'$ and $M = \beta(M) \oplus K'$. By hypothesis

$$\alpha(M) = \alpha(\beta(M) \oplus K') \leq \alpha(\beta(M)) + \alpha(K') \leq \alpha(M) \cap \beta(M) + \alpha(M) \cap K' \leq \alpha(M).$$

Hence

$$\alpha(M) = \alpha(M) \cap \beta(M) \oplus \alpha(M) \cap K'. \tag{1}$$

It follows that $N \cap K = \alpha(M) \cap \beta(M)$ is a direct summand of M. Hence M has the SIP. Then (1) also shows that if $M = A \oplus B$, then

$$\alpha(M) = (\alpha(M) \cap A) \oplus (\alpha(M) \cap B). \tag{2}$$

Since $\alpha(M)$ and $\beta(M)$ are direct summands and M has the SIP, there exists $U \leq M$ such that

$$M = (\alpha(M) \cap \beta(M)) \oplus U. \tag{3}$$

Then $\beta(M) = \alpha(M) \cap \beta(M) \oplus U \cap \beta(M)$ by the modularity. So $\alpha(M) + \beta(M)$ $= \alpha(M) + \alpha(M) \cap \beta(M) + U \cap \beta(M) = \alpha(M) \oplus U \cap \beta(M)$. Since M has the SIP and $\beta(M)$ and U are direct summands, there exists $V \leq M$ such that

$$M = U \cap \beta(M) \oplus V. \tag{4}$$

By (4), (2) and $\alpha(M) \cap U \cap \beta(M) = 0$,

$$\alpha(M) = \alpha(M) \cap U \cap \beta(M) \oplus \alpha(M) \cap V = \alpha(M) \cap V \leq V.$$

Hence $V = \alpha(M) \oplus N' \cap V$, by modularity and $M = \alpha(M) \oplus N'$. So we replace V in $M = U \cap \beta(M) \oplus V$ to obtain

$$M = U \cap \beta(M) + \alpha(M) + N' \cap V = (\alpha(M) + \beta(M)) \oplus (N' \cap V).$$

Thus $N + K = \alpha(M) + \beta(M)$ is direct summand of M. Thus M has the SSP.

We now provide the following example which is an application of our main result.

Example 6. Let Z and Q denote the ring of integers and the field of rational numbers, respectively.

(1) The Z-module $Z \oplus A$ is not duo for any Z-module A.

(2) For any distinct prime integers p_i $(i = 1, 2, \dots, n)$, the Z-module $M = \oplus_{i=1}^{n} Z/Zp_i^{n_i}$ is duo module for any positive integers n_i $(i = 1, 2, \dots, n)$.

(3) The Z-module $Q \oplus A$ is not a duo module for any Z-module A.

Proof. (1) Let A be any nonzero Z-module. Assume that $Z \oplus A$ is duo Z-module. By Theorem 5, $Z \oplus A$ has the SIP and the SSP. Hence by [1] and [4] the image and kernel of any $f \in Hom_Z(Z, A)$ (or $f \in Hom_Z(A, Z)$) are direct summands. It follows that A is isomorphic to a direct summand of Z and Z is isomorphic to a direct summand of A. A contradiction. Hence $Z \oplus A$ is not duo Z-module.

(2) We complete the proof by induction and we may assume $n = 2$. Let p and q be distinct prime integers. Then we prove $M = (Z/Zp^r) \oplus Z/Zq^s$ is duo, where r and s are positive integers, and let r' and s' be integers such that $p^r r' + q^s s' = 1$. Now assume N is any submodule of M and $(a, b) \in N$. Then $p^r r'(a, b) = (0p^r r', b) = (0, b) \in N$ and $q^s s'(a, b) = (q^s s' a, 0) = (a, 0) \in N$. Hence $N = (N \cap (Z/Zp^r)) \oplus (N \cap (Z/Zq^s))$. Let $N_1 = N \cap (Z/Zp^r)$ and $N_2 = N \cap (Z/Zq^s)$. Let $f \in End_Z(M)$. Then $f(a, b) = f(a, 0) + f(0, b) \in N_1 + N_2 = N$ since Z/Zp^r and Z/Zq^s are duo and $(a, 0) \in N_1$ and $(0, b) \in N_2$. Hence M is duo Z-module.

(3) If $Q \oplus A$ is duo module for any Z-module A, then Q will be a duo Z-module by Lemma 2. But this contradicts Example 4. Therefore, $Q \oplus A$ is not duo module for any Z-module A.

References

[1] M. Alkan and A. Harmanci, On summand sum and summand intersection property of modules, Turkish J. Math. 26 (2002), 131-147.

[2] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Springer-Verlag, New York, 1992.

[3] S. Chotchaisthit, When is a quasi-p-injective module continuous?, Southeast Asian Bull. Math. 26 (2002), 391-394.

[4] J. L. Garcia, Properties of direct summands of modules, Comm. Algebra 17 (1989), 73-92.

[5] S. H. Mohamed and B. J. Müller, Continuous and discrete modules, L. M. S. Lecture Notes 147, Cambridge University Press, Cambridge, U. K., 1990.

[6] W. K. Nicholson, J. K. Park and M. F. Yousif, Principally quasi-injective modules, Comm. Algebra 27(4) (1999), 1683-1693.

[7] G. Puninski, R. Wisbauer and M. Yousif, On P-injective rings, Glasg. Math. J. 37 (1995), 373-378.

[8] N. V. Sanh and K. P. Shum, On quasi-principally injective modules, Algebra Colloq. 6(3) (1999), 269-276.

[9] H. Tansee and S. Wongwai, A note on semi-projective modules, Kyungpook Math. J. 42 (2002), 369-380.

[10] R. Wisbauer, Foundations of Module and Ring Theory, Gordon & Breach, Reading, MA, 1991.

[11] S. Wongwai, On the endomorphism ring of a semi-injective module, Acta Math. Univ. Comenian. LXXI(1) (2002), 27-33.

N. Agayev
Department of Mathematics
Gazi University
06500 Ankara, Turkey
e-mail: nagayev@gazi.edu.tr

T. Koşan, A. Leghwel and A. Harmanci
Department of Mathematics
Hacettepe University
06532 Beytepe Campus, Ankara, Turkey
e-mail: tkosan@hacettepe.edu.tr
abdurzak@hacettepe.edu.tr
harmanci@hacettepe.edu.tr