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RELATIVE PROPERTIES AND FUNCTION SPACES

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Abstract

We investigate duality between (covering) properties in topological spaces and the closure-type properties of mappings. We give a version of the Pytkeev property for continuous mappings.

1. Introduction

In this paper all the spaces are Tychonoff. The notation and terminology we follow are standard [6]. Let X be a topological space. Then

(1) the symbol $\mathcal{O}$ denotes the collection of open covers of X;

(2) the symbol Ω denotes the collection of ω-covers of X. An open cover $\mathcal{U}$ of a space X is called an ω-*cover* if X is not a member of $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$ [7];

(3) the symbol $\mathcal{K}$ denotes the collection of k-covers of X. An open cover $\mathcal{U}$ of a space X is called a k-*cover* if X is not a member of $\mathcal{U}$ and

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every compact subset of X is contained in a member of $\mathcal{U}$ [5].

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of a topological space X. Then

(a) the symbol $S_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(b_n : n \in \mathbb{N})$ such that for each n $b_n \in A_n$ and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$;

(b) the symbol $S_{fin}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbb{N})$ such that for each n B_n is a finite subset of A_n and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$.

For a space X by $C_p(X)$ we denote the space of all continuous real-valued functions on X in the pointwise convergence topology. For a function $f \in C_p(X)$, a finite set F in X and a positive real number ε we let

$$W(f;\ F;\ \varepsilon) = \{g \in C_p(X) : |f(x) - g(x)| < \varepsilon,\ \forall x \in F\}.$$

The standard local base of a point $f \in C_p(X)$ consists of the sets $W(f;\ F;\ \varepsilon)$, where F is a finite subset of X and ε is a positive real number.

By $C_k(X)$ we denote the space of all continuous real-valued functions on a space X endowed with the compact-open topology. For a function $f \in C_k(X)$, a compact set $K \subset X$ and a positive real number ε we let

$$W(f;\ K;\ \varepsilon) = \{g \in C_p(X) : |f(x) - g(x)| < \varepsilon,\ \forall x \in K\}.$$

The standard local base of a point $f \in C_k(X)$ consists of the sets $W(f;\ K;\ \varepsilon)$, where K is a compact subset in X and ε is a positive real number.

The symbol $\underline{0}$ denotes the constantly zero function in $C_p(X)$ and in $C_k(X)$. Since $C_p(X)$ and $C_k(X)$ are homogenous spaces we may consider the point $\underline{0}$ when studying local properties of them.

In [1] Arhangel'skiĭ considered the mapping π from $C_p(X)$ (resp.

$C_k(X))$ into $C_p(Y)$ (resp. $C_k(Y))$ defined by $\pi(f) = f_{|Y}$, for each $f \in C_p(X)$ (resp. $f \in C_k(X)$).

Some results in the literature show that there is a duality between relative covering properties of a subspace Y of a Tychonoff space X and the closure-type properties of the mapping π. This sort of duality was documented by Gordienko for the Lindelöf property [8], by Kočinac and Babinkostova for the Menger property and for the Rothberegr property [13], by Guido and Kočinac for the Hurewicz property [9] and by Babinkostova et al. for the γ-sets [2].

This investigation is a part of general idea to transfer properties of topological spaces to continuous mappings [3] and [4].

2. The Countable (Strong) Fan Tightness

For a space X and a point $x \in X$, the symbol Ω_x denotes the set $\{A \subset X \setminus \{x\} : x \in \overline{A}\}$.

Let Y be a subspace of a space X. We denote by $\Omega_X(\mathcal{K}_X)$ the collection of ω-cover (k-cover) of X and by $\Omega_Y(\mathcal{K}_Y)$ the collection of ω-cover (k-cover) of Y, by sets open in X.

A space X has *countable fan tightness* [1] if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that, for each $n \in \mathbb{N}$, $B_n \subset A_n$ and $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$, i.e., if $S_{fin}(\Omega_x, \Omega_x)$ holds for each $x \in X$.

A space X has *countable strong fan tightness* [21] if for each $x \in X$ the selection principle $S_1(\Omega_x, \Omega_x)$ holds.

Definition 2.1. Let f be a continuous mapping from X to Y. Then

(1) f has *countable fan tightness* [13] if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that, for each n, $B_n \subset A_n$ and $f(x) \in \overline{\bigcup_{n\in\mathbb{N}} f(B_n)}$.

(2) f has *countable strong fan tightness* [13] if for each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there exists a sequence $(x_n : n \in \mathbb{N})$ such that $x_n \in A_n$, for each $n \in \mathbb{N}$, and $f(x) \in \overline{\{f(x_n) : n \in \mathbb{N}\}}$.

In the paper [13] it was shown the following theorem.

Theorem 2.1. *For a subspace Y of a space X, the following are equivalent*:

(1) $S_1(\Omega_X, \Omega_Y)$ *holds*;

(2) $\pi : C_p(X) \to C_p(Y)$ *has countable strong fan tightness*.

We show

Theorem 2.2. *For a subspace Y of a space X, the following are equivalent*:

(1) $S_1(\mathcal{K}_X, \mathcal{K}_Y)$ *holds*;

(2) $\pi : C_k(X) \to C_k(Y)$ *has countable strong fan tightness*.

Proof. (2) $\Rightarrow$ (1) Let $\{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of k-covers of X. For each n and a compact subset K of X we denote by $\mathcal{U}_{n,K}$ the set $\{U \in \mathcal{U}_n : K \subset U\}$. If $U \in \mathcal{U}_{n,K}$, let $f_{U,K} : X \to [0, 1]$ be a continuous function satisfying $f_{U,K}(K) = 0$, $f_{U,K}(X \setminus U) = 1$. Let, for each $n \in \mathbb{N}$, $A_n = \{f_{U,K} : U \in \mathcal{U}_{n,K}, K \text{ compact}\}$. Then $\underline{0}$ is in the closure of A_n, for each $n \in \mathbb{N}$. If $W(\underline{0}, K, \varepsilon)$ is a neighborhood of $\underline{0}$ and $U \in \mathcal{U}_{n,K}$, then the function $f_{U,K}$ belongs to $A_n \cap W(\underline{0}, K, \varepsilon)$, for each n. Since $\pi : C_k(X) \to C_k(Y)$ has the countable strong fan tightness, there exists a sequence $(f_{K_n,U_n} : n \in \mathbb{N})$ such that, for each n, $f_{K_n,U_n} \in A_n$ and $\pi(\underline{0})$ belongs to the closure of $\{\pi(f_{K_n,U_n}) : n \in \mathbb{N}\}$. Consider the sets U_n, $n \in \mathbb{N}$. We claim that the sequence $(U_n : n \in \mathbb{N})$ witnesses that $S_1(\mathcal{K}_X, \mathcal{K}_Y)$ holds. Let T be a compact subset of Y. From $\pi(\underline{0}) \in \overline{\{\pi(f_{K_n,U_n}) : n \in \mathbb{N}\}}$ it follows that there is an $i \in \mathbb{N}$ such that $W(\pi(\underline{0}), T, 1)$ contains the function $\pi(f_{K_i,U_i})$. Then $T \subset U_i$. Otherwise, for some $x \in T$ one has that $x \notin U_i$, so that

$\pi(f_{K_i, U_i})(x) = f_{K_i, U_i}(x) = 1$ which contradicts the fact $\pi(f_{K_i, U_i}) \in W(\pi(\underline{0}), T, 1)$.

(1) $\Rightarrow$ (2) Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of $C_k(X)$ the closures of which contain $\underline{0}$. Fix n. For every compact set $K \subset X$ the neighborhood $W\left(\underline{0}, K, \frac{1}{n}\right)$ of $\underline{0}$ intersects A_n, so there exists a continuous function $f_{K,n} \in A_n$ such that $|f_{K,n}(x)| < \frac{1}{n}$, for each $x \in K$. Since $f_{K,n}$ is a continuous function there are neighborhoods U_x of x, $x \in K$, such that, for $U_{K,n} = \bigcup_{x \in K} U_x \supset K$, we have $f_{K,n}(U_{K,n}) \subset \left(-\frac{1}{n}, \frac{1}{n}\right)$. Let $\mathcal{U}_n = \{U_{K,n} : K$ compact subset of $X\}$. For each n, $\mathcal{U}_n$ is a k-cover of X. By the condition 2 one can find a sequence $(U_{K,n} : n \geq m)$ such that, for each n, $U_{K,n} \in \mathcal{U}_n$ and $\{U_{K,n} : n \in \mathbb{N}\}$ is a k-cover for Y. Consider the corresponding functions $f_{K,n}$ in A_n. We verify that the sequence $\{f_{K,n} : n \in \mathbb{N}\}$ witnesses for $(A_n : n \in \mathbb{N})$ that π has the countable strong fan tightness. Let $W(\pi(\underline{0}), T, \varepsilon)$ be a neighborhood of $\pi(\underline{0})$ in $C_k(Y)$ and let m be a natural number such that $\frac{1}{m} < \varepsilon$. Since T is a compact subset of Y and $S_1(\mathcal{K}_X, \mathcal{K}_Y)$ holds, there is an $n_0 \in \mathbb{N}$, $n_0 \geq m$, such that one can find a $U_{K,n_0} \in \mathcal{U}_n$ with $T \subset U_{K,n_0}$. We have

$$\pi(f_{K,n})(T) = f_{K,n}(T) \subset f_{K,n}(U_{K,n_0}) \subset \left(-\frac{1}{n_0}, \frac{1}{n_0}\right) \subset \left(-\frac{1}{m}, \frac{1}{m}\right) \subset (-\varepsilon, \varepsilon),$$

i.e., $\pi(f_{K,n}) \in W(\pi(\underline{0}), T, \varepsilon)$.

In a similar way one can prove:

Theorem 2.3. *For a subspace Y of a space X, the following are equivalent*:

(1) $S_{fin}(\mathcal{K}_X, \mathcal{K}_Y)$ *holds*;

(2) $\pi : C_k(X) \to C_k(Y)$ *has countable fan tightness.*

3. The Pytkeev Property of Continuous Mappings

For a space X and $x \in X$ a family $\mathcal{F}$ of subsets of X is called π-*network at* x if every neighborhood of x contains an element of $\mathcal{F}$.

A space X is called a *Pytkeev space* [16] if $x \in \overline{A} \setminus A$ and $A \subset X$ implies the existence of a countable π-network at x consisting of infinite subsets of A.

Now we transfer this property to the mapping as follows.

Definition 3.1. Let X and Y be topological spaces and let $f : X \to Y$ be a continuous mapping. f has the *Pytkeev property at* $x \in X$ if for each $A \subset X$ and every $x \in X$ such that $x \in \overline{A} \setminus A$ there is a sequence $(B_n : n \in \mathbb{N})$ of infinite subsets of A such that $(f(B_n) : n \in \mathbb{N})$ is π-network at $f(x)$.

If f has this property at all points $x \in X$ we shall say that f has the *Pytkeev property*.

Remark 3.1. Observe that if either X or Y has the Pytkeev property, then f has that property.

In [19] Sakai gave a characterization of the Pytkeev property in the function spaces $C_p(X)$ in terms of shrinkable ω-covers. For a similar investigation see also [17].

Definition 3.2. Let Y be a subspace of a space X. An open ω-cover (k-cover) $\mathcal{U}$ of X is said to be *relatively* ω-*shrinkable* (*relatively* k-*shrinkable*) *with respect to* Y if for each $U \in \mathcal{U}$ there is a closed set $C(U)$ of X such that $C(U) \subset U$ and $\{C(U) : U \in \mathcal{U}\}$ is an ω-cover (k-cover) of Y.

We prove

Theorem 3.1. *Let Y be a subspace of a space X. Then the following are equivalent*:

(1) $\pi : C_p(X) \to C_p(Y)$ *has the Pytkeev property*;

(2) *if* $\mathcal{U}$ *is an* ω-*cover of X relatively* ω-*shrinkable with respect to* Y,

then there is a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ *of subfamilies of* $\mathcal{U}$ *such that, for each* $n \in \mathbb{N}$, $|\mathcal{U}_n| = \omega$ *and* $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ *is an* ω*-cover of* Y.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{U}$ be an ω-cover relatively ω-shrinkable with respect to Y. For each $U \in \mathcal{U}$ we choose a closed set $C(U)$ of X such that $C(U) \subset U$ and $\{C(U) : U \in \mathcal{U}\}$ is a k-cover of Y. So we can take in X, for each $U \in \mathcal{U}$ a zero-set $Z(U)$ and a cozero-set $V(U)$ such that $C(U) \subset Z(U) \subset V(U) \subset U$. Without loss in generality, we may assume that for distinct, U' and U'' in $\mathcal{U}$, $Z(U')$ and $Z(U'')$ are distinct.

For a compact subset K of X let $\mathcal{U}_K = \{U \in \mathcal{U} : K \subset U\}$. For each $U \in \mathcal{U}$, let $f_{K,U}$ be a continuous map from X to $[0, 1]$ such that $f_{K,U}^{-1}(0) = Z(U)$ and $f_{K,U}^{-1}(1) = X \setminus V(U)$. Let $A = \{f_{K,U} : K$ compact subset of X, $U \in \mathcal{U}_k\}$. Note that for distinct U' and $U' \in \mathcal{U}$, $f_{K,U'}$ and $f_{K,U''}$ are distinct, and obviously $\underline{0} \in \overline{A} \setminus A$. By the condition 1 there exists a sequence $(B_n : n \in \mathbb{N})$ such that B_n is infinite subset of A and $(\pi(B_n) : n \in \mathbb{N})$ is a π-network at $\pi(\underline{0})$. For each n let $\mathcal{U}_n$ be a subfamily of $\mathcal{U}$ such that $|\mathcal{U}_n| = \omega$ and $B_n = \{f_U : U \in \mathcal{U}_n\}$. We claim that $(\mathcal{U}_n : n \in \mathbb{N})$ witnesses that $\mathcal{U}$ satisfies the condition 2. Let K be a compact subset of Y and consider the neighborhood $W(\pi(\underline{0}), K, 1)$ of $\pi(\underline{0})$. Then there is an $n_0 \in \mathbb{N}$ such that $\pi(B_{n_0}) \subset W(\pi(\underline{0}), K, 1)$. This means that $K \subset \bigcap\{V(W) : W \in \mathcal{W}_n\} \subset \{U : U \in \mathcal{U}_n\}$. Thus $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is a k-cover of Y.

(2) $\Rightarrow$ (1) Let $\pi : C_k(X) \to C_k(Y)$ and let A be a subset of $C_k(X)$ such that $\underline{0} \in \overline{A} \setminus A$. For each compact set K of X the neighborhood $W\left(\underline{0}, K, \frac{1}{n}\right)$ in $C_k(X)$ of $\underline{0}$ intersects A. So that there exists a continuous map $f_K \in A$ such that $|f_K(x)| < \frac{1}{n}$, for each $x \in K$. Let $U(f_k) = \left\{x \in X : |f_K(x)| < \frac{1}{n}\right\}$ and $\mathcal{U} = \{U(f_K) : f_K \in A\}$. We see that $\mathcal{U}$ is a k-cover of X relative k-shrinkable with respect to Y. Indeed for each

$f_K \in A$ let $Z(f_k) = \left\{x \in X : \ | f_K(x) | \leq \frac{1}{2n}\right\}$. Obviously $Z(f_k)$ is closed in $X, U(f_k) \subset Z(f_k)$ and $K \subset \ Z(f_k)$. Apply the condition 2 to $\mathcal{U}$ there exists a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ such that, for each n, $\mathcal{U}_n$ is subfamily of $\mathcal{U}$ and $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is a k-cover of Y. Then there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, B_n is an infinite subset of A and $\mathcal{U}_n = \{U(f_K) : \ f_K \in B_n\}$. It easy to show that the family $(\pi(B_n) : n \in \mathbb{N})$ is a π-network at $\pi(\underline{0})$. In fact let $W(\pi(\underline{0}), K, \varepsilon)$ be a neighborhood of $\pi(\underline{0})$ in $C_k(Y)$ and m be a natural number such that $\frac{1}{m} < \varepsilon$. Since K is a compact subset of Y there is a $j \in \mathbb{N}$, $j \geq m$, such that $K \subset \bigcap U_j$. Then

$$\pi(f_K(K)) = f_K(K) \subset f_K(\bigcap \mathcal{U}_j) \subset \left(\frac{-1}{j}, \frac{1}{j}\right) \subset \left(\frac{-1}{m}, \frac{1}{m}\right),$$

i.e., $\pi(f_K) \in W(\pi(\underline{0}), K, \varepsilon)$ and $B_j \subset W(\underline{0}, K, \varepsilon)$. □

In a similar way one can show that

Theorem 3.2. *Let Y be a subspace of a space X. The following are equivalent*:

(1) $\pi : C_k(X) \to C_k(Y)$ *has the Pytkeev property*;

(2) *if $\mathcal{U}$ is a k-cover of X relatively k-shrinkable with respect to Y, then there is a sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of subfamilies of $\mathcal{U}$ such that, for each $n \in \mathbb{N}$, $|\mathcal{U}_n| = \omega$ and $\{\bigcap \mathcal{U}_n : n \in \mathbb{N}\}$ is a k-cover of Y.*

The selectively Pytkeev property was studied in [12] (in hyperspaces) and in [17] (in function spaces).

Definition 3.3 [12]**.** Let X be a topological space. We say that X is a *selectively Pytkeev space* if for each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ such that B_n is an infinite subsets of A_n, for each n, and $(B_n : n \in \mathbb{N})$ is π-network at x.

If X has this property at all points $x \in X$ we shall say that X is a *selectively Pytkeev space*.

We introduce the following definition to transfer this property to continuous mappings.

Definition 3.4. Let $f : X \to Y$ be a continuous mapping and let $x \in X$. We say that f has the *selectively Pytkeev property at x* if for each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there is a sequence $(B_n : n \in \mathbb{N})$ such that B_n is an infinite subsets of A_n for each n and $(f(B_n) : n \in \mathbb{N})$ is π-network at $f(x)$.

If f has this property at all points $x \in X$, then we shall say that f has the *selectively Pytkeev property*.

With small modifications in the proof of Theorem 3.1 one can prove the following two theorems.

Theorem 3.3. *Let Y be a subspace of a space X. The following are equivalent*:

(1) $\pi : C_p(X) \to C_p(Y)$ *has the Pytkeev property*;

(2) *if* $(\mathcal{U}_n : n \in \mathbb{N})$ *is a sequence of* ω*-covers of X relatively* ω*-shrinkable with respect to Y, then there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *such that, for each* n, $\mathcal{V}_n$ *is an infinitely countable subset of* $\mathcal{U}_n$ *and* $\{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ *is an* ω*-cover of Y.*

Theorem 3.4. *Let Y be a subspace of a space X. The following are equivalent*:

(1) $\pi : C_k(X) \to C_k(Y)$ *has the selectively Pytkeev property*;

(2) *if* $(\mathcal{U}_n : n \in \mathbb{N})$ *is a sequence of k-covers of X relatively k-shrinkable with respect to Y, then there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *such that, for each* n, $\mathcal{V}_n$ *is an infinitely countable subset of* $\mathcal{U}_n$ *and* $\{\bigcap \mathcal{V}_n : n \in \mathbb{N}\}$ *is a k-cover of Y.*

4. The Reznichenko Property of Continuous Mappings

According to [14] we have the following notions.

Definition 4.1. For a space X and an element $x \in X$ we have:

(1) An ω-cover (a k-cover) $\mathcal{U}$ of X called *groupable* [14] if there is a partition $(\mathcal{U}_n : n \in \mathbb{N})$ of $\mathcal{U}$ into pairwise disjoint finite sets such that for each finite (compact) subset F of X, for all but finitely many n, there is a $U \in \mathcal{U}_n$ such that $F \subset U$.

(2) An element A of Ω_x is groupable [14] if there is a partition $(A_n : n \in \mathbb{N})$ of A into pairwise disjoint finite sets such that each neighborhood of x has nonempty intersection with all but finitely many elements A_n.

We use the following notation:

Ω^{gp} -the collection of all groupable ω-covers of X;

K^{gp} -the collection of all groupable k-covers of a space;

$(\Omega_x)^{gp}$ -the collection of all groupable elements of Ω_x.

A space X is said to have the ω-*grouping property* [15] if each countable ω-cover $\mathcal{U}$ of X is groupable.

We give now the following.

Definition 4.2. Let Y be a subset of a space X. Y is said to have the *relative* ω-*grouping property* in X if for each ω-cover $\mathcal{U}$ of X there is a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of pairwise disjoint finite subfamilies of $\mathcal{U}$ such that each finite subset F of Y is contained in some $U \in \mathcal{U}_n$ for all but finitely many n.

In 1996 Reznichenko introduced the following property: Each countable element of Ω_x is a member of $(\Omega_x)^{gp}$. In [14], the authors defined the *selectively Reznichenko property*: X has selectively Reznichenko property if $S_{fin}(\Omega_x, (\Omega_x)^{gp})$ holds for each $x \in X$.

In [9] the authors transfer this property to mappings and introduce the following definitions.

Definition 4.3. Let $f : X \to Y$ be a continuous mapping and let $x \in X$. We say that f has the *selectively Reznichenko property at x* if for each sequence $(A_n : n \in \mathbb{N})$ of elements of Ω_x there is a collection $\{B_n :$

$n \in \mathbb{N}\}$ such that, for each n, B_n is a finite subset of A_n and for each neighborhood V of $f(x)$, for all but finitely many n, $V \cap f(B_n) \neq \emptyset$.

If f has this property at all points $x \in X$, then we shall say that f has the *selectively Reznichenko property*.

Definition 4.4. Let $f : X \to Y$ be a continuous mapping and let $x \in X$. We say that f has the *Reznichenko property at x* if for each A of elements of Ω_x there is a collection $\{B_n : n \in \mathbb{N}\}$ of finite subset of A and for each neighborhood V of $f(x)$, for all but finitely many $n, V \cap f(B_n) \neq \emptyset$.

If f has this property at all points $x \in X$, then we shall say that f has the *Reznichenko property*.

Remark 4.1. Observe that if either X or Y has the (selectively) Reznichenko property, then f has that property.

Improving a result from [9] and following [19], we prove that

Theorem 4.1. *Let Y be a subspace of a space X. The following are equivalent*:

(1) $\pi : C_p(X) \to C_p(Y)$ *has the Reznichenko property*;

(2) *if* $\mathcal{U}$ *is an* ω*-cover of X relatively* ω*-shrinkable with respect to Y, then there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *of pairwise disjoint finite subsets of* $\mathcal{U}$ *such that, for each finite set F of Y, the set* $\{n \in \mathbb{N} : F \subset U$ *for some* $U \in \mathcal{V}_n\}$ *is cofinite in* $\mathbb{N}$.

Proof. (1) $\Rightarrow$ (2) Let $\mathcal{U}$ be an ω-cover of X relatively ω-shrinkable with respect to Y. For each $U \in \mathcal{U}$ we can take a continuous function $f_U : X \to [0, 1]$ such that $f_U^{-1}(0) = U$, $X \setminus U \subset f_U^{-1}(1)$ and $\{f_U^{-1}(0) : U \in \mathcal{U}\}$ is an ω-cover of X. Since $\{f_U^{-1}(0) : U \in \mathcal{U}\}$ is an ω-cover of X, we can assume that for distinct $U, U' \in \mathcal{U}_n$, f_U and f'_U are distinct. Let $A = \{f_U : U \in \mathcal{U}\}$. Obviously $\underline{0} \in \overline{A} \setminus A$. Since π has the Reznichenko property there is a family $\mathcal{F}$ of pairwise disjoint finite subsets of A such that, for each neighborhood W of $\pi(\underline{0})$, the family $\{\pi(A_n) : \pi(A_n) \cap W = \emptyset\}$ is finite, where $A_n = \{f_U : U \in \mathcal{U}_n\}$. It is easy to show that the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ is a desired one.

(2) $\Rightarrow$ (1) Let A be a subset of $C_p(X)$ and $\underline{0} \in \overline{A} \setminus A$. For each $f \in A$ let $U_n(f) = \left\{x \in X : |f(x)| < \frac{1}{n}\right\}$ and $\mathcal{U} = \{U_n(f) : f \in A\}$. It is easy to prove that $\mathcal{U}$ is an ω-cover of X relatively ω-shrinkable with respect to Y. Then there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of pairwise disjoint finite subsets of $\mathcal{U}$ and for each F finite subset of Y the family $\{n \in \mathbb{N} : F \subset U$ for some $U \in \mathcal{V}_n\}$ is cofinite in $\mathbb{N}$. For each n, we set $\mathcal{V}_n = \{U_n(f) : f \in S_n\}$, where S_n is a finite subset of A. Then the family $(S_n : n \in \mathbb{N})$ is disjoint and for each finite subset F of Y the set $\{n \in \mathbb{N} : W(\pi(\underline{0}), F, 1) \cap \pi(S_n) = \varnothing\}$ is cofinite in $\mathbb{N}$.

We set $S = \bigcup\{\pi(S_n) : n \in \mathbb{N}\}$, $L_0 = \bigcup\{\pi(S_{2n}) : n \in \mathbb{N}\}$ and $L_1 = \bigcup\{\pi(S_{2n+1}) : n \in \mathbb{N}\}$. Obviously $\pi(\underline{0}) \in \overline{L_0 \cup (\pi(A) \setminus \pi(S))}$ or $\pi(\underline{0}) \in \overline{L_1 \cup (\pi(A) \setminus \pi(S))}$. Let $\pi(\underline{0}) \in \overline{L_1 \cup (\pi(A) \setminus \pi(S))}$, and enumerate as $\{S_{2n} : n \in \mathbb{N}\} = \{\pi(A_{1n}) : n \in \mathbb{N}\}$. Since $\mathcal{U}'_n = \{U_n(f) : f \in L_1 \cup (\pi(A) \setminus \pi(S))\}$ is an ω-cover of X relatively ω-shrinkable with respect to Y by the same procedure as above, there exists a disjoint family $\{\pi(A_{2n}) : n \in \mathbb{N}\}$ of finite subset of $L_1 \cup (\pi(A) \setminus \pi(S))$ and for each finite subset F of Y the set $\left\{n \in \mathbb{N} : W\left(\pi(\underline{0}), F, \frac{1}{2}\right) \cap \pi(A_{2n}) = \varnothing\right\}$ is cofinite in $\mathbb{N}$. Then $\pi(\underline{0}) \in \overline{\pi(A) \setminus \bigcup\{\pi(A_{mn}) : n \in \mathbb{N}, m = 1, 2\}}$. By repeating this operation, we have a disjoint family $\{A_{mn} : m \in \mathbb{N}\}$ of finite subsets of A such that for each $m \in \mathbb{N}$ and each F finite in Y the set $\left\{n \in \mathbb{N} : W\left(\pi(\underline{0}), F, \frac{1}{m}\right) \cap \pi(A_{mn}) = \varnothing\right\}$ is cofinite in $\mathbb{N}$. Now let $A_n = \{A_{ij} : i + j = n\}$. It is not difficult to see that $\{A_n : n \in \mathbb{N}\}$ is desired one.

In a similar way one can prove the following assertions.

Theorem 4.2. *For a subset Y of a space X the following are equivalent*:

(1) $\pi : C_k(X) \to C_k(Y)$ *has the Reznichenko property*;

(2) *if* $\mathcal{U}$ *is a k-cover of X relatively k-shrinkable with respect to Y, then there is a sequence* $(\mathcal{U}_n : n \in \mathbb{N})$ *of pairwise disjoint finite subsets of* $\mathcal{U}$ *such that, for each compact K of Y, the set* $\{n \in \mathbb{N} : K \subset U$ *for some* $U \in \mathcal{U}_n\}$ *is cofinite in* $\mathbb{N}$.

Theorem 4.3. *Let Y be a subspace of a space X. The following are equivalent*:

(1) $\pi : C_p(X) \to C_p(Y)$ *has the selectively Reznichenko property*;

(2) *if* $(\mathcal{U}_n : n \in \mathbb{N})$ *is a sequence of* ω*-covers of X relatively* ω*-shrinkable with respect to Y, then there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *of pairwise disjoint sets such that, for each n,* $\mathcal{V}_n$ *is a finite subsets of* $\mathcal{U}_n$ *and for each finite set F of Y, the set* $\{n \in \mathbb{N} : F \subset U$ *for some* $U \in \mathcal{V}_n\}$ *is cofinite in* $\mathbb{N}$.

Theorem 4.4. *Let Y be a subspace of a space X. The following are equivalent*:

(1) $\pi : C_k(X) \to C_k(Y)$ *has the selectively Reznichenko property*;

(2) *if* $(\mathcal{U}_n : n \in \mathbb{N})$ *is a sequence of k-covers of X relative k-shrinkable with respect to Y, then there is a sequence* $(\mathcal{V}_n : n \in \mathbb{N})$ *of pairwise disjoint such that, for each n,* $\mathcal{V}_n$ *is a finite subset of* $\mathcal{U}_n$ *such that, for each compact set K of Y, the set* $\{n \in \mathbb{N} : K \subset U$ *for some* $U \in \mathcal{V}_n\}$ *is cofinite in* $\mathbb{N}$.

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