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# THE ASYMPTOTIC PROPERTIES OF SOME DISCRETE DISTRIBUTIONS GENERATED BY LEVY'S LAW

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## Abstract

There are some frequency distributions in large-scale biomolecular systems having properties like stable densities. It is of interest to construct such frequency distributions. This paper is devoted to a special case of stable densities. The large sample distributions of the maximum-likelihood estimates (M.L.E.) of the scale parameter for some discrete distributions generated by Levy's law are studied. It is shown the existence, strong consistency and asymptotic normality of that.

## 1. Introduction

A basic topic of any statistical inference of biomolecular system is characterization of the distributions of object frequencies for a population so-called frequency distributions.

Based on huge data sets of such systems several common statistical facts on frequency distribution have been discovered. From the mathematical point of view these are: skewness to the right; regular

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variation at infinity; continuity by parameters; unimodality, etc. of distributions (see [1]).

In large-scale biomolecular systems some distributions are widely used. But the variety of such systems requires to generate new ones that satisfy the empirical facts above.

Taking into account statistical facts, new frequency distributions so-called stable laws are recommended for the needs of large-scale biomolecular systems. There are four equivalent definitions for stable laws. One of them is as follows:

**Definition 1.** Non-degenerate random variable $X$ is *stable* iff all $n > 1$, there is constant $d_n \in R$ such that $X_1 + X_2 + \cdots + X_n \overset{d}{=} n^{1/\alpha} X + d_n$, where $X_1, X_2, \ldots, X_n$ are independent identically copies of $X$ and $\alpha \in (0, 2]$. (The symbol $\overset{d}{=}$ means equality by distribution.)

The stable laws are determined by four parameters: an index of stability $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\gamma > 0$, and a location parameter $\delta \in (-\infty, \infty)$. This is the reason to denote stable distributions by $S(\alpha, \beta, \gamma, \delta)$ and to write

$$X \sim S(\alpha, \beta, \gamma, \delta).$$

If $\alpha = \frac{1}{2}$ and $\beta = 1$, that is, $X \sim S\left(\frac{1}{2}, 1, \gamma, \delta\right)$, then $X$ is a *Levy* stable random variable which has the following density (see, for instance, [7, 8]):

$$s(x; \gamma, \delta) = \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x - \delta)^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2(x - \delta)}\right), \quad \delta < x < +\infty. \tag{1.1}$$

Suppose that $\Theta = \{(\gamma, \delta) : 0 < \gamma < \infty, \delta = 0\}$ and $D$ is an arbitrary open subset of $\Theta$, whose closure, $\overline{D}$, is also contained in $\Theta$.

Let us create the following discrete distribution:

$$p(x; \gamma) = c_\gamma^{-1} s(x; \gamma), \quad x = 1, 2, \ldots \quad \text{with} \quad c_\gamma = \sum_{y=1}^{\infty} s(y; \gamma), \tag{1.2}$$

where $s(x; \gamma)$ is the *Levy* density.

## 2. The Asymptotic Properties of the M.L.E.

Let $\hat{\gamma}_n$ be the M.L.E. of $\gamma$ based on the sample $x^n = (x_1, x_2, ..., x_n)$. Our goal is to prove the strong consistency and asymptotic normality of the M.L.E. of the scale parameter $\gamma$ for some discrete distributions generated by *Levy's* law. To do this, taking into account lemma ([3, p. 952]) and Subsections 16.2, 33.3 in [2], it suffices to show that the family of (1.2) is satisfied in regularity conditions (1)-(6) of the following theorem which adjusted somewhat to this case.

**Theorem.** *Let $\gamma_0 \in \overline{D}$ be the true value of $\gamma$, and suppose that the following regularity conditions are met*:

(1) *For each $x = 1, 2, ...,$ $p(x; \gamma)$ is a continuous function of $\gamma$ for $\gamma \in \Theta,$ and has continuous derivatives of first and second orders with respect to $\gamma$ for $\gamma \in \overline{D}$.*

(2) *Let $x^n = (x_1, x_2, ..., x_n),$ $n$ is fixed integer number, be sample from $p(x; \gamma),$ and let $L(x^n; \gamma) = \prod_{i=1}^{n} p(x_i; \gamma).$ Then for all $\gamma_0 \in \overline{D}$,*

$$E_{\gamma_0}\left[\sup_{\gamma \in \Theta - D} \ln \frac{L(x^n; \gamma)}{L(x^n; \gamma_0)}\right] < \infty.$$

(3) *Suppose that* $A(x; \gamma) = \dfrac{\partial \ln p(x; \gamma)}{\partial \gamma}$ *and* $B(x; \gamma) = \dfrac{\partial^2 \ln p(x; \gamma)}{\partial \gamma^2}$.

*Then there exists a function $C(x)$, such that for all $\gamma \in \overline{D}$,*

$$\sum_{x=1}^{\infty} C(x) p(x; \gamma) \leq M < \infty, \tag{2.1}$$

*and $B(x; \gamma)$ is bounded in absolute value by $C(x)$.*

(4) *For every $\gamma \in \overline{D}$, $E_\gamma(A(x; \gamma)) = 0$ and $E_\gamma(A(x; \gamma))^2 = -E_\gamma(B(x; \gamma))$ $= I(\gamma)$, $x = 1, 2, ...,$ where $I(\gamma)$ is the Fisher's information at the point $\gamma$.*

(5) *For every $\gamma \in \Theta$, and for all $\gamma_0 \in \overline{D}$, $\gamma \neq \gamma_0$, the condition*

$$\sum_{x=1}^{\infty} | p(x; \gamma) - p(x; \gamma_0) | > 0$$

*is met.*

(6) *For each* $\gamma \in \overline{D}$, *the Fisher's information function satisfies* $0 < I(\gamma) < \infty$.

*Under these conditions, we have*

(i) *The M.L.E.* $\hat{\gamma}_n$ *based on the first n observations, is strongly consistent, asymptotically normal, and* $\sqrt{n}(\hat{\gamma}_n - \gamma_0) \xrightarrow{d} N(0, I^{-1}(\gamma_0))$, *where* $I(\gamma_0)$ *is Fisher's information function.*

(ii) *For the sample* $x^n = (x_1, x_2, ..., x_n)$, $\hat{\gamma}_n$, *with probability* 1, *is the unique solution of the likelihood equation* $\frac{\partial L(x^n; \gamma)}{\partial \gamma} = 0$ *and also satisfying to condition* $|\hat{\gamma}_n - \gamma_0| < \tau$, *where* $\tau$ *is a positive number independent of* $\gamma_0$.

### 3. The Proofs of Regularity Conditions (1)-(6) of Theorem

**Note.** The proofs of (1) and (5) are obvious and need no explanation.

**Proof.** (2) If $\gamma$ is restricted away from zero and infinity, then $L(x^n; \gamma)$ is uniformly bounded for all $x$ and $\gamma$. Thus, in order to satisfy Condition (2), it remains to investigate it for the cases $\gamma \to 0$ and $\gamma \to \infty$.

Taking into account (1.2), we obtain

$$L(x^n; \gamma) = \prod_{i=1}^{n} \frac{s(x_i; \gamma)}{c_\gamma} = \prod_{i=1}^{n} \frac{\frac{1}{(x_i)^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2x_i}\right)}{\left(\sum_{y=1}^{\infty} \frac{1}{y^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2y}\right)\right)}. \tag{3.1}$$

Now, if $\gamma \to 0$, then from (3.1) we have

$$L(x^n; \gamma) \to \frac{1}{\left(\sum_{y=1}^{\infty} \frac{1}{y^{\frac{3}{2}}}\right)^n} \prod_{i=1}^{n} \frac{1}{x_i^{\frac{3}{2}}},$$

which is always finite.

Similarly, it can be shown that, as $\gamma \to \infty$, $L(x^n; \gamma)$ is bounded, which completes the proof of (2).

(3) If $x$ is fixed and $\gamma \in \overline{D}$, then $|B(x; \gamma)|$ is restricted by a function $C(x)$, which is itself bounded in each $x$ fixed point. It suffices to consider the treatment of $C(x)$ as $x \to \infty$.

Let us denote

$$p_\gamma(x; \gamma) = \frac{\partial}{\partial\gamma}\left(\frac{s(x; \gamma)}{c_\gamma}\right), \qquad p_{\gamma\gamma}(x; \gamma) = \frac{\partial^2}{\partial\gamma^2}\left(\frac{s(x; \gamma)}{c_\gamma}\right). \tag{3.2}$$

In addition, we obtain

$$s_\gamma(x; \gamma) = \frac{\partial s(x; \gamma)}{\partial\gamma} = \frac{1}{2x^{\frac{3}{2}}}\exp\left(-\frac{\gamma}{2x}\right)\left[-\frac{1}{x}\sqrt{\frac{\gamma}{2\pi}} + \frac{1}{\sqrt{2\pi\gamma}}\right] \tag{3.3}$$

and

$$s_{\gamma\gamma}(x; \gamma) = \frac{\partial^2 s(x; \gamma)}{\partial\gamma^2}$$

$$= \frac{1}{4x^{\frac{3}{2}}}\exp\left(-\frac{\gamma}{2x}\right)\left[-\frac{1}{\sqrt{2\pi}\gamma^{\frac{3}{2}}} - \frac{1}{x}\frac{1}{\sqrt{2\pi\gamma}} + \frac{1}{x^2}\sqrt{\frac{\gamma}{2\pi}} - \frac{1}{x}\frac{1}{\sqrt{2\pi\gamma}}\right]. \tag{3.4}$$

Now, using (3.2)-(3.4) and doing some calculations we get

$$C(x) = O(f(x)), \quad \text{as} \quad x \to \infty,$$

where $f(x) = \frac{1}{x} + k$, $k$ is some positive constant.

In accord with the value of $f(x)$ it is easily seen that (2.1) is met.

Condition (3) is proved.

(4) Taking into account that $\sum_{x=1}^{\infty} p_\gamma(x; \gamma) = 0$, the proof of Condition (4) is evident.

(6) According to the definition $I(\gamma)$, we have

$$I(\gamma) = \sum_{x=1}^{\infty} \frac{(p_\gamma(x;\ \gamma))^2}{p(x;\ \gamma)}. \tag{3.5}$$

To establish $I(\gamma) > 0$, it is sufficient to prove that there exists, at least, an $x$ such that $p_\gamma(x;\ \gamma) \neq 0$, or, equivalently,

$$\exists x \quad \text{s.t.} \quad \left[ s_\gamma(x;\ \gamma) c_\gamma - s(x;\ \gamma) \frac{\partial c_\gamma}{\partial \gamma} \right] \neq 0,$$

or

$$\exists x \quad \text{s.t.} \quad s_\gamma(x;\ \gamma) \sum_{y=1}^{\infty} s(y;\ \gamma) \neq s(x;\ \gamma) \sum_{y=1}^{\infty} s_\gamma(y;\ \gamma). \tag{3.6}$$

From (3.6) we have

$$\frac{1}{\sqrt{2\pi\gamma}} \sum_{y=1}^{\infty} \frac{1}{y^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2y}\right) - \frac{1}{x} \sqrt{\frac{\gamma}{2\pi}} \sum_{y=1}^{\infty} \frac{1}{y^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2y}\right)$$

$$\neq \frac{1}{\sqrt{2\pi\gamma}} \sum_{y=1}^{\infty} \frac{1}{y^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2y}\right) - \sqrt{\frac{\gamma}{2\pi}} \sum_{y=1}^{\infty} \frac{1}{y^{\frac{5}{2}}} \exp\left(-\frac{\gamma}{2y}\right) \tag{3.7}$$

by putting $x = 1$ in (3.7), the proof is finished.

In order to satisfy $I(\gamma) < \infty$, from (3.5) we obtain

$$I(\gamma) = \frac{1}{c_\gamma} \sum_{x=1}^{\infty} \frac{(s_\gamma(x;\ \gamma))^2}{s(x;\ \gamma)} - 2 \frac{\frac{\partial c_\gamma}{\partial \gamma}}{(c_\gamma)^2} \sum_{x=1}^{\infty} s_\gamma(x;\ \gamma) + \frac{\left(\frac{\partial c_\gamma}{\partial \gamma}\right)^2}{(c_\gamma)^3} \sum_{x=1}^{\infty} s(x;\ \gamma) \tag{3.8}$$

which implies

$$I(\gamma) = \sqrt{\frac{2\pi}{\gamma}} \frac{1}{c_\gamma} \sum_{x=1}^{\infty} \frac{1}{4x^{\frac{3}{2}}} \exp\left(-\frac{\gamma}{2x}\right) \left[ -\frac{1}{x} \sqrt{\frac{\gamma}{2\pi}} + \frac{1}{\sqrt{2\pi\gamma}} \right]^2$$

$$-2\frac{\frac{\partial c_\gamma}{\partial\gamma}}{(c_\gamma)^2}\sum_{x=1}^{\infty}\frac{1}{2x^{\frac{3}{2}}}\exp\left(-\frac{\gamma}{2x}\right)\left[-\frac{1}{x}\sqrt{\frac{\gamma}{2\pi}}+\frac{1}{\sqrt{2\pi\gamma}}\right]$$

$$+\sqrt{\frac{\gamma}{2\pi}}\frac{\left(\frac{\partial c_\gamma}{\partial\gamma}\right)^2}{(c_\gamma)^3}\sum_{x=1}^{\infty}\frac{1}{x^{\frac{3}{2}}}\exp\left(-\frac{\gamma}{2x}\right). \tag{3.9}$$

We need to verify all of series in (3.9) are uniformly convergent. The proof of this statement, after doing some simplifications, is not difficult.

Condition (6) is verified.

Now, based on Theorem of Section 2 and the proofs of Conditions (1)-(6) in Section 3, the following conclusion is proved.

## Conclusion

When sampling from $p(x;\,\gamma)$, $\gamma \in \Theta$, the M.L.E. $\hat{\gamma}_n$ for $\gamma$ based on the first $n$ observations, is strongly consistent and asymptotically normal if $\gamma_0$ (the true value of $\gamma$) is in the interior of parameter space $\Theta$.

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## References

[1] J. Astola and E. Danielian, Frequency Distributions in Biomolecular Systems and Growing Network, TICSP Series No. 31, Tampere, Finland, 2006.

[2] A. A. Borovkov, Mathematical Statistics, Gordon & Breach Science Publishers, 1998 (translated from Russian into English).

[3] W. DuMouchel, On the asymptotic normality of the maximum likelihood estimate when sampling from a stable distribution, Ann. Statist. 1 (1973), 948-957.

[4] D. Farbod, The asymptotic properties of some discrete distributions generated by standard symmetric stable laws, J. Inform. Tech. Manag., Yerevan 10 (2007), 49-55.

[5] W. Feller, An Introduction to Probability Theory and its Applications, Vol. 2, John Wiley & Sons, New York, 1971.

[6] L. LeCam, On some asymptotic properties of maximum likelihood estimates and related Bayes estimates, Ph.D. Dissertation, University of California, Berkeley, 1952.

[7] J. P. Nolan, Stable Distributions - Models for Heavy Tailed Data, Birkhäuser, Boston, 2007.

[8] V. M. Zolotarev, One-dimensional Stable Distributions, American Mathematical Society, Vol. 65 of Translations of Mathematical Monographs, 1986 (translated from the original 1983 Russian edition).

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