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BIVARIATE GENERALIZATION OF THE HYPERGEOMETRIC FUNCTION TYPE I DISTRIBUTION

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Abstract

The bivariate generalization of the hypergeometric function type I distribution is defined by the probability density function proportional to $x_1^{\nu_1-1}x_2^{\nu_2-1}(1-x_1-x_2)^{\gamma-1}{}_2F_1(\alpha, \beta; \gamma; 1-x_1-x_2)$, $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 < 1$, where ν_1, ν_2, α, β and γ are suitably chosen constants and ${}_2F_1$ is the Gauss hypergeometric function. In this article, we study several properties of this distribution and derive density functions of X_1/X_2, $X_1/(X_1+X_2)$, X_1+X_2 and X_1X_2.

1. Introduction

The random variable X is said to have a *hypergeometric function type I distribution*, denoted as $X \sim H^I(\nu, \alpha, \beta, \gamma)$, if its p.d.f. is given by (Gupta and Nagar [1], Nagar and Alvarez [4])

$$\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)} x^{\nu-1}(1-x)^{\gamma-1}{}_2F_1(\alpha, \beta; \gamma; 1-x), \tag{1}$$

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where $0 < x < 1$, $\nu > 0$, $\gamma > 0$, $\gamma + \nu > \alpha + \beta$ and ${}_2F_1$ is the Gauss hypergeometric function (Luke [2]). The hypergeometric function type I distribution occurs as the distribution of the product of two independent beta variables (Gupta and Nagar [1], Pham-Gia and Turkkan [6], Nagar and Alvarez [4]). For $\alpha = \gamma$, the density (1) reduces to a beta type I density given by

$$\frac{\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu - \beta)} x^{\nu-\beta-1}(1 - x)^{\gamma-1}, \quad 0 < x < 1,$$

and for $\beta = \gamma$, the hypergeometric function type I density slides to

$$\frac{\Gamma(\gamma + \nu - \alpha)}{\Gamma(\gamma)\Gamma(\nu - \alpha)} x^{\nu-\alpha-1}(1 - x)^{\gamma-1}, \quad 0 < x < 1.$$

Further, for $\alpha = 0$ or $\beta = 0$ the hypergeometric function type I density simplifies to a beta type I density with parameters ν and γ.

Recently, Nagar and Alvarez [4, 5] have studied several properties and stochastic representations of the hypergeometric function type I distribution. They have also derived the density function of the product of two independent random variables having hypergeometric function type I distribution.

The bivariate generalization of the hypergeometric function type I distribution, denoted by $(X_1, X_2) \sim H^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, is defined by the density

$$C(\nu_1, \nu_2; \alpha, \beta, \gamma) x_1^{\nu_1-1} x_2^{\nu_2-1}(1 - x_1 - x_2)^{\gamma-1} {}_2F_1(\alpha, \beta; \gamma; 1 - x_1 - x_2), \quad (2)$$

where $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 < 1$ and $C(\nu_1, \nu_2; \alpha, \beta, \gamma)$ is the normalizing constant. From (2), the constant $C(\nu_1, \nu_2; \alpha, \beta, \gamma)$ is derived as

$$\begin{aligned}\{C(\nu_1, \nu_2; \alpha, \beta, \gamma)\}^{-1} &= \int_0^1 \int_0^{1-x_1} x_1^{\nu_1-1} x_2^{\nu_2-1}(1 - x_1 - x_2)^{\gamma-1} \\ &\quad {}_2F_1(\alpha, \beta; \gamma; 1 - x_1 - x_2) dx_2 dx_1 \\ &= \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1 + \nu_2)} \int_0^1 x^{\nu_1+\nu_2-1}(1 - x)^{\gamma-1} \\ &\quad {}_2F_1(\alpha, \beta; \gamma; 1 - x) dx, \qquad (3)\end{aligned}$$

where the last line has been obtained by substituting $y = x_1/x$ and $x = x_1 + x_2$ with the Jacobian $J(x_1, x_2 \to y, x) = x$ and integrating y. Now, evaluating the above integral and simplifying the resulting expression using (A.3) and (A.5), we obtain

$$C(\nu_1, \nu_2; \alpha, \beta, \gamma) = \frac{\Gamma(\nu_1 + \nu_2 + \gamma - \alpha)\Gamma(\nu_1 + \nu_2 + \gamma - \beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu_1 + \nu_2 + \gamma - \alpha - \beta)},$$

where $\nu_1 > 0$, $\nu_2 > 0$, $\gamma > 0$ and $\nu_1 + \nu_2 + \gamma > \alpha + \beta$.

For $\alpha = 0$ or $\beta = 0$, the density (2) slides to a Dirichlet type I density with parameters ν_1, ν_2 and γ. In Bayesian analysis the Dirichlet distribution is used as a conjugate prior distribution for the parameters of a multinomial distribution. However, the Dirichlet family is not sufficiently rich in scope to represent many important distributional assumptions, because the Dirichlet distribution has few number of parameters. The bivariate distribution given by the density (2) is a generalization of the Dirichlet distribution with added number of parameters.

It can also be observed that bivariate generalization of the hypergeometric function type I distribution defined by the density (2) belongs to the Liouville family of distributions proposed by Marshall and Olkin [3] and Sivazlian [7].

In this article, in Section 2 and Section 3, we show that if $(X_1, X_2) \sim H^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, then $X_1 \sim H^I(\nu_1, \alpha, \beta, \nu_2 + \gamma)$ and $X_2 \sim H^I(\nu_2, \alpha, \beta, \nu_1 + \gamma)$. Further, $X_1 + X_2 \sim H^I(\nu_1 + \nu_2, \alpha, \beta, \gamma)$, which is independent of $X_1/(X_1 + X_2) \sim B^I(\nu_1, \nu_2)$ and $X_1/X_2 \sim B^{II}(\nu_1, \nu_2)$. We also derive the density of the product $Y = X_1 X_2$. Finally, in the Appendix, we give some well known results and definitions that are used in this article.

2. Properties

In this section we study several properties of the bivariate distribution defined in Section 1. We first derive marginal and conditional distributions.

Theorem 2.1. *If* $(X_1, X_2) \sim H^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, *then* $X_1 \sim H^I(\nu_1, \alpha\ \beta, \nu_2 + \gamma)$ *and* $X_2 \sim H^I(\nu_2, \alpha, \beta, \nu_1 + \gamma)$.

Proof. By integrating x_2 in (2), we get the marginal p.d.f. of X_1 as

$$C(\nu_1, \nu_2; \alpha, \beta, \gamma) x_1^{\nu_1 - 1} \int_0^{1-x_1} x_2^{\nu_2 - 1}(1 - x_1 - x_2)^{\gamma - 1}$$

$$\quad {}_2F_1(\alpha, \beta; \gamma; 1 - x_1 - x_2) \mathrm{d}x_2$$

$$= C(\nu_1, \nu_2; \alpha, \beta, \gamma) x_1^{\nu_1 - 1}(1 - x_1)^{\nu_2 + \gamma - 1}$$

$$\times \int_0^1 z^{\nu_2 - 1}(1 - z)^{\gamma - 1} {}_2F_1(\alpha, \beta; \gamma; (1 - x_1)(1 - z)) \mathrm{d}z,$$

where we have used the substitution $z = x_2/(1 - x_1)$. Now, the desired result is obtained by using (A.3).

Using the above theorem, the conditional density function of X_1 given $X_2 = x_2 > 0$ is obtained as

$$\frac{\Gamma(\gamma + \nu_1)}{\Gamma(\nu_1)\Gamma(\gamma)} \frac{x_1^{\nu_1 - 1}(1 - x_1 - x_2)^{\gamma - 1}}{(1 - x_2)^{\gamma + \nu_1 - 1}} \frac{{}_2F_1(\alpha, \beta; \gamma; 1 - x_1 - x_2)}{{}_2F_1(\alpha, \beta; \nu_1 + \gamma; 1 - x_2)},$$

where $0 < x_1 < 1 - x_2$. Further, using (2) and (3), the joint (r, s)-th moment is obtained as

$$E(X_1^r X_2^s) = C(\nu_1, \nu_2; \alpha, \beta, \gamma) \int_0^1 \int_0^{1-x_1} x_1^{\nu_1 + r - 1} x_2^{\nu_2 + s - 1}$$

$$\times (1 - x_1 - x_2)^{\gamma - 1} {}_2F_1(\alpha, \beta; \gamma; 1 - x_1 - x_2) \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \frac{C(\nu_1, \nu_2; \alpha, \beta, \gamma)}{C(\nu_1 + r, \nu_2 + s; \alpha, \beta, \gamma)}$$

$$= \frac{\Gamma(\nu_1 + \nu_2 + \gamma - \alpha)\Gamma(\nu_1 + \nu_2 + \gamma - \beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_1 + \nu_2 + \gamma - \alpha - \beta)}$$

$$\times \frac{\Gamma(\nu_1 + r)\Gamma(\nu_2 + s)\Gamma(\nu_1 + \nu_2 + \gamma + r + s - \alpha - \beta)}{\Gamma(\nu_1 + \nu_2 + \gamma + r + s - \alpha)\Gamma(\nu_1 + \nu_2 + \gamma + r + s - \beta)},$$

where $\nu_1 > 0$, $\nu_2 > 0$, $\gamma > 0$, $\nu_1 + \nu_2 + \gamma + r + s > \alpha + \beta$. Now, substituting appropriately, we obtain

$$\mathrm{E}(X_i) = \frac{\nu_i(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)(\nu + \gamma - \beta)},$$

$$\mathrm{E}(X_i^2) = \frac{\nu_i(\nu_i + 1)(\nu + \gamma - \alpha - \beta)(\nu + \gamma - \alpha - \beta + 1)}{(\nu + \gamma - \alpha)(\nu + \gamma - \alpha + 1)(\nu + \gamma - \beta)(\nu + \gamma - \beta + 1)},$$

$$\mathrm{E}(X_1 X_2) = \frac{\nu_1 \nu_2(\nu + \gamma - \alpha - \beta)(\nu + \gamma - \alpha - \beta + 1)}{(\nu + \gamma - \alpha)(\nu + \gamma - \alpha + 1)(\nu + \gamma - \beta)(\nu + \gamma - \beta + 1)},$$

$$\mathrm{Var}(X_i) = \frac{\nu_i(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)(\nu + \gamma - \beta)} \times \left[\frac{(\nu_i + 1)(\nu + \gamma - \alpha - \beta + 1)}{(\nu + \gamma - \alpha + 1)(\nu + \gamma - \beta + 1)} - \frac{\nu_i(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)(\nu + \gamma - \beta)}\right],$$

and

$$\mathrm{Cov}(X_1, X_2) = \frac{\nu_1 \nu_2(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)^2(\nu + \gamma - \beta)^2} \times \frac{[-(\nu + \gamma - \alpha - \beta)(\nu + \gamma - \alpha - \beta + 1) + \alpha\beta]}{(\nu + \gamma - \alpha + 1)(\nu + \gamma - \beta + 1)},$$

where $\nu_1 + \nu_2 = \nu$.

In the next theorem we derive the bivariate generalization of the hypergeometric function type I distribution using independent beta and Dirichlet variables.

Theorem 2.2. *Let* $(U_1, U_2) \sim D^I(c_1, c_2; d)$ *and* $Z \sim B^I(a, b)$ *be independent. Then* $(ZU_1, ZU_2) \sim H^I(c_1, c_2; b, c_1 + c_2 + d - a, b + d)$.

Proof. The joint density of Z and (U_1, U_2) is given by

$$K u_1^{c_1 - 1} u_2^{c_2 - 1}(1 - u_1 - u_2)^{d-1} z^{a-1}(1 - z)^{b-1}, \tag{4}$$

where $0 < z < 1$, $u_1 > 0$, $u_2 > 0$, $u_1 + u_2 < 1$ and

$$K = \frac{\Gamma(c_1 + c_2 + d)\Gamma(a + b)}{\Gamma(c_1)\Gamma(c_2)\Gamma(d)\Gamma(a)\Gamma(b)}.$$

Transforming $X_i = ZU_i$, $i = 1, 2$ with the Jacobian $J(u_1, u_2, z \to x_1, x_2, z) = z^{-2}$ in (4) and integrating out z, we get the marginal density of (X_1, X_2) as

$$Kx_1^{c_1-1}x_2^{c_2-1}\int_{x_1+x_2}^{1} \frac{(z - x_1 - x_2)^{d-1}(1 - z)^{b-1}\mathrm{d}z}{z^{c_1+c_2+d-a}}, \tag{5}$$

where $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 < 1$. Now, substituting $v = (1 - z)/(1 - x_1 - x_2)$ with the Jacobian $J(z \to v) = 1 - x_1 - x_2$ in (5), we obtain

$$Kx_1^{c_1-1}x_2^{c_2-1}(1 - x_1 - x_2)^{b+d-1}\int_0^1 \frac{v^{b-1}(1 - v)^{d-1}\mathrm{d}v}{[1 - (1 - x_1 - x_2)v]^{c_1+c_2+d-a}}.$$

Finally, evaluation of the above integral using (A.4) yields the desired result.

Corollary 2.2.1. *Let* $Z \sim B^I(a, b)$ *and* $(U_1, U_2) \sim D^I(c_1, c_2; d)$ *be independent. Then* $((1 - Z)U_1, (1 - Z)U_2) \sim H^I(c_1, c_2; a, c_1 + c_2 + d - b, a + d)$. *Further,* $(ZU_1, ZU_2) \sim D^I(c_1, c_2; b + d)$ *if* $a = c_1 + c_2 + d$ *and* $((1 - Z)U_1, (1 - Z)U_2) \sim D^I(c_1, c_2; a + d)$ *if* $b = c_1 + c_2 + d$.

Corollary 2.2.2. *Let* $V \sim B^{II}(a, b)$ *and* $(U_1, U_2) \sim D^I(c_1, c_2; d)$ *be independent. Then* $(VU_1/(1 + V), VU_2/(1 + V)) \sim H^I(c_1, c_2; b, c_1 + c_2 + d - a, b + d)$ *and* $(U_1/(1 + V), U_2/(1 + V)) \sim H^I(c_1, c_2; a, c_1 + c_2 + d - b, a + d)$. *Further,* $(VU_1/(1 + V), VU_2/(1 + V)) \sim D^I(c_1, c_2; b + d)$ *if* $a = c_1 + c_2 + d$ *and* $(U_1/(1 + V), U_2/(1 + V)) \sim D^I(c_1, c_2; a + d)$ *if* $b = c_1 + c_2 + d$.

3. Distributions of Sum and Quotients

It is well known that if $(X_1, X_2) \sim D^I(\nu_1, \nu_2; \nu_3)$, then X_1/X_2 and $X_1/(X_1 + X_2)$ are independent of $X_1 + X_2$. Further, $X_1/X_2 \sim B^{II}(\nu_1, \nu_2)$, $X_1/(X_1 + X_2) \sim B^I(\nu_1, \nu_2)$ and $X_1 + X_2 \sim B^I(\nu_1 + \nu_2, \nu_3)$. In this section we derive similar results when X_1 and X_2 have the bivariate hypergeometric function type I distribution.

Theorem 3.1. *Let* $(X_1, X_2) \sim H^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$. *Then* $Z = X_1/(X_1 + X_2)$ *and* $S = X_1 + X_2$ *are independent,* $Z \sim B^I(\nu_1, \nu_2)$ *and* $S \sim H^I(\nu_1 + \nu_2, \alpha, \beta, \gamma)$.

Proof. Transforming $Z = X_1/(X_1 + X_2)$ and $S = X_1 + X_2$ with the Jacobian $J(x_1, x_2 \to z, s) = s$ in (2), we obtain the joint p.d.f. of Z and S as

$$C(\nu_1, \nu_2; \alpha, \beta, \gamma) z^{\nu_1 - 1}(1 - z)^{\nu_2 - 1} s^{\nu_1 + \nu_2 - 1}(1 - s)^{\gamma - 1} {}_2F_1(\alpha, \beta; \gamma; 1 - s),$$

where $0 < z < 1$ and $0 < s < 1$. Now, from the above factorization it is clear that Z and S are independent, $Z \sim B^I(\nu_1, \nu_2)$ and $S \sim H^I(\nu_1 + \nu_2, \alpha, \beta, \gamma)$.

Corollary 3.1.1. *Let* $(X_1, X_2) \sim H^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$. *Then* X_1/X_2 *and* $X_1 + X_2$ *are independent. Further,* $X_1/X_2 \sim B^{II}(\nu_1, \nu_2)$.

Theorem 3.2. *If* $(X_1, X_2) \sim H^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, *then the p.d.f. of* $Y = X_1 X_2$ *is given by*

$$\frac{\sqrt{\pi} C(\nu_1, \nu_2; \alpha, \beta, \gamma)\Gamma(\gamma)}{2^{\nu_1 - \nu_2 + \gamma - 1}\Gamma(\gamma + 1/2)} \frac{y^{\nu_2 - 1}(1 - 4y)^{\gamma - 1/2}}{(1 + \sqrt{1 - 4y})^{\nu_2 + \gamma - \nu_1}}$$

$$\times \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma + 1/2)_k 2^k k!} \left(\frac{1 - 4y}{1 + \sqrt{1 - 4y}} \right)^k$$

$$\times {}_2F_1\left(\gamma + k, \nu_2 - \nu_1 + \gamma + k; 2\gamma + 2k; \frac{2\sqrt{1 - 4y}}{1 + \sqrt{1 - 4y}} \right), \quad 0 < y < \frac{1}{4}.$$

Proof. Making the transformation $Y = X_1 X_2$ with the Jacobian $J(x_1, x_2 \to x_1, y) = x_1^{-1}$ in (2), we obtain the joint density of X_1 and Y as

$$C(\nu_1, \nu_2; \alpha, \beta, \gamma) \frac{y^{\nu_2 - 1}(-x_1^2 + x_1 - y)^{\gamma - 1}}{x_1^{\nu_2 + \gamma - \nu_1}} {}_2F_1\left(\alpha, \beta; \gamma; \frac{-x_1^2 + x_1 - y}{x_1} \right), \quad (6)$$

where $p < x_1 < q$ with

$$p = \frac{1 - \sqrt{1 - 4y}}{2}, \qquad q = \frac{1 + \sqrt{1 - 4y}}{2}$$

and $0 < y < 1/4$. Now, integrating x_1 in (6), we obtain the marginal density of Y as

$$C(\nu_1, \nu_2; \alpha, \beta, \gamma) y^{\nu_2 - 1} \int_p^q \frac{[(x_1 - p)(q - x_1)]^{\gamma - 1}}{x_1^{\nu_2 + \gamma - \nu_1}}$$

$$\times {}_2F_1\left(\alpha, \beta; \gamma; \frac{(x_1 - p)(q - x_1)}{x_1}\right) \mathrm{d}x_1$$

$$= C(\nu_1, \nu_2; \alpha, \beta, \gamma) \frac{y^{\nu_2 - 1}(q - p)^{2\gamma - 1}}{q^{\nu_2 + \gamma - \nu_1}} \int_0^1 \frac{t^{\gamma - 1}(1 - t)^{\gamma - 1}}{[1 - t(1 - p/q)]^{\nu_2 + \gamma - \nu_1}}$$

$$\times {}_2F_1\left(\alpha, \beta; \gamma; \frac{t(1 - t)(q - p)^2}{q[1 - t(1 - p/q)]}\right) \mathrm{d}t,$$

where we have used the substitution $t = (q - x_1)/(q - p)$. Now, expanding ${}_2F_1$ in series form, integrating t using (A.4) and simplifying the resulting expression, we get the desired result.

Appendix

The Pochammer symbol $(a)_n$ is defined by $(a)_n = a(a + 1) \cdots (a + n - 1) = (a)_{n-1}(a + n - 1)$ for $n = 1, 2, \ldots,$ and $(a)_0 = 1$. The generalized hypergeometric function of scalar argument is defined by

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{A.1}$$

where a_i, $i = 1, \ldots, p$; b_j, $j = 1, \ldots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (A.1) are available in the literature, see Luke [2]. From (A.1) it is easy to see that

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \tag{A.2} $$

Also, under suitable conditions, we have (Luke [2, Eq. 3.6(10)])

$$ \int_0^1 z^{\alpha-1}(1-z)^{\beta-1} {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; zy) \mathrm{d}z $$

$$ = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} {}_{p+1}F_{q+1}(a_1, \ldots, a_p, \alpha; b_1, \ldots, b_q, \alpha+\beta; y). \tag{A.3} $$

The integral representation of the Gauss hypergeometric function is given as

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} \mathrm{d}t, $$

$$ \operatorname{Re}(c) > \operatorname{Re}(a) > 0, \ |\arg(1-z)| < \pi, \tag{A.4} $$

respectively. Note that, the series expansion for ${}_2F_1$ given in (A.2) can be obtained by expanding $(1-zt)^{-b}$, $|zt| < 1$, in (A.4) and integrating t. Substituting $z = 1$ in (A.4) and integrating, we obtain

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \operatorname{Re}(c-a-b) > 0, \tag{A.5} $$

where $c \neq 0, -1, -2, \ldots$. For properties and further results on these functions the reader is referred to Luke [2].

Finally, we define the beta type I, beta type II and Dirichlet distributions.

Definition A.1. The random variable X is said to have a *beta type I distribution* with parameters (a, b), $a > 0$, $b > 0$, denoted as $X \sim B^I(a, b)$, if its p.d.f. is given by

$$ \{B(a, b)\}^{-1} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1, $$

where $B(a, b)$ is the beta function given by

$$ B(a, b) = \Gamma(a)\Gamma(b)\{\Gamma(a+b)\}^{-1}. $$

Definition A.2. The random variable X is said to have a *beta type II distribution* with parameters (a, b), denoted as $X \sim B^{II}(a, b)$, $a > 0$, $b > 0$, if its p.d.f. is given by

$$\{B(a, b)\}^{-1} x^{a-1} (1 + x)^{-(a+b)}, \quad x > 0.$$

The bivariate generalization of the beta type I density is defined by

$$\frac{\Gamma(\nu_1 + \nu_2 + \nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)} x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} (1 - x_1 - x_2)^{\nu_3 - 1},$$

$$x_1 > 0,\ x_2 > 0,\ x_1 + x_2 < 1, \tag{A.6}$$

where $\nu_i > 0$, $i = 1, 2, 3$. This distribution has been considered by several authors and is well known in the scientific literature as the Dirichlet (type I) distribution. We will write $(X_1, X_2) \sim D^I(\nu_1, \nu_2; \nu_3)$ if the joint density of X_1 and X_2 is given by (A.6).

The matrix variate generalizations of the beta type I, beta type II and Dirichlet distributions have been defined and studied extensively. For example, see Gupta and Nagar [1].

References

[1] A. K. Gupta and D. K. Nagar, Matrix Variate Distributions, Chapman & Hall/CRC, Boca Raton, 2000.

[2] Y. L. Luke, The Special Functions and their Approximations, Vol. 1, Academic Press, New York, 1969.

[3] Albert W. Marshall and Ingram Olkin, Inequalities: Theory of Majorization and Its Applications, Mathematics in Science and Engineering 143, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, London, 1979.

[4] Daya K. Nagar and José A. Alvarez, Properties of the hypergeometric function type I distribution, Adv. & Appl. in Stat. 5(3) (2005), 341- 351.

[5] Daya K. Nagar and José A. Alvarez, Distribution of the product of independent hypergeometric function type I variables, Int. J. Appl. Math. Stat. 13(M08) (2008), 46-56.

[6] T. Pham-Gia and N. Turkkan, The product and quotient of general beta distributions, Statistical Papers 43 (2002), 537-550.

[7] B. D. Sivazlian, On a multivariate extension of the gamma and beta distributions, SIAM J. Appl. Math. 41(2) (1981), 205-209.

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