ZEROS AND STIELTJES CONTINUED FRACTION
FOR $J(z)$

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Abstract

For an orthogonal set of polynomials $p_r(z)$ relating to a kernel weight function, we consider the zeros of $p_r(z)$, focusing attention on the largest. Systems such as binomial in its three forms (Poisson, binomial, negative binomial), the normal, and the approach using a Maple code for zeros of functions expressed in factorial form are described. In some cases a nearly linear form for the largest zero is $z_r = A + B\sqrt{r} + Cr$. Particular attention is given to the third degree zeros relating to the general case.

1. Introduction

We have from Wall [8]

$$\ln \Gamma(z) = \left(z - \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + J(z) \quad (\Re(z) > 0),$$

where the last term is the continued fraction $J(z) = \dfrac{a_1}{z+} \dfrac{a_2}{z+} \dfrac{a_3}{z+} \cdots$ and $a_1, a_2, ..., a_{40}$ are given in Char [3].

The associated set of orthogonal polynomials $p_r(z)$ follows the recursion

$$p_r(z) = z p_{r-1}(z) - a_r p_{r-2}(z) \quad (r = 1, 2, ...)$$

with $p_0 = 1$, $p_1 = z$, $p_s = 0$ if $s < 0$. The first few cases of $p_r(z)$ are given in Table 1.

Table 1. The orthogonal polynomial $p_r(z)$

r	$p_r(z)$
1	z
2	$z^2 - a_2$
3	$z^3 - z(a_2 + a_3)$
4	$z^4 - z^2(a_2 + a_3 + a_4) + a_4 a_2$
5	$z^5 - z^3(a_2 + a_3 + a_4 + a_5) + z(a_4 a_2 + a_5 a_2 + a_5 a_3)$

A graphical representation is shown in Figure 1.

It is well known that the zeros of $p_r(z)$ are real, distinct, and between a consecutive pair p_{r-1}, p_r there is a zero of $p_{r+1}(z)$. What is the form of the largest zero in $p_r(z)$; can it be approximated?
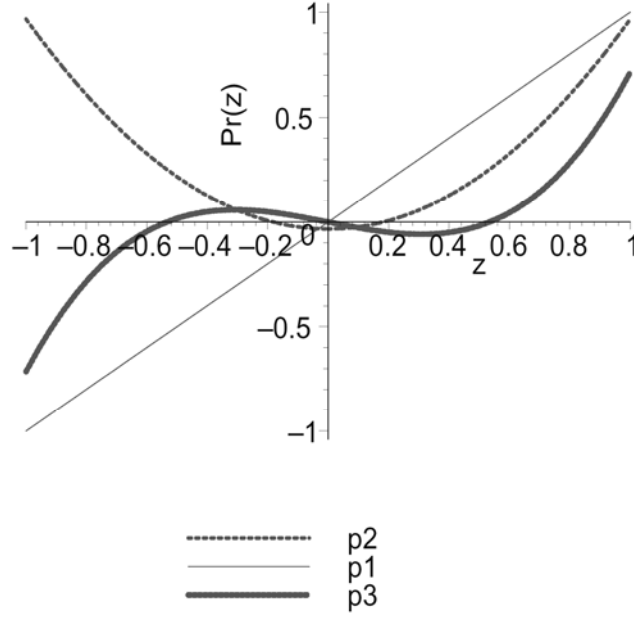
Figure 1. Orthogonal polynomials p_1, p_2 and p_3.

2. The Zeros of $p_r(z)$, and $J(z)$

Using the Maple symbolic code we have the zeros (z_r):

$z_1 = 0.0833333333$, $z_2 = 0.0333333333$, $z_3 = 0.2523809524$,

$z_4 = 0.5256064590$, $z_5 = 1.0115230681$, $z_6 = 1.5174736492$,

$z_7 = 2.2694889742$, $z_8 = 3.0099173833$, $z_9 = 4.0268871923$,

$z_{10} = 5.002768081$.

The orthogonal polynomials are:

$p_1(z) = z,$

$p_2(z) = z^2 - 0.033333,$

$p_3(z) = z^3 - 0.285714z,$

$p_4(z) = z^4 - 0.811321z^2 + 0.017520,$

$p_5(z) = z^5 - 1.822844z^3 + 0.306527z,$

$$p_6(z) = z^6 - 3.340317z^4 + 1.537685z^2 - 0.026586,$$

$$p_7(z) = z^7 - 5.609806z^5 + 5.674609z^3 - 0.722246z,$$

$$p_8(z) = z^8 - 8.619724z^6 + 15.728688z^4 - 5.350549z^2 + 0.080023,$$

$$p_9(z) = z^9 - 12.646611z^7 + 38.318746z^5 - 28.201558z^3 + 2.988425z,$$

$$p_{10}(z) = z^{10} - 17.649379z^8 + 81.441225z^6 - 106.888537z^4$$
$$+ 29.755983z^2 - 0.400337.$$

The solutions to $p_r(z) = 0$ are

Table 2. Zeros of $p_r(z)$

r	Zeros
3	0, ±0.5345224838
4	±0.8883234111, ±0.1490042347
5	0, ±1.278832666, ±0.4329328185
6	±0.1341235375, ±0.7273992128, ±1.671292531
7	0, ±0.3853571018, ±1.058497207, ±2.083480262
8	±0.1251872687, ±0.6481101603, ±1.396413273, ±2.496804990
9	0, ±0.3563135901, ±0.9453299115, ±0.755931799, ±2.922790877
10	±0.1190261337, ±0.5985892571, ±1.250580285, ±2.120209256, ±3.349278196

Now set up the zeros when we take the Stieltjes approximation (see Char [3]) $a_s^* = s^2/16$ (Stieltjes [5, letter 173, p. 354]) and also see Stieltjes [6].

There should be a very close set of zeros corresponding to using the correct values of a_s.

Table 3. Comparison of the largest zeros for a_r and a_r^*

r	11	12	13	14	15	16
a_r	4.22	4.66	5.11	5.55	6.00	6.46
a_r^*	4.22	4.67	5.11	5.56	6.01	6.46

Table 3 indicates the largest zeros are practically the same for the partial numerators relating to the Stieltjes continued fraction form and the Stieltjes conjecture $(a_r \sim r^2/16)$.

z_r is linear with a small gradient. This suggests the model

$$z_r = A + B\sqrt{r} + Cr$$

which we fitted using a least square Maple program, the second term in z_r could be Br^λ, where $0 < \lambda \leq 1$.

Note that for higher curvature a possible model for zeros might be

$$z_r = A + B\sqrt{r} + Cr + Dr^2,$$

where A, B, C and D are constants.

3. The Normal Density and the Well Known Laplace Continued Fraction

Laplace continued fraction:

$$f(z) = \frac{1}{z+} \frac{1}{z+} \frac{2}{z+} \frac{3}{z+} \cdots \quad \Re(z) > 0.$$

The orthogonal set $p_r(z)$

$$p_0(z) = 1,$$

$$p_1(z) = z,$$

$$p_2(z) = z^2 - 1,$$

$$p_3(z) = z^3 - 3z,$$

$$p_4(z) = z^4 - 6z^2 + 3,$$

$$p_r(z) = zp_{r-1}(z) - (r-1)p_{r-2}(z)$$

and so on. The linear form with slight slope is an approximant to z_r.

4. Zeros of Orthogonal Polynomials Related to the Binomial Distribution

4.1. The Poisson case

Using finite difference expressions, such as

$$x^{(r)} = x(x-1)\cdots(x-r+1) \quad (r = 1, 2, \ldots),$$

$$\Delta x^{(r)} = x^{(r+1)} - x^{(r)} = rx^{(r-1)},$$

consider the orthogonal system related to the Poisson case $p_r(x)$ for which

$$p_r(x) = e^{-\theta\Delta}x^{(r)} \quad (r = 1, 2, \ldots, \theta > 0)$$

$$= \left(1 - \theta\Delta + \frac{\theta^2}{2!}\Delta^2 - \frac{\theta^3}{3!}\Delta^3 \cdots\right)x^{(r)}$$

$$= \left(\sum_{s=0}^{\infty} \frac{(-\theta)^s \Delta^s}{s!}\right)x^{(r)}.$$

Thus

$$p_r(x) = x^{(r)} - \binom{r}{1}\theta x^{(r-1)} + \binom{r}{2}\theta^2 x^{(r-2)} - \binom{r}{3}\theta^3 x^{(r-3)} \cdots$$

$$(r = 1, 2, \ldots, \theta > 0).$$

4.2. The binomial case

The generating function of the binomial being $(pt + q)^n$, $0 < p \le 1$, $p + q = 1$ and $n = 1, 2, \ldots$.

The orthogonal set $[G_r(x)]$ is defined as

$$G_r(x) = (1 + p\Delta)^{-(n-r+1)}x^{(r)}$$

and we consider the cases $r = 1, 2, \ldots, n-1$. See Aitken and Gonin [1, expression (11)]; expanding

$$G_r(x) = x^{(r)} - \binom{r}{1}(n - r + 1)\,px^{(r-1)} + \binom{r}{2}(n - r + 2)^{(2)}\,p^2 x^{(r-2)}$$

$$- \binom{r}{3}(n - r + 3)^{(3)}\,p^3 x^{(r-3)} \cdots .$$

The negative binomial is set up for its orthogonal system by setting $p = -p$, and $n = -k$, where $p > 0$, and $k > 0$.

There are problems involved in the binomial case since the orthogonal set is finite - hence only approximations are available.

4.3. The negative binomial distribution and largest zeros

The orthogonal set $\{H_r(x)\}$ is given by

$$H_r(x) = (1 - p\Delta)^{k+r-1} x^{(r)} \quad (p > 0,\ k > 0)$$

$$= x^{(r)} - \binom{k + r - 1}{1} prx^{(r-1)} + \binom{k + r - 1}{2} p^2 r^{(2)} x^{(r-2)}$$

$$- \binom{k + r - 1}{3} p^3 r^{(3)} x^{(r-3)} + \cdots .$$

When $k = 1$, the negative binomial distribution reduces to the geometric distribution, the form essentially being $A + (AB)t + (AB^2)t \cdots$ for its probability generating function with $0 < A \le 1$, $A + B = 1$. The corresponding negative binomial is

$$H_r^*(x) = (1 - p\Delta)^r x^{(r)}$$

$$= x^{(r)} + \binom{r}{1} prx^{(r-1)} + \binom{r}{2} p^2 r^{(2)} x^{(r-2)} + \cdots .$$

4.4. The Stieltjes continued fraction form under equivalence transformations

The continued fraction

$$\frac{p_1}{z+}\,\frac{q_1}{1+}\,\frac{p_2}{z+}\,\frac{q_2}{1+}\,\frac{p_3}{z+}\,\frac{q_3}{1+}\cdots$$

(p's and q's are positive and real) may be expressed as

$$\frac{1}{\alpha_1 z +}\ \frac{1}{\alpha_2 +}\ \frac{1}{\alpha_3 z +}\ \frac{1}{\alpha_4 +}\cdots.$$

The parameters α_1, α_2, ..., being given by Stieltjes [5, letter 177, p. 365]. A first condition for the moment problem to have a solution is for the α's to be positive. Using successive equivalence transformations we have

Standard terms: $\dfrac{1}{p_1}$, $\dfrac{q_1}{p_1 p_2}$, $\dfrac{q_1 q_2}{p_1 p_2 p_3}$ for α_1, α_3, α_5.

New terms: $\dfrac{p_1}{q_1}$, $\dfrac{p_1 p_2}{q_1 q_2}$, $\dfrac{p_1 p_2 p_3}{q_1 q_2 q_3}$ for α_2, α_4, α_6.

The pattern is clear, and

$$\alpha_{2s+1} = \frac{q_1 q_2 \cdots q_s}{p_1 p_2 \cdots p_{s+1}} \quad (s = 0, ...;\ q_0 = 1)$$

and

$$\alpha_{2s} = \frac{p_1 p_2 \cdots p_s}{q_1 q_2 \cdots q_s} \quad (s = 1, 2, ...).$$

4.5. Examples

Example 1. The $p_s = q_s$, $s = 1, 2, ...$, then $\alpha_{2s} = 1$, $\alpha_{2s+1} = 1/p_{s+1}$ so $\sum \alpha_s = \infty$ satisfying the 2nd condition for the existence of a solution to the moment problem.

Example 2.

$$p_r = (2r - 1)^3, \quad q_r = (2r)^3,$$

$$\alpha_{2r+1} = \left\{(1 + 1)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\cdots\left(1 + \frac{1}{2r - 1}\right)\left(1 + \frac{1}{2r + 1}\right)\right\}^3,$$

$$\alpha_{2r} = \left\{\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2r}\right)\right\}^3 \quad (r = 1, 2, ...).$$

Finite and infinite products are needed to come up with satisfactory bounds (see Bromwich [2, Chapter VI]).

5. The Zeros of the Poisson Case $(\theta = 1)$

We have always pointed out that the largest zeros for a Poisson system are approximately linear. This property also holds for the complete set of zeros for $p_r(x)$, r being fixed (i.e., the internal zeros are nearly linear deviating somewhat in the vicinity of the largest). Evidence of this is clear from the listing below.

Complete list of zeros for Poisson $(\theta = 1)$

$(2) = 0.3820,\ 2.6180,$

$(3) = 0.1392,\ 1.7459,\ 4.1149,$

$(4) = 0.0440,\ 1.3320,\ 3.0797,\ 5.5443,$

$(5) = 0.0114,\ 1.1307,\ 2.5406,\ 4.3884,\ 6.9288,$

$(6) = 0.0024,\ 1.0429,\ 2.2457,\ 3.7514,\ 5.6768,\ 8.2807,$

$(7) = 0.0004,\ 1.0114,\ 2.0956,\ 3.3771,\ 4.9597,\ 6.9485,\ 9.6073,$

$(8) = 0.0001,\ 1.0024,\ 2.0304,\ 3.1662,\ 4.5174,\ 6.1636,\ 8.2063,\ 10.9137,$

$(9) = 0.0000,\ 1.0004,\ 2.0077,\ 3.0612,\ 4.2504,\ 5.6623,\ 7.3625,\ 9.4521,$
     $12.2033,$

$(10) = 0.0000,\ 1.0001,\ 2.0016,\ 3.0181,\ 4.1036,\ 5.3445,\ 6.8093,\ 8.5564,$
      $10.6877,\ 13.4787,$

$(11) = 0.0000,\ 1.0000,\ 2.0003,\ 3.0043,\ 4.0350,\ 5.1564,\ 6.4455,\ 7.9569,$
      $9.7454,\ 11.9143,\ 14.7419,$

$(12) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0008,\ 4.0095,\ 5.0592,\ 6.2181,\ 7.5513,$
      $9.1040,\ 10.9297,\ 13.1329,\ 15.9945,$

$(13) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0001,\ 4.0021,\ 5.0181,\ 6.0907,\ 7.2869,$
      $8.6603,\ 10.2502,\ 12.1096,\ 14.3443,\ 17.2376,$

$(14) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0004,\ 5.0044,\ 6.0308,\ 7.1292,$
      $8.3615,\ 9.7715,\ 11.3951,\ 13.2853,\ 15.5493,\ 18.4724,$

$(15) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0001,\ 5.0009,\ 6.0084,\ 7.0483,$

$8.1741,\ 9.4405,\ 10.8841,\ 12.5384,\ 14.4572,\ 16.7485,\ 19.6996,$

$(16) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0000,\ 5.0001,\ 6.0018,\ 7.0145,$

$8.0706,\ 9.2245,\ 10.5231,\ 11.9975,\ 13.6802,\ 15.6254,$

$17.9423,\ 20.9201,$

$(17) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0000,\ 5.0000,\ 6.0003,\ 7.0035,$

$8.0233,\ 9.0979,\ 10.2797,\ 11.6084,\ 13.1112,\ 14.8202,\ 16.7901,$

$19.1312,\ 22.1342,$

$(18) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0000,\ 5.0000,\ 6.0001,\ 7.0007,$

$8.0062,\ 9.0351,\ 10.1299,\ 11.3389,\ 12.6958,\ 14.2249,\ 15.9585,$

$17.9517,\ 20.3156,\ 23.3427,$

$(19) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0000,\ 5.0000,\ 6.0000,$

$7.0001,\ 8.0013,\ 9.0102,\ 10.0503,\ 11.1662,\ 12.4016,\ 13.7848,$

$15.3385,\ 17.0950,\ 19.1102,\ 21.4958,\ 24.5459,$

$(20) = 0.0000,\ 1.0000,\ 2.0000,\ 3.0000,\ 4.0000,\ 5.0000,\ 6.0000,\ 7.0000,$

$8.0002,\ 9.0024,\ 10.0158,\ 11.0690,\ 12.2066,\ 13.4672,\ 14.8750,$

$16.4518,\ 18.2299,\ 20.2658,\ 22.6721,\ 25.7443.$

As r increases, roots become $0, 1, 2, 3, \ldots$. In the three cases the largest zeros are approximately linear, see Figure 2. Note that the zeros are found using Maple code for polynomials with terms x, $x^{(2)}$, $x^{(3)}$, etc. Moreover we expect the zeros to be real.
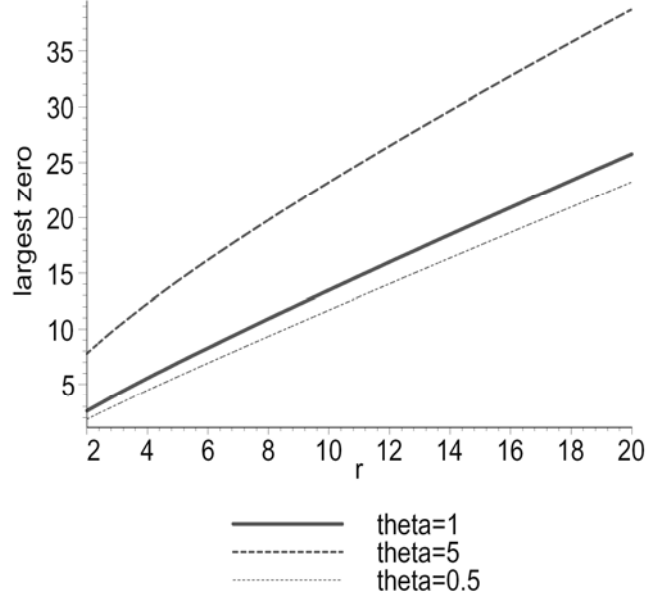
Figure 2. Largest zeros for Poisson with $\theta = 0.5,\ 1.0$ and 5.0.

The solving of the cubic equation is described by Nickalls [4] in details. We have

$$p_3(x) = ax^3 + bx^2 + c + d, \tag{1}$$

and in our case, $a = 1$, and the relations

$$x_N = -\frac{b}{3}, \quad \delta = \sqrt{b^2 - 3c}/3, \quad h = 2\delta^3,$$

produce y_N by replacing x in (1) by x_N, followed by

$$\cos(3\phi) = -y_N/h, \quad \text{and} \quad \phi = \arccos\left(-y_N/h\right)/3,$$

and

$$x_1 = x_N + 2\delta \cos \phi,$$

$$x_2 = x_N + 2\delta \cos(2\pi/3 + \phi),$$

$$x_3 = x_N + 2\delta \cos(4\pi/3 + \phi).$$

For the Poisson case the orthogonal polynomial for the cubic is

$$p_3(x) = x^3 - 3x^2(1 + \theta) + x(2 + 3\theta + 3\theta^2) - \theta^3,$$

and solutions are

$$x = 1 + \theta + 2\sqrt{\frac{1 + 3\theta}{3}} \cos\left(\phi + \frac{2\pi k}{3}\right) \quad (k = -1, 0, 1),$$

where

$$\phi = \frac{1}{3} \arccos\left(\frac{\theta}{2\sqrt{\frac{(1 + 3\theta)^3}{27}}}\right) \quad (\theta > 0).$$

For the case of $\theta = 1$, three roots are 4.1149, 0.1392 and 1.7459 as expected.

The essence of the solution to the trigonometrical cubic lies in the equivalence of the two equations

$$z^3 = Az + B \quad \text{and} \quad 4\cos^3\phi = 3\cos\phi + \cos(3\phi);$$

i.e.,

$$\frac{z^3}{4\cos^3\phi} = \frac{Az}{3\cos\phi} = \frac{B}{\cos(3\phi)}.$$

6. Concluding Remarks

For a classical account of the general properties of zeros of functions see Szegö [7]. Simple formulas for zeros in general are rare: however, some progress is possible using symbolic programs, such as Maple, Mathematica, etc. Our study includes equations of zeros, the equations being of order 40 to 50.

An unusual model emerges, as

$$z_r \sim A + B\sqrt{r} + Cr,$$

and in some cases this appears to be a linear form. Generalizations of this suggest new problems in elementary geometry.

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