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ANALYTIC DENSITIES IN NUMBER THEORY. PART I: ANALYTIC DENSITIES OF SUBSETS

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Abstract

In this paper, we present a detailed study of the analytic, conditional analytic and derived analytic densities and give some applications to classical number theory. Some new existence criteria [A. Fuchs and R. A. Giuliano, Théorie Générale des Densités, Pub. I.R.M.A., Strasbourg. I, 1989] are established. Certain results obtained generalize those obtained in ([JP Jour. Algebra, Number Theory & Appl. 5(3) (2005), 513-533], [Far East J. Math. Sci. (FJMS) 18(1) (2005), 31-48]).

1. Prelude

We consider a family $\Re = \{\boldsymbol{\mu}_\alpha, \alpha \in T\}$ of $\boldsymbol{\sigma}$-finitely additive probability measures on the set $\wp(\mathbb{N}^*)$ of subsets E of $\mathbb{N}^*$. We examine the convergence, when α tends to α_0, of

$$\boldsymbol{\mu}_\alpha(E) := \sum_{n=1}^{\infty} I_E(n)\boldsymbol{\mu}_\alpha(\{n\}).$$

If the limit, $\lim\limits_{\alpha} \boldsymbol{\mu}_\alpha(E)$, when $\alpha \to \alpha_0$, exists, then we say that E has a *density* in the sense of the family $\Re$.

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If we take, for example, $\alpha = s$, $T =]1, +\infty[$, then we obtain the zeta-family

$$\zeta := \{\zeta_s,\ s > 1\},$$

where for all subsets E of $\mathbb{N}^*$,

$$\boldsymbol{\mu}_s(E) := \zeta_s(E) := \frac{1}{\zeta(s)} \sum_n \frac{I_E(n)}{n^s}$$

and I_E is the indicator function of the subset E.

By taking the limit when s tends to 1^+, we diffuse the considered measure, and we obtain that we call an *analytic density*.

We prove that the latter gives a generalization to the asymptotic density [1, 2].

More precisely, analytic density is an extension of the asymptotic density. Notably, the class $\mathcal{E}$ of subsets E of $\mathbb{N}^*$, for which $\lim_s \boldsymbol{\mu}_s(E)$ exists contains, strictly, the class $\mathfrak{D}$ of subsets of $\mathbb{N}^*$, for which $\lim_n \mathbf{v}_n(E)$ exists.

We recall that for all real numbers $s > 1$, the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^s}$$

converges, and its sum is noted $\zeta(s)$. Thus

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

Definition 1.1. The *Riemann's zeta function* ζ is the function defined, for all real numbers $s > 1$, by

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

Proposition 1.1. *The function* ζ *defined on* $]+1, +\infty[$, *is continuous, derivable and decreasing*.

For ulterior needs, we look how the zeta function ζ and its derivation ζ' behave in a neighborhood of 1 (the asymptotic behaviour of $\zeta(s)$, $Log\,\zeta(s)$ and $\zeta'(s)$, as $s \to 1^+$).

Theorem 1.2 [1, 6]**.** *We have*

(a) $\zeta(s) = \dfrac{1}{s-1} + O(1), \ (s \to 1^+),$

(b) $Log\,\zeta(s) = Log\,\dfrac{1}{s-1} + O(s-1), \ (s \to 1^+).$

Theorem 1.3 [1, 6]**.** *We have*

$$\zeta'(s) = -\frac{1}{(s-1)^2} + O(1), \ as \ (s \to 1^+).$$

2. Main Results

2.1. Analytic densities

A generalization of asymptotic density [2] is the density introduced by use of Riemann's zeta function given previously.

We begin by introducing on $(\mathbb{N}^*, \wp(\mathbb{N}^*))$ a family of laws of probability indexed by a real number $s > 1$, as in [3].

Definition 2.1. Let E be a subset of $\mathbb{N}^*$. We put, for all $s > 1$,

$$\boldsymbol{\mu}_s(E) := \frac{1}{\zeta(s)} \sum_{n \geq 1} \frac{I_E(n)}{n^s},$$

where $I_E(n)$ is the indicator function of the subset E.

We say that E has the number ℓ as an *analytic density*, if $\ell = \lim_s \boldsymbol{\mu}_s(E)$, when s tends to 1^+. (Notice that this limit belongs to $[0, 1]$.)

We denote this limit by $\delta(E)$, and we call $\delta(E)$ to be the *analytic density* of E. We write $\mathcal{E}$ to be a class of subsets of $\mathbb{N}^*$ which has an analytic density.

Proposition 2.1. *Analytic density* δ *is invariant under translation. More precisely, if* $E \subset \mathbb{N}^*$ *and* $k \in \mathbb{N}$, *then*

$$\lim_{(s \to 1^+)} (\boldsymbol{\mu}_s(E + k) - \boldsymbol{\mu}_s(E)) = 0$$

uniformly on E.

Proof. We prove this property by increasing recurrence: (1) For $k = 1$, we prove that $\boldsymbol{\mu}_s(E)$ and $\boldsymbol{\mu}_s(E + 1)$ have the same asymptotic comportment when s tends to 1^+. Indeed, we put

$$e_1^s = \boldsymbol{\mu}_s(E) - \boldsymbol{\mu}_s(E + 1) = \frac{1}{\zeta(s)} \sum_n \frac{I_E(n)}{n^s} - \frac{1}{\zeta(s)} \sum_n \frac{I_{E+1}(n)}{n^s}$$

$$= \frac{1}{\zeta(s)} \sum_{n \in E} \frac{1}{n^s} - \frac{1}{\zeta(s)} \sum_{n \in E} \frac{1}{(n+1)^s} = \frac{1}{\zeta(s)} \sum_{n \in E} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right).$$

And then

$$0 \leq e_1^s \leq \frac{1}{\zeta(s)} \sum_{n=1}^{+\infty} \left(\frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = \frac{1}{\zeta(s)}.$$

Let s tend to 1^+. Then we see that $e_1^s \to 0$ uniformly in E. Otherwise, $\boldsymbol{\mu}_s(E)$ and $\boldsymbol{\mu}_s(E + 1)$ have the same asymptotic comportment, and these two limits are equal if these exist. In another way,

$$\delta(E) = \delta(E + 1),$$

so, invariance by translation of analytic density.

(2) We suppose

$$\boldsymbol{\mu}_s(E), \boldsymbol{\mu}_s(E + 1), \ldots, \boldsymbol{\mu}_s(E + k - 1)$$

have the same asymptotic comportment when s tends to 1^+ and we prove that

$$\boldsymbol{\mu}_s(E) \text{ and } \boldsymbol{\mu}_s(E + k)$$

have the same property. Indeed,

$$\boldsymbol{\mu}_s(E) = \frac{1}{\zeta(s)} \sum_{n \in E} \frac{1}{n^s}; \qquad \boldsymbol{\mu}_s(E + 1) = \frac{1}{\zeta(s)} \sum_{n \in E} \frac{1}{(n + k)^s}.$$

We put

$$e_{k-1}^s = \boldsymbol{\mu}_s(E) - \boldsymbol{\mu}_s(E + k - 1).$$

Then

$$e_k^s = \boldsymbol{\mu}_s(E) - \boldsymbol{\mu}_s(E + k) = (\boldsymbol{\mu}_s(E) - \boldsymbol{\mu}_s(E + 1)) + (\boldsymbol{\mu}_s(E + 1) - \boldsymbol{\mu}_s(E + 2))$$

$$+ \cdots + (\boldsymbol{\mu}_s(E + k - 2) - \boldsymbol{\mu}_s(E + k - 1)) + (\boldsymbol{\mu}_s(E + k - 1) - \boldsymbol{\mu}_s(E + k)).$$

The second member is the sum of finite number of terms which tends to 0 uniformly in E, so

$$e_k^s \to 0 \text{ uniformly in } E.$$

Then

$$\boldsymbol{\mu}_s(E) \text{ and } \boldsymbol{\mu}_s(E + k)$$

have the same asymptotic comportment when s tends to 1^+, and these two limits are equal if there exist, otherwise,

$$\delta(E) = \delta(E + k) \quad \forall k.$$

Proposition 2.2. (a) *All finite subsets* $E \in \wp(\mathbb{N}^*)$ *belong to* $\mathcal{E}$ *and* $\delta(E) = 0$.

(b) *All cofinite subsets* $E \in \wp(\mathbb{N}^*)$ *belong to* $\mathcal{E}$ *and* $\delta(E) = 1$.

Proposition 2.3. $\mathcal{E}$ *contains the algebra of finite and cofinite subsets of E.*

Proposition 2.4. *For all m of* $\mathbb{N}^*$, *the class* $m\mathbb{N}^*$ *of multiples of m belongs to* $\mathcal{E}$ *and*

$$\delta(m\mathbb{N}^*) = \frac{1}{m}.$$

Proof. Noting that

$$\boldsymbol{\mu}_s(m\mathbb{N}^*) = \frac{1}{m^s}$$

and letting $(s \to 1^+)$, we have

$$\delta(m\mathbb{N}^*) = \frac{1}{m}.$$

Proposition 2.5. *The set* $\mathbb{P}$ *of prime numbers belongs to* $\mathcal{E}$ *and* $\delta(\mathbb{P}) = 0$. *In another way, prime numbers are rare.*

Proof. We have

$$\boldsymbol{\mu}_s(\mathbb{P}) = \frac{1}{\zeta(s)} \sum_{p \in \mathbb{P}} \frac{1}{p^s}$$

or, by Theorem 1.2(a),

$$\frac{1}{\zeta(s)} \sim (s-1), \text{ as } (s \to 1^+).$$

Also,

$$\sum_{p \in \mathbb{P}} \frac{1}{p^s} \sim Log \frac{1}{s-1}, \text{ as } (s \to 1^+).$$

Thus

$$\boldsymbol{\mu}_s(\mathbb{P}) \sim (s-1) Log \frac{1}{s-1}, \text{ as } (s \to 1^+).$$

This tends to 0, when $(s \to 1^+)$, so $\delta(\mathbb{P}) = 0$.

Proposition 2.6. *The set* E_2 *of square-free integers belongs to* $\mathcal{E}$ *and*

$$\delta(E_2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Proof. By Proposition 2.10 in [3] and the fact that $\frac{1}{\zeta(2s)}$ is continuous, it follows that

$$\boldsymbol{\mu}_s(E_2) = \frac{1}{\zeta(2s)} \to \frac{1}{\zeta(2)} = \frac{6}{\pi^2}, \text{ as } (s \to 1^+).$$

So

$$\delta(E_2) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Proposition 2.7. *The sets* E_k $(k \geq 2)$ *of integers without divisors of the form* n^k *belong to* $\mathcal{E}$ *and*

$$\delta(E_k) = \frac{1}{\zeta(k)}.$$

Proof. Since

$$\boldsymbol{\mu}_s(E_k) = \frac{1}{\zeta(ks)},$$

on taking the limit when $(s \to 1^+)$, it follows that

$$\delta(E_k) = \frac{1}{\zeta(k)}.$$

Theorem 2.8. *Let E be a subset of* $\mathbb{N}^*$ *such that*

$$\sum_{n \in E} \frac{1}{n} < +\infty.$$

Then E has an analytic density $\delta(E)$ *and* $\delta(E) = 0$. *The converse is not necessarily true.*

Proof. We have

$$\boldsymbol{\mu}_s(E) = \frac{1}{\zeta(s)} \sum_{n \in E} \frac{1}{n^s} < \frac{1}{\zeta(s)} \sum_{n \in E} \frac{1}{n} \quad (s > 1).$$

Then

$$\boldsymbol{\mu}_s(E) \to 0, \text{ as } (s \to 1^+).$$

For the converse see Theorem 4.2.

Before giving other applications, we require the following result:

Theorem 2.9. (Criterion). *Let E be a subset of* $\mathbb{N}^*$ *neither finite, nor cofinite, written in the form*

$$E = \bigcup_{n \geq 1} [p_n, q_n[,$$

where $(p_n)_{n\geq 1}$ *and* $(q_n)_{n\geq 1}$ *are two sequences of integers such that*

$$0 < p_n < q_n < p_{n+1} \quad \forall n \geq 1.$$

Put

$$\rho_n = Log\, q_n - Log\, p_n \text{ and } \sigma_n = Log\, q_n - Log\, q_{n-1}, \quad n \geq 1 \quad (q_0 = 1).$$

Let ℓ *be a real number in* $]0, 1[$ *and suppose the following two hypotheses hold*:

(H_1) : $Log\, p_n \sim Log\, q_{n-1}$, *as* $(n \to +\infty)$.

(H_2) : $\dfrac{\rho_n}{\sigma_n} \to \ell$, *as* $(n \to +\infty)$.

Then the set E has an analytic density $\delta(E)$ *and* $\delta(E) = \ell$. *If* $\ell = 0$, *then* $(H_2) \Rightarrow \delta(E) = 0$.

For the proof of this theorem, we use the following result:

Lemma 2.10 ([1, Théo. VII.9, p. 168], [5, Théo. 8.2, p. 25]). *Let* $E = \bigcup_{n\geq 1} [p_n, q_n[$ *be a subset of* $\mathbb{N}^*$, *neither finite, nor cofinite. Let* μ *be a positive measure on* $(\mathbb{N}^*, \wp(\mathbb{N}^*))$, *with total mass* $+\infty$ *and support* $\mathbb{N}^*$ *and F be its a distribution function. We put*

$$\rho_k = F(q_k) - F(p_k), \quad \sigma_k = F(q_k) - F(q_{k-1}), \quad k \geq (q_0 = 1).$$

Then

$$\overline{\delta}_\mu = \limsup_{(n\to+\infty)} \frac{\sum_{k=1}^{n} \rho_k}{\sum_{k=1}^{n} \sigma_k}, \quad \underline{\delta}_\mu = \liminf_{(n\to+\infty)} \frac{F(q_{n-1})}{F(p_n)} \frac{\sum_{k=1}^{n-1} \rho_k}{\sum_{k=1}^{n-1} \sigma_k}.$$

We give a direct application of this theorem to the first-digit problem.

Definition 2.2. We suppose that we adopt the base b $(b \geq 2)$ as a numeration base; a digit is then a number $k \in \{0, 1, ..., b-1\}$ and the set

E_k of strictly positive integers which admits a development in the base b, with first-digit $k \in \{0, 1, ..., b-1\}$, is given by

$$E_k = \bigcup_{n \geq 0} [kb^n, (k+1)b^n[,$$

the disjoint union of its connected components in second member. A solution of the first-digit problem is independent of the numeration base.

Proposition 2.11. *Let k be a given integer, with $1 \leq k \leq 9$. We consider the set E formed by strictly positive integers with development in base b has a significantly first-digit equal to k. Then the set E has an analytic density $\delta(E)$ and*

$$\delta(E) = Log_b\left(1 + \frac{1}{k}\right).$$

Proof. Indeed E takes the form

$$E = \bigcup_{n \geq 0} [p_n, q_n[,$$

where

$$p_n = kb^n, \quad q_n = (k+1)b^n, \quad k \in \{1, 2, ..., 9\}.$$

(a) We have

$$Log\, p_n = nLog\, b + Log\, k,$$

$$Log\, q_{n-1} = (n-1)Log\, b + Log(k+1).$$

For fixed k, we have

$$Log\, p_n \sim Log\, q_{n-1}.$$

(b) Then

$$\frac{q_n}{p_n} = \frac{(k+1)b^n}{kb^n} = \left(1 + \frac{1}{k}\right) \text{ and } \frac{q_n}{q_{n-1}} = \frac{(k+1)b^n}{(k+1)b^{n-1}} = b.$$

We put

$$\rho_n = Log\,\frac{q_n}{p_n} = Log\left(1+\frac{1}{k}\right) \text{ and } \sigma_n = Log\,\frac{q_n}{q_{n-1}} = Log\,b.$$

Then

$$\frac{\rho_n}{\sigma_n} = \frac{Log\left(1+\frac{1}{k}\right)}{Log\,b} = Log_b\left(1+\frac{1}{k}\right).$$

So, by Theorem 2.9, we shall have

$$\delta(E) = Log_b\left(1+\frac{1}{k}\right).$$

This result will be obtained in another way in an ulterior theorem.

Proposition 2.12. *Let E be the set of natural integers with development in base $b \geq 2$ containing an odd number of digits. Then, E has an analytic density $\delta(E)$ and $\delta(E) = \frac{1}{2}$.*

Proof. We write E in the form of disjoint union of its connected components

$$E = \bigcup_{k\geq 0} [b^{2k}, b^{2k+1}[.$$

We put

$$p_k = b^{2k}, \quad q_k = b^{2k+1}, \quad k \geq 0.$$

Then

(a) $Log\,p_k = 2k\,Log\,b$; $Log\,q_{k-1} = (2k-1)Log\,b.$

We have

$$\frac{Log\,p_k}{Log\,q_{k-1}} = \frac{2k\,Log\,b}{(2k-1)Log\,b} = \frac{2k}{2k-1} \to 1, \text{ as } (k \to +\infty).$$

In other words

$$Log\, p_k \sim Log\, q_{k-1}.$$

(b) $\dfrac{q_k}{p_k} = \dfrac{b^{2k+1}}{b^{2k}} = b$ and $\dfrac{q_k}{q_{k-1}} = \dfrac{b^{2k+1}}{b^{2k-1}} = b^2.$

We put

$$\rho_k = Log\, \frac{q_k}{p_k} = Log\, b$$

and

$$\sigma_k = Log\, \frac{q_k}{q_{k-1}} = 2\, Log\, b.$$

Then

$$\frac{\rho_k}{\sigma_k} = \frac{Log\, b}{2\, Log\, b} = \frac{1}{2}.$$

So, by Theorem 2.9, we shall have, $\delta(E) = \dfrac{1}{2}$.

Corollary 2.13. *We suppose that sequences* $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$ *satisfy*

$$\frac{q_n}{p_n} \to \ell_1, \quad \frac{p_n}{q_{n-1}} \to \ell_2, \ as\ (n \to +\infty)$$

with $\ell_1, \ell_2 \in [1, +\infty[$, *one at least of those limits is different of* 1. *Then*

(a) $Log\, p_n \sim Log\, q_{n-1}$, *as* $(n \to +\infty)$;

(b) $$\frac{\rho_n}{\sigma_n} = \frac{Log\, \frac{q_n}{p_n}}{Log\, \frac{q_n}{q_{n-1}}} = \frac{Log\, \frac{q_n}{p_n}}{Log\left(\frac{q_n}{p_n}\frac{p_n}{q_{n-1}}\right)} \to \frac{Log\, \ell_1}{Log(\ell_1 \ell_2)},\ as\ (n \to +\infty).$$

It results that E admits an analytic density

$$\delta(E) = \frac{Log\, \ell_1}{Log(\ell_1 \ell_2)}.$$

Theorem 2.14. (Existence criterion). *Let ℓ be a real number such that $0 < \ell \leq 1$. Then the following two properties are equivalent*:

(p_1) : *E admits ℓ as an analytic density.*

(p_2) : $(p_2)_1$: *Log* $p_n \sim$ *Log* q_{n-1}.

$(p_2)_2$: *For*

$$\rho_n = Log\, q_n - Log\, p_n, \quad \sigma_n = Log\, q_n - Log\, q_{n-1}, \quad n \geq 1, \quad (q_0 = 1),$$

we have

$$\frac{\sum_{k=1}^{n} \rho_k}{\sum_{k=1}^{n} \sigma_n} \to \ell,$$

in other words

$$\frac{\frac{1}{n}\sum_{k=1}^{n} \rho_k}{\frac{1}{n}\sum_{k=1}^{n} \sigma_n} \to \ell, \ as \ (n \to +\infty).$$

If $\ell = 0$, then a condition (p_1) amounts to $(p_2)_2$.

In particular, if these two sequences $(\rho_n)_{n\geq 1}$, $(\sigma_n)_{n\geq 1}$ converge in the sense of Césaro to two limits ℓ_1, ℓ_2 (with $\ell_2 > 0$), then property $(p_2)_2$ is verified with $\ell = \frac{\ell_1}{\ell_2}$.

It results the following corollary:

Corollary 2.15. *We suppose that two sequences $(p_n)_{n\geq 1}$, $(q_n)_{n\geq 1}$ satisfy the following two properties*:

(p_1) : $\frac{q_n}{p_n} \to r$, *as* $(n \to +\infty)$.

(p_2) : $(q_n)^{\frac{1}{n}} \to \rho$, *as* $(n \to +\infty)$.

Then, E *admits an analytic density* $\delta(E)$ *and*

$$\delta(E) = \frac{Log\, r}{Log\, \rho}.$$

Proof. It is enough to prove that under the hypothesis properties $(p_2)_1$ and $(p_2)_2$ above hold.

(1) It holds from (p_1) that

$$Log\, p_n \sim Log\, q_n$$

and from (p_2) that

$$Log\, q_n \sim n\, Log\, \rho.$$

Also

$$Log\, q_n \sim Log\, q_{n-1}.$$

It results that

$$Log\, p_n \sim Log\, q_{n-1}.$$

(2) $\rho_n = Log\left(\frac{q_n}{p_n}\right) \to Log\, r,$ also $\rho_n \to Log\, r$ in the sense of Césaro,

$$\sigma_n = Log\, q_n - Log\, q_{n-1},$$

$$\frac{1}{n}\sum_{k=1}^{n} \sigma_k = \frac{1}{n} Log\, q_n = Log((q_n)^{\frac{1}{n}}) \to Log\, \rho,$$

in other words, $\sigma_n \to Log\, \rho$ in the sense of Césaro so the result.

Remark 2.1. By virtue of (p_1), condition (p_2) can be replaced by the following:

$$(p_2)' : (p_n)^{\frac{1}{n}} \to \rho, \quad (\rho > 1).$$

Indeed,

$$(p_n)^{\frac{1}{n}} \to \left(\frac{p_n}{q_n}\right)^{\frac{1}{n}} (q_n)^{\frac{1}{n}},$$

or, by (p_1),

$$\left(\frac{p_n}{q_n}\right)^{\frac{1}{n}} \sim 1,$$

then

$$(p_n)^{\frac{1}{n}} \sim (q_n)^{\frac{1}{n}}.$$

2.2. Applications

Proposition 2.16. *Let E be a subset of $\mathbb{N}^*$ given by*

$$E = \bigcup_{k \geq 1} [p_k, q_k[,$$

where

$$\begin{cases} p_k = b^{P(k)}, & P(k) = ak + d, \\ q_k = b^{Q(k)}, & Q(k) = ak + d^*. \end{cases}$$

We suppose that a, d, d^* are real numbers such that:

(H_1) : $a > 0$.

(H_2) : *For all $k \geq 1$, p_k and q_k are integers ≥ 1.*

(H_3) : $0 < \dfrac{d^* - d}{a} < 1$.

Then E admits an analytic density $\delta(E)$ and $\delta(E) = \dfrac{d^* - d}{a}$.

Proof. Indeed, we put

$$\begin{cases} \rho_k = Log\, q_k - Log\, p_k = (d^* - d) Log\, b, \\ \sigma_k = Log\, q_k - Log\, q_{k-1} = a Log\, b. \end{cases}$$

We verify

(a) $\dfrac{Log\, p_k}{Log\, q_{k-1}} = \dfrac{ak + d}{a(k-1) + d^*} \to 1$, as $(k \to +\infty)$;

in other words

$$Log\, p_k \sim Log\, q_{k-1}.$$

(b) $\dfrac{\rho_k}{\sigma_k} = \dfrac{d^* - d}{a}$.

So, by Theorem 2.9, it holds that E admits an analytic density $\delta(E)$ and $\delta(E) = \dfrac{d^* - d}{a}$.

Proposition 2.17. *Let E be a subset of* $\mathbb{N}^*$ *given by*

$$E = \bigcup_{k \geq 1} [p_k, q_k[,$$

where

$$\begin{cases} p_k = b^{P(k)}, & P(k) = ak^n + dk^{n-1} + o(k^{n-1}), \\ q_k = b^{Q(k)}, & Q(k) = ak^n + d^* k^{n-1} + o(k^{n-1}). \end{cases}$$

We suppose that a, d, d^*, n *are numbers such that*:

(H_1) : *n is an integer* ≥ 2.

(H_2) : $a > 0$.

(H_3) : *For all* $k \geq 1$, *p_k and q_k are integers* ≥ 1.

(H_4) : $0 < \dfrac{d^* - d}{na} < 1$.

Then E admits an analytic density $\delta(E)$ *and* $\delta(E) = \dfrac{d^* - d}{na}$.

Proof. Indeed, we put

$$\begin{cases} \rho_k = Log\, q_k - Log\, p_k = ((d^* - d)k^{n-1} + o(k^{n-1}))Log\, b, \\ \sigma_k = Log\, q_k - Log\, q_{k-1} = (nak^{n-1} + o(k^{n-1}))Log\, b. \end{cases}$$

We verify

(a) $\frac{Log\, p_k}{Log\, q_{k-1}} = \frac{P(k)}{Q(k-1)} \to 1$, as $(k \to +\infty)$;

in other words

$$Log\, p_k \sim Log\, q_{k-1}.$$

(b) $\frac{\rho_k}{\sigma_k} \to \frac{d^* - d}{na}$, as $(k \to +\infty)$.

So, by Theorem 2.9, it results that E admits an analytic density $\delta(E)$ and $\delta(E) = \frac{d^* - d}{na}$.

Proposition 2.18. *Let E be a subset of $\mathbb{N}^*$ given by*

$$E = \bigcup_{k \geq 0} [c^{2k}, c^{2k+1}[,$$

where integers $c \geq 2$,

$$a = Log\, c, \quad d = 0, \quad d^* = Log_b\, c,$$

in Proposition 2.16.

Then E admits an analytic density $\delta(E)$ and $\delta(E) = \frac{1}{2}$.

3. Conditional Analytic Density

It is natural to consider which we call the *conditional analytic density* on the prime numbers.

We start with the relation

$$\boldsymbol{\mu}_s(A \mid B) = \frac{\boldsymbol{\mu}_s(A \cap B)}{\boldsymbol{\mu}_s(B)},$$

where $\boldsymbol{\mu}_s(B) > 0$ and $A = E$, $B = \mathbb{P}$, for all $s > 1$. We have the following definition:

Definition 3.1. Let E be a subset of $\mathbb{N}^*$ and we consider, for all $s > 1$, the following expression:

$$\boldsymbol{\mu}_s(E \mid \mathbb{P}) = \frac{\boldsymbol{\mu}_s(E \cap \mathbb{P})}{\boldsymbol{\mu}_s(\mathbb{P})} = \frac{\sum_{p \in E} \frac{1}{p^s}}{\sum_{p \in \mathbb{P}} \frac{1}{p^s}}.$$

Then we say that E admits the number ℓ as a *conditional analytic density* related to $\mathbb{P}$, if $\lim_s \boldsymbol{\mu}_s(E \mid \mathbb{P})$ exists and equals ℓ, when s tends to 1^+.

(Notice that this limit belongs to $[0, 1]$.) We shall denote this conditional density by $\delta_c(E)$.

Or, we know that

$$Log\, \zeta(s) \sim \sum_{p \in \mathbb{P}} \frac{1}{p^s}, \text{ as } (s \to 1^+).$$

So, E admits a conditional analytic density ℓ, conditionally to $\mathbb{P}$, if and only if

$$\lim_{(s \to 1^+)} \frac{1}{Log\, \zeta(s)} \sum_{p \in E} \frac{1}{p^s}$$

exists and equals ℓ.

This density has been used, first of all, by Dirichlet, who proved in the first third of 19th century that there are infinitely many prime numbers of the form:

$$p \equiv k \pmod m,$$

where k and m are two relatively prime numbers.

Theorem 3.1. *Let k and m be two relatively prime integers and let $E_{k,m}$ be the set of prime numbers of the form $p \equiv k \pmod m$. Then $E_{k,m}$*

admits a conditional analytic density $\delta_c(E_{k,m})$ *and*

$$\delta_c(E_{k,m}) = \frac{1}{\varphi(m)},$$

where φ *is the Euler function. In other words,*

$$\lim_{(s\to 1^+)} \boldsymbol{\mu}_s(m\mathbb{N}^* + k \,|\, \mathbb{P}) = \frac{1}{\varphi(m)}.$$

4. Comparison between Asymptotic and Analytic Densities

Before we give two general theorems which characterize analytic density of subsets of $\mathbb{N}^*$, we obtain the following theorem which states one result of comparison between asymptotic density and analytic density for a subset of $\mathbb{N}^*$.

Theorem 4.1. *Let* E *be a subset of* $\mathbb{N}^*$. *Consider the following two properties*:

$$\begin{cases} (p_1) : \lim\limits_{(n\to+\infty)} \mathbf{v}_n(E) = \lim\limits_{(n\to+\infty)} \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} I_E(k) \text{ exists } (= d(E)), \\ (p_2) : \lim\limits_{(s\to 1^+)} \boldsymbol{\mu}_s(E) = \lim\limits_{(s\to 1^+)} \dfrac{1}{\zeta(s)}\displaystyle\sum_{n=1}^{+\infty} \dfrac{I_E(n)}{n^s} \text{ exists } (= \delta(E)). \end{cases}$$

Then $(p_1) \Rightarrow (p_2)$ *and we have*, $d(E) = \delta(E)$.

The converse of above is false.

In other words, if E admits an asymptotic density $d(E)$, then it admits an analytic density $\delta(E)$ and these two densities are equal $(d(E) = \delta(E))$. The converse is false.

So, if we denote a class of subsets of $\mathbb{N}^*$ which admits an asymptotic density by $\mathfrak{D}$ and a class of subsets of $\mathbb{N}^*$ which admits an analytic density by $\mathcal{E}$, then we obtain a strict inclusion $\mathfrak{D} \subset \mathcal{E}$.

Moreover, analytic density δ on $\mathcal{E}$ is an extension of asymptotic density d on $\mathfrak{D}$.

Proof. For a direct proof see [4]. For the converse an example is given by Theorem 2.6, Corollary 2.7 and Corollary 2.8 in [2].

Theorem 4.2. *Let E be a subset of* $\mathbb{N}^*$ *such that*

$$\sum_{n \in E} \frac{1}{n} < +\infty.$$

Then E admits an analytic density $\delta(E)$ *and* $\delta(E) = 0$.

The converse is not necessarily true.

Proof. The direct conclusion follows by noting that for all $s > 1$, we have

$$0 \leq \boldsymbol{\mu}_s(E) = \frac{1}{\zeta(s)} \sum_{n=1}^{+\infty} \frac{I_E(n)}{n^s} \leq \frac{1}{\zeta(s)} \sum_{n \in E} \frac{I_E(n)}{n} \leq \frac{C}{\zeta(s)} \to 0, \text{ as } s \to 1^+.$$

For the converse, we take $f(p) = I_{\mathbb{P}}(p)$, where

$$I_{\mathbb{P}}(p) = \begin{cases} 1, & \text{if } p \in \mathbb{P}, \\ 0, & \text{if } p \notin \mathbb{P}, \end{cases}$$

is the indicator function of the set of prime numbers $\mathbb{P}$.

Then, by Proposition 2.5, $\delta(\mathbb{P})$ exists and $\delta(\mathbb{P}) = 0$. But

$$\sum_{p \in \mathbb{P}} \frac{1}{p}$$

diverges.

5. Derived Analytic Density

A subset E of $\mathbb{N}^*$ admits an analytic density $\delta(E)$ equal to ℓ, $\ell \in [0, 1]$, if

$$\sum_{k=1}^{+\infty} \frac{I_E(k)}{k^s} \sim \frac{\ell}{(s-1)}, \text{ as } s \text{ tends to } 1^+.$$

Definition 5.1. Let E be a subset of $\mathbb{N}^*$. We say that E admits a *derived analytic density* ℓ, ($\ell \in [0, 1]$) or a $\boldsymbol{\mu}'_s(E)$-*density* ℓ, if

$$\sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k^s} \sim \frac{\ell}{(s-1)^2}, \text{ as } s \text{ tends to } 1^+.$$

Corollary 5.1. *$\boldsymbol{\mu}'_s(E)$-density ℓ means that*

$$\sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k k^{s-1}} \sim \frac{\ell}{(s-1)^2}, \text{ as } s \text{ tends to } 1^+,$$

or

$$\sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k k^t} \sim \frac{\ell}{t^2}, \text{ as } t \text{ tends to } 0^+,$$

so, this is a density related to the sequence $(\boldsymbol{\mu}_t)_{t>0}$ of measures defined by

$$\boldsymbol{\mu}_t := \sum_{k=1}^{+\infty} \frac{Log\, k}{k^{t+1}} \varepsilon_k,$$

where ε_k is the Dirac measure on $\mathbb{N}^*$, defined by the unit mass placed at the point k.

Its discrete Laplace transform is of the form

$$g(t) = \sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k} \exp(-t\, Log\, k) = \sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k k^t},$$

and the distribution function F is given by

$$F(x) = \sum_{Log\, k \le x} \frac{I_E(k) Log\, k}{k}.$$

By Tauberian theorem [1, (I.6), p. 30],

$$g(t) \sim \frac{\ell}{t^2}, \text{ as } t \text{ tends to zero,}$$

if and only if

$$F(x) \sim \frac{\ell}{2} x^2, \text{ as } x \text{ tends to infinity.}$$

If

$$x = Log\, n, \quad t = s - 1,$$

then

$$\sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k^s} \sim \frac{\ell}{(s-1)^2},$$

if and only if

$$\sum_{k=1}^{n} \frac{I_E(k) Log\, k}{k} \sim \frac{\ell\, Log^2 n}{2}.$$

Corollary 5.2. *Let E be a subset of* $\mathbb{N}^*$ *and let* ℓ *be a real number in* $[0, 1]$. *Then the following three properties are equivalent*:

$$\begin{cases} (p_1): E \text{ admits a } \boldsymbol{\mu}'_s\text{-density } \ell, \\ (p_2): \displaystyle\sum_{k=1}^{+\infty} \frac{I_E(k) Log\, k}{k^s} \sim \frac{\ell}{(s-1)^2}, \text{ as } (s \to 1^+), \\ (p_3): \displaystyle\sum_{k=1}^{n} \frac{I_E(k) Log\, k}{k} \sim \frac{\ell Log^2 n}{2}, \text{ as } (n \to +\infty). \end{cases}$$

Definition 5.2. Let E be a subset of $\mathbb{N}^*$. For a real number $s > 1$,

$$\mathbb{E}_s(E) := \frac{1}{Log\, \zeta(s)} \sum_{k \geq 2} \frac{I_E(k)}{(k\, Log\, k)^s}.$$

We call *iterated analytic density* of the set E to be of order 2 if $\lim_s \mathbb{E}_s(E)$ exists when s tends to 1^+.

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References

[1] N. Daili, Contributions à l'étude des densités, Thèse de Doctorat des Mathématiques, Pub(443/TS-22), I.R.M.A.- C.N.R.S., Strasbourg. I, France, 1991.

[2] N. Daili, Asymptotic densities in number theory. Part I: A survey, JP Jour. Algebra, Number Theory & Appl. 5(3) (2005), 513-533.

[3] N. Daili, Probabilistic zeta law, Far East J. Math. Sci. (FJMS) 18(1) (2005), 31-48.

[4] N. Daili, Analytic densities in number theory. Part II: Analytic densities of arithmetic functions, 2005, pp. 1-16, preprint.

[5] A. Fuchs and R. A. Giuliano, Théorie Générale des Densités, Pub. I.R.M.A., Strasbourg. I, 1989.

[6] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford Univ. Press, London, New York, 1949.

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