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ON A SUBCLASS OF CERTAIN HARMONIC MEROMORPHIC FUNCTIONS

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Abstract

In this paper, we introduce a new subclass of harmonic meromorphic functions. Coefficient bounds, distortion bounds, extreme points, convolution conditions and convex linear combinations for the functions belonging to this class are obtained.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a domain $D \subset C$ if both u and v are real harmonics in D. In any simply connected domain we write $f = h + \overline{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'| > |g'|$ in D (see [1]). Hengartner and Schober [2] investigated functions harmonic in the exterior of the unit disk $\widetilde{U} = \{z : |z| > 1\}$, among other things they showed that complex valued, harmonic, orientation preserving univalent mapping

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f, defined in $\widetilde{U}$ and satisfying $f(\infty) = \infty$, must admit the representation

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where $h(z)$ and $g(z)$ are defined by

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^{-n}, \quad |z| \in \widetilde{U}. \tag{1.2}$$

For $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $k \geq 0$ and $0 \leq \gamma \leq 2\pi$, let $M_H(\alpha, \lambda, k)$ consist of functions f satisfying the conditions

$$\operatorname{Re}\left\{(1 + ke^{i\gamma})\left(\frac{zf'(z)}{\lambda zf'(z) + (1-\lambda)f(z)}\right) - ke^{i\gamma}\right\} \geq \alpha. \tag{1.3}$$

Also let $M_{\overline{H}}(\alpha, \lambda, k)$ be the subclass of $M_H(\alpha, \lambda, k)$ consisting of functions $f = h + \overline{g}$ for which

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^{-n}, \quad g(z) = -\sum_{n=1}^{\infty} b_n z^{-n}, \quad a_n \geq 0, \quad b_n \geq 0. \tag{1.4}$$

Note that the class of harmonic meromorphic starlike functions has been studied by Jahangiri [3] and Jahangiri and Silverman [4]. We also note the class $M_H(\alpha, 0, 1)$ of harmonic meromorphic starlike functions studied by Rosy et al. [5].

Here we state a result due to Jahangiri [3], which we will use throughout this paper.

Theorem 1.1. *Let* $f = h + \overline{g}$ *with h and g of the form* (1.2). *If*

$$\sum_{n=1}^{\infty} [(n + \alpha)|a_n| + (n - \alpha)|b_n|] \leq 1 - \alpha, \tag{1.5}$$

then f is harmonic, orientation preserving and univalent in $\widetilde{U}$.

2. Coefficient Bounds

First we prove a sufficient coefficient bound.

Theorem 2.1. *Let* $f = h + \overline{g}$ *with h and g be given by* (1.2). *If*

$$\sum_{n=1}^{\infty} (\{n(1+k) - (\alpha + k)(\lambda(n+1) - 1)\} \mid a_n \mid$$

$$+ \{n(1+k) - (\alpha + k)(\lambda(n-1) + 1)\} \mid b_n \mid) \leq 1 - \alpha, \tag{2.1}$$

then f is harmonic, orientation preserving and univalent in $\widetilde{U}$ *and* $f \in M_H(\alpha, \lambda, k)$.

Proof. Consider the function $f = h + \overline{g}$, where h and g are given by (1.2). In [4] it has been proved that if $\sum_{n=1}^{\infty} n(\mid a_n \mid + \mid b_n \mid) \leq 1$, then f is harmonic, orientation preserving and univalent in $\widetilde{U}$. For $0 \leq \lambda < 1$, note that

$$n \leq (n(1+k) - (\lambda(n+1) - 1)(\alpha + k))/1 - \alpha$$

and

$$n \leq (n(1+k) - (\lambda(n-1) + 1)(\alpha + k))/1 - \alpha.$$

Suppose that (2.1) holds. Then we have

$$\operatorname{Re}\left\{\left[\frac{zh'(z) - \overline{zg'(z)}}{\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g}(z))}\right](1 + ke^{i\gamma}) - ke^{i\gamma}\right\} \geq \alpha, \tag{2.2}$$

where $z = re^{i\gamma}$, $0 \leq \gamma \leq 2\pi$, $0 \leq r < 1$, $k \geq 0$, $0 \leq \alpha < 1$ and $0 \leq \lambda \leq 1$.

Now let

$$A(z) = (1 + ke^{i\gamma})[zh'(z) - \overline{zg'(z)}]$$

$$- ke^{i\gamma}[\lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g}(z))] \tag{2.3}$$

and

$$B(z) = \lambda(zh'(z) - \overline{zg'(z)}) + (1-\lambda)(h(z) + \overline{g}(z)). \tag{2.4}$$

For $0 \leq \alpha < 1$, we observe that

$$\mid A(z) + (1-\alpha)B(z) \mid - \mid A(z) - (1+\alpha)B(z) \mid \geq 0. \tag{2.5}$$

From (2.3) and (2.4), we note that

$$|A(z) + (1-\alpha)B(z)|$$

$$= |((1-\alpha) - ke^{i\lambda})(\lambda z h'(z) + (1-\lambda)h(z)) + (1 + ke^{i\gamma})(zh'(z))$$

$$+ \overline{((1-\alpha) - ke^{i\lambda})(-\lambda z g'(z) + (1-\lambda)g(z)) + (1 + ke^{i\gamma})(-zg'(z))}|$$

$$\left| (2-\alpha)z - \sum_{n=1}^{\infty} (n(1 + ke^{i\lambda}) + (\lambda(n+1) - 1)(1 - \alpha - ke^{i\lambda}))a_n z^{-n} \right.$$

$$\left. - \sum_{n=1}^{\infty} (n(1 + ke^{i\lambda}) + (\lambda(n-1) + 1)(1 - \alpha - ke^{i\lambda}))b_n z^{-n} \right|$$

$$\geq (2-\alpha)|z| - \sum_{n=1}^{\infty} (n(1+k) + (\lambda(n+1) - 1)(1 - \alpha - k))|a_n||z|^{-n}$$

$$- \sum_{n=1}^{\infty} (n(1+k) + (\lambda(n-1) + 1)(1 - \alpha - k))|b_n||z|^{-n}$$

and

$$|A(z) - (1+\alpha)B(z)|$$

$$= |((1+\alpha) + ke^{i\lambda})(\lambda z h'(z) + (1-\lambda)h(z)) - (1 + ke^{i\gamma})(zh'(z))$$

$$+ \overline{((1+\alpha) + ke^{i\lambda})(-\lambda z g'(z) + (1-\lambda)g(z)) - (1 + ke^{i\gamma})(-zg'(z))}|$$

$$\left| \alpha z + \sum_{n=1}^{\infty} (n(1 + ke^{i\lambda}) - (\lambda(n+1) - 1)(1 + \alpha + ke^{i\lambda}))a_n z^{-n} \right.$$

$$\left. + \sum_{n=1}^{\infty} (n(1 + ke^{i\lambda}) - (\lambda(n-1) + 1)(1 + \alpha + ke^{i\lambda}))b_n z^{-n} \right|$$

$$\geq \alpha|z| + \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n+1) - 1)(1 + \alpha + k))|a_n||z|^{-n}$$

$$+ \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n-1) + 1)(1 + \alpha + k))|b_n||z|^{-n},$$

then

$$| A(z) + (1-\alpha) B(z) | - | A(z) - (1+\alpha) B(z) |$$

$$\geq 2(1-\alpha)| z | - 2 \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n+1)-1)(-\alpha - k)) | a_n || z |^{-n}$$

$$- 2 \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n-1)+1)(\alpha + k)) | b_n || z |^{-n}$$

$$= 2| z | \left\{ 1 - \alpha - \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n+1)-1)(-\alpha - k)) | a_n || z |^{-n-1} \right.$$

$$\left. - \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n-1)+1)(\alpha + k)) | b_n || z |^{-n-1} \right\}$$

$$\geq 2 \left\{ 1 - \alpha - \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n+1)-1)(\alpha + k)) | a_n | \right.$$

$$\left. - \sum_{n=1}^{\infty} (n(1+k) - (\lambda(n-1)+1)(\alpha + k)) | b_n | \right\} \geq 0,$$

by (2.1).

We next show that the condition (2.1) is also necessary for functions in $M_{\overline{H}}(\alpha, \lambda, k)$.

Theorem 2.2. *Let* $f = h + \overline{g}$ *with* h *and* g *be given by* (1.4). *Then* $f \in M_{\overline{H}}(\alpha, \lambda, k)$ *if and only if the inequality* (2.1) *holds for the coefficient of* $f = h + \overline{g}$.

Proof. In view of Theorem 2.1, we need only to show that $f \notin M_{\overline{H}}(\alpha, \lambda, k)$ if the condition (2.1) does not hold. We note that for $f \in M_{\overline{H}}(\alpha, \lambda, k)$, by (1.3) the condition (2.2) must be satisfied for all values of z in $\widetilde{U}$. Substituting for h, g, h', g', h'' and g'' in (2.2) and choosing values of z on the real axis where $0 \leq z = r > 1$, we are required to have $\operatorname{Re}\{A(z)/B(z)\} \geq 0$, where

$$A(z) = (1 - \alpha) - \sum_{n=1}^{\infty} (\{n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})[(\lambda(n+1) - 1)]\} | a_n |$$

$$+ \{n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})[(\lambda(n-1) + 1)]\} | b_n || z |^{-n-1})$$

and

$$B(z) = 1 + \sum_{n=1}^{\infty} (1 - \lambda(n+1)) | a_n || z |^{-n-1}$$

$$- \sum_{n=1}^{\infty} (1 - \lambda(n-1)) | b_n || z |^{-n-1}.$$

For $\operatorname{Re}(e^{i\gamma}) \leq | e^{i\gamma} | = 1$ the required condition $\operatorname{Re}\{A(z)/B(z)\} \geq 0$ is equivalent to

$$\frac{(1 - \alpha) - \sum_{n=1}^{\infty} (\{n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n+1) - 1)\} | a_n | + \{n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n-1) + 1)\} | b_n |) r^{-n-1}}{1 + \sum_{n=1}^{\infty} (1 - \lambda(n+1)) | a_n || r |^{-n-1} - \sum_{n=1}^{\infty} (1 - \lambda(n-1)) | b_n | r^{-n-1}}. \tag{2.6}$$

If the condition (2.1) does not hold, then the numerator of (2.6) is negative for z sufficiently close to 1. Thus there exists a $z_0 = r_0 > 1$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f \in M_{\overline{H}}(\alpha, \lambda, k)$ and so the proof is complete.

3. Distortion Bounds and Extreme Points

In this section, we shall obtain distortion bounds for functions in $M_{\overline{H}}(\alpha, \lambda, k)$ and also provide extreme points for the class $M_{\overline{H}}(\alpha, \lambda, k)$.

Theorem 3.1. *If* $f \in M_{\overline{H}}(\alpha, \lambda, k)$, *for* $0 \leq \alpha < 1$ *and* $|z| = r > 1$, *then*

$$r - (1-\alpha)r^{-1} \leq |f(z)| \leq r + (1-\alpha)r^{-1}.$$

Proof. We only prove the right hand inequality. The argument for left hand inequality is similar and will be omitted. Let $f \in M_{\overline{H}}(\alpha, \lambda, k)$. Taking the absolute value of f, we obtain

$$|f(z)| = \left| z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z}^{-n} \right| \leq r + \sum_{n=1}^{\infty} (a_n + b_n) r^{-n}$$

$$\leq r + \sum_{n=1}^{\infty} (a_n + b_n) r^{-1}$$

$$\leq r + \sum_{n=1}^{\infty} (\{n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n+1) - 1)\} |a_n|$$

$$+ \{n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n-1) + 1) |b_n|\}) r^{-1}$$

$$\leq r + (1-\alpha) r^{-1}.$$

Theorem 3.2. $f \in M_{\overline{H}}(\alpha, \lambda, k)$ *if and only if f can be expressed as*

$$f(z) = \sum_{n=0}^{\infty} (x_n h_n + y_n g_n), \tag{3.1}$$

where

$$z \in \widetilde{U},\ h_0(z) = z,\ h_n(z) = z + \frac{(1-\alpha)}{(n(1+k) - (\lambda(n+1) - 1)(\alpha + k))} z^{-n},$$

$$g_0(z) = z,\ g_n(z) = z - \frac{(1-\alpha)}{(n(1+k) - (\lambda(n-1) + 1)(\alpha + k))} \overline{z}^{-n},\ (n = 1, 2, \ldots),$$

$$\sum_{n=0}^{\infty} (x_n + y_n) = 1,\ x_n \geq 0\ \text{and}\ y_n \geq 0.$$

Proof. Note that for f we may write

$$f(z) = \sum_{n=0}^{\infty}(x_n h_n + y_n g_n)$$

$$= x_0 h_0 + y_0 g_0 + \sum_{n=1}^{\infty}\Bigg[x_n\left(z + \frac{(1-\alpha)}{(n(1+k)-(\lambda(n+1)-1)(\alpha+k))}z^{-n}\right)$$

$$+ y_n\left(z - \frac{(1-\alpha)}{(n(1+k)-(\lambda(n-1)+1)(\alpha+k))}\overline{z}^{-n}\right)\Bigg]$$

$$= \sum_{n=0}^{\infty}(x_n + y_n)z + \sum_{n=1}^{\infty}\frac{(1-\alpha)}{(n(1+k)-(\lambda(n+1)-1)(\alpha+k))}x_n z^{-n}$$

$$-\sum_{n=1}^{\infty}\frac{(1-\alpha)}{(n(1+k)-(\lambda(n-1)+1)(\alpha+k))}y_n\overline{z}^{-n}.$$

Now the first part of the proof is complete, since by Theorem 2.2,

$$\sum_{n=1}^{\infty}\Bigg[(n(1+k)-(\lambda(n+1)-1)(\alpha+k))\left(\frac{(1-\alpha)}{(n(1+k)-(\lambda(n+1)-1)(\alpha+k))}x_n\right)$$

$$+(n(1+k)-(\lambda(n-1)+1)(\alpha+k))\left(\frac{(1-\alpha)}{(n(1+k)-(\lambda(n-1)+1)(\alpha+k))}y_n\right)\Bigg]$$

$$= (1-\alpha)\sum_{n=1}^{\infty}(x_n + y_n) \le 1-\alpha.$$

Conversely, suppose that $f \in M_{\overline{H}}(\alpha, \lambda, k)$. Then

$$\sum_{n=1}^{\infty}\Bigg[\frac{(n(1+k)-(\lambda(n+1)-1)(\alpha+k))}{1-\alpha}a_n$$

$$+\frac{(n(1+k)-(\lambda(n-1)+1)(\alpha+k))}{1-\alpha}b_n\Bigg] \le 1.$$

Setting

$$x_n = \frac{(n(1+k) - (\lambda(n+1) - 1)(\alpha + k))}{1 - \alpha} a_n,$$

$$y_n = \frac{(n(1+k) - (\lambda(n-1) + 1)(\alpha + k))}{1 - \alpha} b_n, \quad 0 \le x_0 \le 1$$

and

$$y_0 = 1 - x_0 - \sum_{n=1}^{\infty} (x_n + y_n),$$

we obtain $f(z) = \sum_{n=0}^{\infty} (x_n h_n + y_n g_n)$ as required.

4. Convolution and Convex Linear Combination

In this section, we show that the class $M_{\overline{H}}(\alpha, \lambda, k)$ is invariant under convolution and convex linear combination of its members.

For harmonic functions

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^{-n} - \sum_{n=1}^{\infty} b_n \overline{z}^{-n}$$

and

$$F(z) = z + \sum_{n=1}^{\infty} A_n z^{-n} - \sum_{n=1}^{\infty} B_n \overline{z}^{-n},$$

the convolution of f and F is given by

$$(f * F)(z) = f(z) * F(z) = z + \sum_{n=1}^{\infty} a_n A_n z^{-n} - \sum_{n=1}^{\infty} b_n B_n \overline{z}^{-n}. \qquad (4.1)$$

Theorem 4.1. *For* $0 \le \beta \le \alpha \le 1$, *let* $f \in M_{\overline{H}}(\alpha, \lambda, k)$ *and* $F \in M_{\overline{H}}(\beta, \lambda, k)$. *Then* $f * F \in M_{\overline{H}}(\alpha, \lambda, k) \subset M_{\overline{H}}(\beta, \lambda, k)$.

Proof. Suppose f and F are in $M_{\overline{H}}(\alpha, \lambda, k)$ so that $f * F$ is given by above convolution. Since $f \in M_{\overline{H}}(\alpha, \lambda, k)$ and $F \in M_{\overline{H}}(\beta, \lambda, k)$, the coefficient of f and F must satisfy conditions given in Theorem 2.2. So for the coefficient of $f * F$ we can write

$$\sum_{n=1}^{\infty}(\{n(1+ke^{i\lambda})-(\alpha+ke^{i\lambda})(\lambda(n+1)-1)\}a_nA_n$$

$$+\{n(1+ke^{i\lambda})-(\alpha+ke^{i\lambda})(\lambda(n-1)+1)\}b_nB_n)$$

$$\leq \sum_{n=1}^{\infty}(\{n(1+ke^{i\lambda})-(\alpha+ke^{i\lambda})(\lambda(n+1)-1)\}a_n$$

$$+\{n(1+ke^{i\lambda})-(\alpha+ke^{i\lambda})(\lambda(n-1)+1)\}b_n).$$

The right hand side of the above inequality is bounded by $1-\alpha$ because $f \in M_{\overline{H}}(\alpha, \lambda, k)$. Thus $f * F \in M_{\overline{H}}(\alpha, \lambda, k) \subset M_{\overline{H}}(\beta, \lambda, k)$.

Finally, we examine the convex combination of $M_{\overline{H}}(\alpha, \lambda, k)$.

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \ldots,$ by

$$f_j(z) = z + \sum_{n=1}^{\infty} a_{n,j}z^{-n} - \sum_{n=1}^{\infty} b_{n,j}\overline{z}^{-n}, \quad a_{n,j} \geq 0, \ b_{n,j} \geq 0. \tag{4.2}$$

Theorem 4.2. *Let the functions* $f_j(z)$ *defined by* (4.2) *be in the class* $M_{\overline{H}}(\alpha, \lambda, k)$ *for every* $j = 1, 2, \ldots$. *Then the functions* $t(z)$ *defined by*

$$t(z) = \sum_{j=1}^{\infty} c_j f_j(z), \quad (0 \leq c_j \leq 1) \tag{4.3}$$

are also in the class $M_{\overline{H}}(\alpha, \lambda, k)$, *where* $\sum_{j=1}^{m} c_j = 1$.

Proof. According to the definition of t, we can write

$$t(z) = z + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j a_{n,j}\right) z^{-n} - \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} c_j b_{n,j}\right)\overline{z}^{-n}. \tag{4.4}$$

Further, since $f_j(z)$ are in $M_{\overline{H}}(\alpha, \lambda, k)$ for every $(j = 1, 2, \ldots)$, by (2.2) we have

$$\sum_{n=1}^{\infty}\Bigg\{(n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n+1) - 1)) \sum_{j=1}^{\infty} c_j a_{n,j}$$

$$+ (n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n-1) + 1)) \sum_{j=1}^{\infty} c_j b_{n,j}\Bigg\}$$

$$= \sum_{j=1}^{\infty} c_j \Bigg\{\sum_{n=1}^{\infty}[(n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n+1) - 1))|\, a_{n,j}\,|$$

$$+ (n(1 + ke^{i\lambda}) - (\alpha + ke^{i\lambda})(\lambda(n-1) + 1))|\, b_{n,j}\,|]\Bigg\}$$

$$\leq \sum_{j=1}^{\infty} c_j (1 - \alpha) \leq (1 - \alpha).$$

Hence the theorem follows.

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