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THE GROWTH SERIES OF THE WEYL GROUP OF TYPE $A_1^{(1,1,1,1)*}(12)$

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Abstract

We calculate the growth series of the Weyl group of the 4-extended affine root system of type A_1 (of index 15), which is denoted by $A_1^{(1,1,1,1)*}(12)$. The 4-extended affine root systems of type A_1 are classified into 12 types similarly to [1, 2].

1. Introduction

In 1985, Saito [5] introduced the notion of an extended affine root system and especially classified 2-extended affine root systems associated to the elliptic singularities. In 1997, Allison et al. [1] also introduced the extended affine root systems associated to the extended affine Lie algebras and gave a complete description of them by using the concept of a semilattice. In the cases of n-extended affine root systems, Azam and Shahsanaei [2, 3] have given a presentation of the corresponding Weyl groups. After that in the cases of the 3 and 4-extended affine root systems, the author described them in terms of the 3 and 4-extended

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affine diagrams [7, 8]. The 4-extended affine root systems of type A_1 are classified into 12 types, from index 4 to index 15. The growth series (also called *Poincaré series*) $W(t)$ of a group W with respect to a given generator system is defined by $W(t) = \sum_{w \in W} t^{l(w)}$ [4], where t is an indeterminate and $l(w)$ is the length of a minimal expression of an element w in W in terms of the given generator system. In the previous papers [9, 10, 11], we examined the growth series of the Weyl groups of the cases of index 4, 5, 6 and 7. In this paper, in the case of the 4-extended affine root system of type A_1 (of index 15) $(:= A_1^{(1,1,1,1)*}(12))$, we calculate the growth series of the corresponding Weyl group.

2. The Weyl Group of Type $A_1^{(1,1,1,1)*}(12)$

The 4-extended affine root systems of type A_1 are classified into 12 types similarly to [1, 2], from index 4 to index 15. The root system of type $A_1^{(1,1,1,1)*}(12)$ (of index 15) is given as follows:

Type $A_1^{(1,1,1,1)*}(12)$

$$R : \pm\varepsilon + nb + ma + kc + ld, \ (n, m, k, l \in Z),$$

where $\varepsilon := \varepsilon_1 - \varepsilon_2$.

The Weyl group of the 4-extended affine root system is defined as follows [1, 5]: Let V be an $(l+4)$-dimensional real vector space equipped with a positive semi-definite bilinear form. Let V^0 be the 4-dimensional radical of the form $\langle\ ,\ \rangle$ and $(V^0)^*$ be the dual space of V^0. Set $V = \dot{V} \oplus V^0$, and $\widetilde{V} = \dot{V} \oplus V^0 \oplus (V^0)^*$. Let $\{\varepsilon_1, \ldots, \varepsilon_l\}$ be the standard basis of $\dot{V}$ satisfying $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$ for all $i, j = 1, \ldots, l$. We define the bilinear form $\langle\ ,\ \rangle$ on $\widetilde{V}$, so that $\langle\ ,\ \rangle$ extends the form on V and $\langle\ ,\ \rangle$ is nondegenerate on $\widetilde{V}$. For $\alpha \in R$, we define the reflection $w_\alpha \in GL(\widetilde{V})$ by $w_\alpha(u) = u - \langle u, \alpha^\vee \rangle \alpha \ (u \in \widetilde{V})$ with $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$. Set $\widetilde{W}_R = \langle w_\alpha \mid \alpha \in R \rangle$

$\subseteq GL(\widetilde{V})$. Then $\widetilde{W}_R$ is the Weyl group of the 4-extended affine root system R. Further, let us denote $\dot{w}_\alpha$ be the reflection in $GL(V)$ such that $w_\alpha|_V = \dot{w}_\alpha$, and set $W_R = \langle \dot{w}_\alpha \mid \alpha \in R \rangle$. In this paper, we call W_R the *Weyl group of* R and $\widetilde{W}_R$ the *central extension of* W_R. In the sequel, we set $\alpha_1 = \varepsilon$, $\alpha_0 = -\alpha_1 + b$, $\alpha_i^* = \alpha_i + a$, $\widetilde{\alpha}_i = \alpha_i + c$, $\underline{\alpha}_i = \alpha_i + d$, $\widetilde{\alpha}_i^* = \alpha_i + a + c$, $\underline{\alpha}_i^* = \alpha_i + a + d$, $\underline{\widetilde{\alpha}}_i = \alpha_i + c + d$, $\underline{\widetilde{\alpha}}_i^* = \alpha_i + a + c + d$ $(i = 0, 1)$, and set the corresponding reflections $w_i := w_{\alpha_i}$, $w_i^* := w_{\alpha_i^*}$, $\widetilde{w}_i := w_{\widetilde{\alpha}_i}$, $\underline{w}_i := w_{\underline{\alpha}_i}$, $\widetilde{w}_i^* = w_{\widetilde{\alpha}_i^*}$, $\underline{w}_i^* = w_{\underline{\alpha}_i^*}$, $\underline{\widetilde{w}}_i = w_{\underline{\widetilde{\alpha}}_i}$, $\underline{\widetilde{w}}_i^* = w_{\underline{\widetilde{\alpha}}_i^*}$ $(i = 0, 1)$.

Proposition 2.1. *The Weyl group* W_R *of type* $A_1^{(1,1,1,1)*}(12)$ *is described as follows*:

Generators: $w_i, w_i^*, \widetilde{w}_i, \underline{w}_i, \widetilde{w}_i^*, \underline{w}_i^*, \underline{\widetilde{w}}_i, \underline{\widetilde{w}}_i^*$ $(i = 0, 1)$.

Relations: $w_i^2 = w_i^{*2} = \widetilde{w}_i^2 = \underline{w}_i^2 = \widetilde{w}_i^{*2} = \underline{w}_i^{*2} = \underline{\widetilde{w}}_i^2 = \underline{\widetilde{w}}_i^{*2} = 1$ $(i = 0, 1)$,

$A^*B^* = AB$, *for* $A, B \in \{w_i, \widetilde{w}_i, \underline{w}_i, \underline{\widetilde{w}}_i\}$, $\widetilde{A}\widetilde{B} = AB$, *for* $A, B \in \{w_i, w_i^*, \underline{w}_i, \underline{w}_i^*\}$, $\underline{A}\underline{B} = AB$, *for* $A, B \in \{w_i, w_i^*, \widetilde{w}_i, \widetilde{w}_i^*\}$ $(i = 0, 1)$ *and* $(uvw)^2 = 1$, *for all distinct* u, v *and* w *in the above generators*.

Proof. It is easily checked by direct calculations.

3. The Growth Series of $W(A_1^{(1,1,1,1)*}(12))$

Our main result is the following:

Theorem 3.1. *The growth series of the Weyl group* $W(A_1^{(1,1,1,1)*}(12))$ *with the above generator system is given as follows*:

$$\sum_{w \in W(A_1^{(1,1,1,1)*}(12))} t^{l(w)} = \frac{1 + 16t + 76t^2 + 176t^3 + 230t^4 + 166t^5 + 70t^6 + 2t^7 - t^8}{(1-t)^4(1+t)^4}.$$

In the sequel, by using the result [6], we prove the theorem. We set $T = w_1 w_0$, $R = w_1^* w_1$, $S = \tilde{w}_1 w_1$, $U = \underline{w}_1 w_1$, then $W_R \cong \langle w_1, T, R, S, U \,|\, w_1^2 = (w_1 T)^2 = (w_1 R)^2 = (w_1 S)^2 = (w_1 U)^2 = 1$, and T, R, S, U are all commutative to each other$\rangle$, and $W_R = \{S^k R^m T^n U^l,\ S^k R^m T^n U^l w_1\ (k, m, n, l \in \mathbb{Z})\}$. To calculate the growth series of $W(A_1^{(1,1,1,1)*}(12))$, similarly to the case of $W(A_1^{(1,1,1,1)*}(4))$, we divide $gp\langle w_1, T\rangle$ into four cases (I)~(IV) ([11]) and the same method [6]. From the generator and its expression, w_1, $w_1^* = Rw_1$, $\tilde{w}_1 = Sw_1$, $\underline{w}_1 = Uw_1$, $\tilde{w}_1^* = RSw_1$, $\underline{w}_1^* = RUw_1$, $\underline{\tilde{w}}_1 = SUw_1$, $\underline{\tilde{w}}_1^* = RSUw_1$, $w_0 = T^{-1}w_1$, $w_0^* = T^{-1}R^{-1}w_1$, $\tilde{w}_0 = T^{-1}S^{-1}w_1$, $\underline{w}_0 = T^{-1}U^{-1}w_1$, $\tilde{w}_0^* = T^{-1}R^{-1}S^{-1}w_1$, $\underline{w}_0^* = T^{-1}R^{-1}U^{-1}w_1$, $\underline{\tilde{w}}_0 = T^{-1}S^{-1}U^{-1}w_1$, $\underline{\tilde{w}}_0^* = T^{-1}R^{-1}S^{-1}U^{-1}w_1$. We see that the part sum $W(\mathrm{I})\ (n \ge 1) := \sum_{w \in W(\mathrm{I})(n \ge 1)} t^{l(w)} = W(\mathrm{II})$ and $W(\mathrm{III}) = W(\mathrm{IV})$.

The case of $W(\mathrm{I})\ (n = 0)$ is easily calculated, so we show the cases of (II) and (IV).

(II) $T^{-n} = (w_0 w_1)^n\ (n \ge 1)$.

(i) $R^{-l}T^{-n} = (w_0 w_1^*)^l (w_0 w_1)^{n-l}\ (0 \le l \le n)$.

$$U^p S^k R^{-l} T^{-n}\ (k \ge 1,\ p \ge 0) = (\underline{w}_1 w_1)^p (\tilde{w}_1 w_1)^k (w_0 w_1^*)^l (w_0 w_1)^{n-l}$$

$$= \begin{cases} p \ge k \Rightarrow 2(p+n) \\ p \le k-1 \Rightarrow 2(k+n). \end{cases}$$

$$U^p S^{-k} R^{-l} T^{-n}\ (k \ge 0,\ p \ge 0) = (\underline{w}_1 w_1)^p (w_1 \tilde{w}_1)^k (w_0 w_1^*)^l (w_0 w_1)^{n-l}$$

$$\Leftrightarrow (\underline{w}_1 w_1)^p (w_1 \tilde{w}_1)^k (w_0 w_1)^n$$

$$\Leftrightarrow \begin{cases} (\underline{w}_1 w_1)^p (w_1 \tilde{w}_1)^{k-2n} (\tilde{w}_0 \tilde{w}_1)^n\ (k \ge 2n) \\ \begin{cases} p \ge k-2n \Rightarrow 2(p+n) \\ p \le k-2n-1 \Rightarrow 2(k-n) \end{cases} \\ (\underline{w}_1 w_1)^p (w_0 w_1)^n (k \le 2n-1) \Rightarrow 2(p+n). \end{cases}$$

$$U^{-p}S^kR^{-l}T^{-n}\ \ (k \geq 1,\ p \geq 1) = (w_1\underline{w}_1)^p(\widetilde{w}_1w_1)^k(w_0^*w_1)^l(w_0w_1)^{n-l}$$

$$\Leftrightarrow (\widetilde{w}_1w_1)^k(w_1\underline{w}_1)^p(w_0w_1)^n$$

$$= \begin{cases} \text{(i) } p \geq 2n \\ k \geq p - 2n \Rightarrow 2(k+n) \\ k \leq p - 2n - 1 \Rightarrow 2(p-n) \\ \text{(ii) } p \leq 2n - 1 \Rightarrow 2(k+n). \end{cases}$$

$$U^{-p}S^{-k}R^{-l}T^{-n}\ \ (k \geq 0,\ p \geq 1) = (w_1\underline{w}_1)^p(w_1\widetilde{w}_1)^k(w_0^*w_1)^l(w_0w_1)^{n-l}$$

$$\Leftrightarrow (w_1\underline{w}_1)^p(w_1\widetilde{w}_1)^k(w_0w_1)^n$$

$$= \begin{cases} k \geq 2n,\ p \geq k \Rightarrow 2(p-n) \\ k \geq 2n,\ p \leq k-1 \Rightarrow 2(k-n) \\ k \leq 2n-1,\ p \geq 2n \Rightarrow 2(p-n) \\ k \leq 2n-1,\ p \leq 2n-1 \Rightarrow 2n. \end{cases}$$

(ii) $R^mT^{-n} = (w_1^*w_1)^m(w_0w_1)^n\ \ (m \geq 1)$.

$$U^pS^kR^mT^{-n}\ \ (k \geq 0,\ p \geq 0) = (\underline{w}_1w_1)^p(\widetilde{w}_1w_1)^k(w_1^*w_1)^m(w_0w_1)^n$$

$$= \begin{cases} k \geq m,\ p \geq k \Rightarrow 2(p+n) \\ k \geq m,\ p \leq k-1 \Rightarrow 2(k+n) \\ k \leq m-1,\ p \geq m \Rightarrow 2(p+n) \\ k \leq m-1,\ p \leq m-1 \Rightarrow 2(m+n). \end{cases}$$

$$U^pS^{-k}R^mT^{-n}\ \ (k \geq 1,\ p \geq 0) = (\underline{w}_1w_1)^p(w_1\widetilde{w}_1)^k(w_1^*w_1)^m(w_0w_1)^n$$

$$= \begin{cases} p \geq m,\ k \leq 2n+p \Rightarrow 2(p+n) \\ p \geq m,\ k \geq 2n+p+1 \Rightarrow 2(k-n) \\ p \leq m-1,\ k \leq 2n+m \Rightarrow 2(m+n) \\ p \leq m-1,\ k \geq 2n+m+1 \Rightarrow 2(k-n). \end{cases}$$

$$U^{-p}S^kR^mT^{-n}\ \ (k \geq 0,\ p \geq 1) = (w_1\underline{w}_1)^p(\widetilde{w}_1w_1)^k(w_1^*w_1)^m(w_0w_1)^n$$

$$\Leftrightarrow (\widetilde{w}_1w_1)^{k+m}(w_1\underline{w}_1)^p(w_0w_1)^n.$$

This is the same as the above case.

$$U^{-p}S^{-k}R^{m}T^{-n}\ (k \geq 1,\ p \geq 1) = (w_1\underline{w}_1)^p(w_1\tilde{w}_1)^k(w_1^*w_1)^m(w_0w_1)^n$$

$$= \begin{cases} k \leq p,\ 2n \leq p \leq 2n+m \Rightarrow 2(n+m) \\ k \leq p,\ p \geq 2n+m+1 \Rightarrow 2(p-n) \\ p \leq k-1,\ 2n \leq k \leq 2n+m \Rightarrow 2(m+n) \\ p \leq k-1,\ k \geq 2n+m+1 \Rightarrow 2(k-n) \\ p \leq 2n-1,\ k \leq 2n-1 \Rightarrow 2(m+n). \end{cases}$$

(iii) $R^{-m-n}T^{-n}\ (m \geq 1) = \begin{cases} (w_0w_1^*)^{m-n}(w_0^*w_1^*)^n\ (m \geq n+1) \\ (w_0^*w_1^*)^m(w_0w_1^*)^{n-m}\ (m \leq n). \end{cases}$

(a) When $m \geq n+1$,

$$U^{p}S^{k}R^{-m-n}T^{-n}\ (k \geq 0,\ p \geq 0) = (\underline{w}_1w_1)^p(\tilde{w}_1w_1)^k(w_1w_1^*)^{m-n}(w_0^*w_1^*)^n$$

$$= \begin{cases} \text{(i) } p \geq k \\ p \geq m-n \Rightarrow 2(p+n) \\ p \leq m-n-1 \Rightarrow 2m \\ \text{(ii) } p \leq k-1 \\ k \geq m-n \Rightarrow 2(k+n) \\ k \leq m-n-1 \Rightarrow 2m. \end{cases}$$

$$U^{p}S^{-k}R^{-m-n}T^{-n}\ (k \geq 1,\ p \geq 0) = (\underline{w}_1w_1)^p(w_1\tilde{w}_1)^k(w_1w_1^*)^{m-n}(w_0^*w_1^*)^n$$

$$\Leftrightarrow \begin{cases} (\underline{w}_1w_1)^p(w_1w_1^*)^{m-n}(w_1\tilde{w}_1)^{k-2n}(\tilde{w}_0^*\tilde{w}_1^*)^n\ (k \geq 2n) \\ = \begin{cases} \text{(i) } k \leq m+n \\ p \geq m-n \Rightarrow 2(p+n) \\ p \leq m-n-1 \Rightarrow 2m \\ \text{(ii) } k \geq m+n+1 \\ p \geq k-2n \Rightarrow 2(p+n) \\ p \leq k-2n-1 \Rightarrow 2(k-n). \end{cases} \\ (\underline{w}_1w_1)^p(w_1w_1^*)^{m-n}(w_0^*w_1^*)^n\ (k \leq 2n-1) \\ = \begin{cases} p \geq m-n \Rightarrow 2(p+n) \\ p \leq m-n-1 \Rightarrow 2m. \end{cases} \end{cases}$$

$$U^{-p}S^{k}R^{-m-n}T^{-n}\ (k \geq 0,\ p \geq 1) = \left(w_1\underline{w}_1\right)^p\left(\widetilde{w}_1 w_1\right)^k\left(w_1 w_1^*\right)^{m-n}\left(w_0^* w_1^*\right)^n.$$

This is the same as the above case.

$$U^{-p}S^{-k}R^{-m-n}T^{-n}\ (k \geq 1,\ p \geq 1) = \left(w_1\underline{w}_1\right)^p\left(w_1\widetilde{w}_1\right)^k\left(w_1 w_1^*\right)^{m-n}\left(w_0^* w_1^*\right)^n$$

$$= \begin{cases} 1 \leq p \leq m+n,\ 2n \leq k \leq p \Rightarrow 2m \\ p \geq m+n+1,\ 2n \leq k \leq p \Rightarrow 2(p-n) \\ 2n \leq k \leq m+n,\ p \leq k-1 \Rightarrow 2m \\ k \geq m+n+1,\ p \leq k-1 \Rightarrow 2(k-n) \\ 1 \leq k \leq 2n-1,\ 2n \leq p \leq m+n \Rightarrow 2m \\ 1 \leq k \leq 2n-1,\ p \geq m+n+1 \Rightarrow 2(p-n) \\ 1 \leq k \leq 2n-1,\ 1 \leq p \leq 2n-1 \Rightarrow 2m. \end{cases}$$

(b) When $m \leq n$,

$$U^{p}S^{k}R^{-m-n}T^{-n}\ (k \geq 0,\ p \geq 0) = \left(\underline{w}_1 w_1\right)^p\left(\widetilde{w}_1 w_1\right)^k\left(w_0^* w_1^*\right)^m\left(w_0 w_1^*\right)^{n-m}$$

$$= \begin{cases} p \geq k \Rightarrow 2(p+n) \\ p \leq k-1 \Rightarrow 2(k+n). \end{cases}$$

$$U^{p}S^{-k}R^{-m-n}T^{-n}\ (k \geq 1,\ p \geq 0) = \left(\underline{w}_1 w_1\right)^p\left(w_1\widetilde{w}_1\right)^k\left(w_0^* w_1^*\right)^m\left(w_0 w_1^*\right)^{n-m}$$

$$\Leftrightarrow \left(\underline{w}_1 w_1\right)^p\left(w_1\widetilde{w}_1\right)^k\left(w_0 w_1^*\right)^n$$

$$\Leftrightarrow \begin{cases} \left(\underline{w}_1 w_1\right)^p\left(w_1\widetilde{w}_1\right)^{k-2n}\left(\widetilde{w}_0\widetilde{w}_1^*\right)^n\ (k \geq 2n) \\ = \begin{cases} p \geq k-2n \Rightarrow 2(p+n) \\ p \leq k-2n-1 \Rightarrow 2(k-n) \end{cases} \\ \left(\underline{w}_1 w_1\right)^p\left(w_0 w_1^*\right)^n (k \leq 2n-1) \Rightarrow 2(p+n). \end{cases}$$

$$U^{-p}S^{k}R^{-m-n}T^{-n}\ (k \geq 0,\ p \geq 1) = \left(w_1\underline{w}_1\right)^p\left(\widetilde{w}_1 w_1\right)^k\left(w_0^* w_1^*\right)^m\left(w_0^* w_1\right)^{n-m}.$$

This is the same as the above case.

$$U^{-p}S^{-k}R^{-m-n}T^{-n}\ (k \geq 1,\ p \geq 1) = \left(w_1\underline{w}_1\right)^p\left(w_1\widetilde{w}_1\right)^k\left(w_0^* w_1^*\right)^m\left(w_0 w_1^*\right)^{n-m}$$

$$\Leftrightarrow \left(w_1\underline{w}_1\right)^p\left(w_1\widetilde{w}_1\right)^k\left(w_0 w_1^*\right)^n$$

$$= \begin{cases} k \geq 2n,\ p \geq k \Rightarrow 2(p-n) \\ k \geq 2n,\ p \leq k-1 \Rightarrow 2(k-n) \\ k \leq 2n-1,\ p \geq 2n \Rightarrow 2(p-n) \\ k \leq 2n-1,\ p \leq 2n-1 \Rightarrow 2n. \end{cases}$$

(IV) $T^{-n}w_1 = (w_0w_1)^{n-1}w_0\ (n \geq 1)$.

(i) $R^{-l}T^{-n}w_1 = (w_0w_1^*)^l(w_0w_1)^{n-l-1}w_0\ (0 \leq l \leq n-1)$.

$$U^pS^kR^{-l}T^{-n}w_1\ (k \geq 1,\ p \geq 0) = (\underline{w}_1w_1)^p(\widetilde{w}_1w_1)^k(w_0w_1^*)^l(w_0w_1)^{n-l-1}w_0$$

$$= \begin{cases} p \geq k \Rightarrow 2(p+n)-1 \\ p \leq k-1 \Rightarrow 2(k+n)-1. \end{cases}$$

$$U^pS^{-k}R^{-l}T^{-n}w_1\ (k \geq 0,\ p \geq 0) = (\underline{w}_1w_1)^p(w_1\widetilde{w}_1)^k(w_0w_1^*)^l(w_0w_1)^{n-l-1}w_0$$

$$\Leftrightarrow (\underline{w}_1w_1)^p(w_1\widetilde{w}_1)^k(w_0w_1)^{n-1}w_0$$

$$\Leftrightarrow \begin{cases} (\underline{w}_1w_1)^p(w_1\widetilde{w}_1)^{k-2n+1}(\widetilde{w}_0\widetilde{w}_1)^{n-1}\widetilde{w}_0\ (k \geq 2n-1) \\ = \begin{cases} p \geq k-2n+1 \Rightarrow 2(p+n)-1 \\ p \leq k-2n \Rightarrow 2(k-n)+1 \end{cases} \\ (\underline{w}_1w_1)^p(w_0w_1)^{n-1}w_0(k \leq 2n-2) \Rightarrow 2(p+n)-1. \end{cases}$$

$$U^{-p}S^kR^{-l}T^{-n}w_1\ (k \geq 1,\ p \geq 1) = (w_1\underline{w}_1)^{p-1}(\widetilde{w}_1w_1)^k(w_0^*w_1)^l(w_0w_1)^{n-l-1}\underline{w}_0$$

$$\Leftrightarrow (\widetilde{w}_1w_1)^k(w_1\underline{w}_1)^{p-1}(w_0w_1)^{n-1}\underline{w}_0$$

$$\Leftrightarrow \begin{cases} (\widetilde{w}_1w_1)^k(w_1\underline{w}_1)^{p-2n+1}(\underline{w}_0\underline{w}_1)^{n-1}\underline{w}_0\ (p \geq 2n-1) \\ = \begin{cases} k \geq p-2n+1 \Rightarrow 2(k+n)-1 \\ k \leq p-2n \Rightarrow 2(p-n)+1 \end{cases} \\ (\widetilde{w}_1w_1)^k(w_0w_1)^{n-1}\underline{w}_0\ (p \leq 2n-2) \Rightarrow 2(k+n)-1. \end{cases}$$

$$U^{-p}S^{-k}R^{-l}T^{-n}w_1\ (k \geq 0,\ p \geq 1) = (w_1\underline{w}_1)^{p-1}(w_1\widetilde{w}_1)^k(w_0w_1^*)^l(w_0w_1)^{n-l-1}\underline{w}_0$$

$$\Leftrightarrow (w_1\underline{w}_1)^{p-1}(w_1\widetilde{w}_1)^k(w_0w_1)^{n-1}\underline{w}_0$$

$$= \begin{cases} (w_1\underline{w}_1)^{p-1}(w_1\widetilde{w}_1)^{k-1}(w_0w_1)^{n-1}\underline{\widetilde{w}}_0 \ (k \geq 1) \\ = \begin{cases} k \geq 2n-1,\ p \geq k \Rightarrow 2(p-n)+1 \\ k \geq 2n-1,\ p \leq k-1 \Rightarrow 2(k-n)+1 \\ k \leq 2n-2,\ p \geq 2n-1 \Rightarrow 2(p-n)+1 \\ k \leq 2n-2,\ p \leq 2n-2 \Rightarrow 2n-1 \end{cases} \\ (w_1\underline{w}_1)^{p-1}(w_0w_1)^{n-1}\underline{w}_0 \ (k = 0) \\ = \begin{cases} p \geq 2n-1 \Rightarrow 2(p-n)+1 \\ p \leq 2n-2 \Rightarrow 2n-1. \end{cases} \end{cases}$$

(ii) $R^mT^{-n}w_1 = (w_1^*w_1)^m(w_0w_1)^{n-1}w_0 \ (m \geq 1)$.

$$U^pS^kR^mT^{-n}w_1 \ (k \geq 0,\ p \geq 0) = (\underline{w}_1w_1)^p(\widetilde{w}_1w_1)^k(w_1^*w_1)^m(w_0w_1)^{n-1}w_0.$$

$$= \begin{cases} k \geq m,\ p \geq k \Rightarrow 2(p+n)-1 \\ k \geq m,\ p \leq k-1 \Rightarrow 2(k+n)-1 \\ k \leq m-1,\ p \geq m \Rightarrow 2(p+n)-1 \\ k \leq m-1,\ p \leq m-1 \Rightarrow 2(m+n)-1. \end{cases}$$

$$U^pS^{-k}R^mT^{-n}w_1 \ (k \geq 1,\ p \geq 0) = (\underline{w}_1w_1)^p(w_1\widetilde{w}_1)^{k-1}(w_1^*w_1)^m(w_0w_1)^{n-1}\widetilde{w}_0$$

$$= \begin{cases} \text{(i) } k \geq 2n-1 \\ p \geq m,\ p \geq k-2n+1 \Rightarrow 2(p+n)-1 \\ p \geq m,\ p \leq k-2n \Rightarrow 2(k-n)+1 \\ p \leq m-1,\ k \leq m+2n-1 \Rightarrow 2(m+n)-1 \\ p \leq m-1,\ k \geq m+2n \Rightarrow 2(k-n)+1 \\ \text{(ii) } k \leq 2n-2 \\ p \geq m \Rightarrow 2(p+n)-1 \\ p \leq m-1 \Rightarrow 2(m+n)-1. \end{cases}$$

$$U^{-p}S^kR^mT^{-n}w_1 \ (k \geq 0,\ p \geq 1) = (w_1\underline{w}_1)^p(\widetilde{w}_1w_1)^k(w_1^*w_1)^m(w_0w_1)^{n-1}w_0.$$

This is the same as the above case.

$$U^{-p}S^{-k}R^mT^{-n}w_1 \ (k \geq 1,\ p \geq 1) = (w_1\underline{w}_1)^p(w_1\widetilde{w}_1)^{k-1}(w_1^*w_1)^m(w_0w_1)^{n-1}\underline{\widetilde{w}}_0$$

$$
\Leftrightarrow \begin{cases}
(w_1^* w_1)^m (w_1 \underline{w}_1)^{p-2n+1} (w_1 \tilde{w}_1)^{k-2n+1} (\underline{\tilde{w}}_0 \underline{\tilde{w}}_1)^{n-1} \underline{\tilde{w}}_0 \ (p \geq 2n-1,\ k \geq 2n-1) \\
= \begin{cases}
\text{(i) } p \geq k \\
p \leq m + 2n - 1 \Rightarrow 2(m+n) - 1 \\
p \geq m + 2n \Rightarrow 2(p-n) + 1 \\
\text{(ii) } p \leq k - 1 \\
k \leq m + 2n - 1 \Rightarrow 2(m+n) - 1 \\
k \geq m + 2n \Rightarrow 2(k-n) + 1
\end{cases} \\
(w_1^* w_1)^m (w_1 \underline{w}_1)^{p-2n+1} (\underline{w}_0 \underline{w}_1)^{n-1} \underline{\tilde{w}}_0 \ (p \geq 2n-1,\ k \leq 2n-2) \\
= \begin{cases}
p \leq 2n + m - 1 \Rightarrow 2(m+n) - 1 \\
p \geq 2n + m \Rightarrow 2(p-n) + 1
\end{cases} \\
(w_1^* w_1)^m (w_1 \tilde{w}_1)^{k-2n+1} (\tilde{w}_0 \tilde{w}_1)^{n-1} \underline{\tilde{w}}_0 \ (p \leq 2n-2,\ k \geq 2n-1) \\
= \begin{cases}
k \leq 2n + m - 1 \Rightarrow 2(n+m) - 1 \\
k \geq 2n + m \Rightarrow 2(k-n) + 1
\end{cases} \\
(w_1^* w_1)^m (w_0 w_1)^{n-1} \underline{\tilde{w}}_0 \ (p \leq 2n-2,\ k \leq 2n-2) \Rightarrow 2(m+n) - 1.
\end{cases}
$$

(iii) $R^{-m-n+1} T^{-n} w_1 \ (m \geq 1) = \begin{cases} (w_0^* w_1^*)^{m-1} (w_0 w_1^*)^{n-m} w_0^* \ (1 \leq m \leq n) \\ (w_1 w_1^*)^{m-n} (w_0^* w_1^*)^{n-1} w_0^* \ (m \geq n+1). \end{cases}$

(a) When $1 \leq m \leq n$,

$$U^p S^k R^{-m-n+1} T^{-n} w_1 \ (k \geq 0,\ p \geq 0) = (\underline{w}_1 w_1)^p (\tilde{w}_1 w_1)^k (w_0^* w_1^*)^{m-1} (w_0 w_1^*)^{n-m} w_0^*$$

$$= \begin{cases} p \geq k \Rightarrow 2(p+n) - 1 \\ p \leq k - 1 \Rightarrow 2(k+n) - 1. \end{cases}$$

$$U^p S^{-k} R^{-m-n+1} T^{-n} w_1 \ (k \geq 1,\ p \geq 0) = (\underline{w}_1 w_1)^p (w_1 \tilde{w}_1)^k (w_0^* w_1^*)^{m-1} (w_0 w_1^*)^{n-m} w_0^*$$

$$\Leftrightarrow (\underline{w}_1 w_1)^p (w_1 \tilde{w}_1)^{k-1} (w_0 w_1^*)^{n-1} \tilde{w}_0^*$$

$$
\Leftrightarrow \begin{cases}
(\underline{w}_1 w_1)^p (w_1 \tilde{w}_1)^{k-2n+1} (\tilde{w}_0 \tilde{w}_1^*)^{n-1} \tilde{w}_0^* \ (k \geq 2n-1) \\
= \begin{cases}
p \geq k - 2n + 1 \Rightarrow 2(p+n) - 1 \\
p \leq k - 2n \Rightarrow 2(k-n) + 1
\end{cases} \\
(\underline{w}_1 w_1)^p (w_0 w_1^*)^{n-1} \tilde{w}_0^* \ (k \leq 2n-2) \Rightarrow 2(p+n) - 1.
\end{cases}
$$

$$U^{-p} S^k R^{-m-n+1} T^{-n} w_1 \ (k \geq 0,\ p \geq 1)$$

$$= (w_1 \underline{w}_1)^p (\tilde{w}_1 w_1)^k (w_0^* w_1^*)^{m-1} (w_0 w_1^*)^{n-m} w_0^*.$$

This is the same as the above case.

$$U^{-p}S^{-k}R^{-m-n+1}T^{-n}w_1 \;\; (k \geq 1,\; p \geq 1)$$

$$= (w_1\underline{w}_1)^p(w_1\widetilde{w}_1)^k(w_0^*w_1^*)^{m-1}(w_0w_1^*)^{n-m}w_0^*$$

$$\Leftrightarrow (w_1\underline{w}_1)^{p-1}(w_1\widetilde{w}_1)^{k-1}(w_0w_1^*)^{n-1}\underline{\widetilde{w}}_0^*$$

$$= \begin{cases} k \geq 2n-1,\; p \geq k \Rightarrow 2(p-n)+1 \\ k \geq 2n-1,\; p \leq k-1 \Rightarrow 2(k-n)+1 \\ k \leq 2n-2,\; p \geq 2n-1 \Rightarrow 2(p-n)+1 \\ k \leq 2n-2,\; p \leq 2n-2 \Rightarrow 2n-1. \end{cases}$$

(b) When $m \geq n+1$,

$$U^pS^kR^{-m-n+1}T^{-n}w_1 \;\; (k \geq 0,\; p \geq 0)$$

$$= (\underline{w}_1w_1)^p(\widetilde{w}_1w_1)^k(w_1w_1^*)^{m-n}(w_0^*w_1^*)^{n-1}w_0^*$$

$$= \begin{cases} \text{(i) } p \geq k \\ p \geq m-n \Rightarrow 2(p+n)-1 \\ p \leq m-n-1 \Rightarrow 2m-1 \\ \text{(ii) } p \leq k-1 \\ k \geq m-n \Rightarrow 2(k+n)-1 \\ k \leq m-n-1 \Rightarrow 2m-1. \end{cases}$$

$$U^pS^{-k}R^{-m-n+1}T^{-n}w_1 \;\; (k \geq 1,\; p \geq 0)$$

$$= (\underline{w}_1w_1)^p(w_1\widetilde{w}_1)^{k-1}(w_1w_1^*)^{m-n}(w_0^*w_1^*)^{n-1}\widetilde{w}_0^*$$

$$\Leftrightarrow \begin{cases} (\underline{w}_1w_1)^p(w_1w_1^*)^{m-n}(w_1\widetilde{w}_1)^{k-2n+1}(\widetilde{w}_0^*\widetilde{w}_1^*)^{n-1}\widetilde{w}_0^* \;\; (k \geq 2n-1) \\ = \begin{cases} \text{(i) } k \leq m+n-1 \\ p \geq m-n \Rightarrow 2(p+n)-1 \\ p \leq m-n-1 \Rightarrow 2m-1 \\ \text{(ii) } k \geq m+n \\ p \geq k-2n+1 \Rightarrow 2(p+n)-1 \\ p \leq k-2n \Rightarrow 2(k-n)+1 \end{cases} \\ (\underline{w}_1w_1)^p(w_1w_1^*)^{m-n}(w_0^*w_1^*)^{n-1}\widetilde{w}_0^* \;\; (k \leq 2n-2) \\ = \begin{cases} p \geq m-n \Rightarrow 2(p+n)-1 \\ p \leq m-n-1 \Rightarrow 2m-1. \end{cases} \end{cases}$$

$$U^{-p}S^{k}R^{-m-n+1}T^{-n}w_1\ (k \geq 0,\ p \geq 1) = (\widetilde{w}_1 w_1)^{k}(w_1 w_1^*)^{m-n}(w_1 \underline{w}_1)^{p-1}(w_0^* w_1^*)^{n-1}\underline{w}_0^*.$$

This is the same as the above case.

$$U^{-p}S^{-k}R^{-m-n+1}T^{-n}w_1(k \geq 1,\ p \geq 1) = (w_1 \underline{w}_1)^{p-1}(w_1 \widetilde{w}_1)^{k-1}(w_1 w_1^*)^{m-n}(w_0^* w_1^*)^{n-1}\underline{\widetilde{w}}_0^*.$$

$$= \begin{cases} p \leq m+n-1,\ 2n-1 \leq k \leq p \Rightarrow 2m-1 \\ p \geq m+n,\ 2n-1 \leq k \leq p \Rightarrow 2(p-n)+1 \\ 2n-1 \leq k \leq m+n-1,\ p \leq k-1 \Rightarrow 2m-1 \\ k \geq m+n,\ p \leq k-1 \Rightarrow 2(k-n)+1 \\ k \leq 2n-2,\ 2n-1 \leq p \leq m+n-1 \Rightarrow 2m-1 \\ k \leq 2n-2,\ p \geq m+n \Rightarrow 2(p-n)+1 \\ k \leq 2n-2,\ p \leq 2n-2 \Rightarrow 2m-1. \end{cases}$$

From the above, we have $W(\mathrm{I})(n = 0) = \dfrac{(1+t)^2(1+22t^2+t^4)}{(1-t)^3(1+t)^3}$, $W(\mathrm{II}) = \dfrac{t^2(27+115t^2+46t^4)}{(1-t)^4(1+t)^4}$, $W(\mathrm{IV}) = \dfrac{t(8+88t^2+83t^4+t^6)}{(1-t)^4(1+t)^4}$ and from these, we obtain the result.

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