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NUMERICAL QUENCHING FOR A QUASILINEAR PARABOLIC EQUATION

LOUIS A. ASSALÉ, THÉODORE K. BONI and DIABATE NABONGO*

Institut National Polytechnique Houphouët-Boigny de Yamoussoukro
BP 1093 Yamoussoukro, Côte d'Ivoire
e-mail: assale@inphb.edu.ci; theokboni@yahoo.fr

*Département de Mathématiques et Informatiques
Université d'Abobo-Adjamé
UFR-SFA, 16 BP 372 Abidjan 16, Côte d'Ivoire
e-mail: nabongo_diabate@yahoo.fr

Abstract

This paper concerns the study of the numerical approximation for the following initial-boundary value problem

$$\begin{cases} u_t = (\varphi(u_x))_x - u^{-p}, & x \in (0, 1), \quad t \in (0, T), \\ u_x(0, t) = 0, \quad u_x(1, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x) > 0, & x \in [0, 1], \end{cases}$$

where $p > 0$, $\varphi(s)$ is positive, increasing and $s\varphi''(s) \leq 0$ for positive values of s. We show that the solution of a semidiscrete form of the above problem quenches in a finite time and estimate its semidiscrete quenching time. We also prove that, under some assumptions, the semidiscrete quenching time converges to the real one when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate our analysis.

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1. Introduction

Consider the following initial-boundary value problem

$$u_t = (\varphi(u_x))_x - u^{-p}, \quad x \in (0, 1), \quad t \in (0, T),$$

$$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t \in (0, T),$$

$$u(x, 0) = u_0(x) > 0, \quad x \in [0, 1],$$

where $p > 0$, $u_0'(0) = 0$, $u_0'(1) = 0$, $\varphi(s)$ is positive, increasing and $s\varphi''(s) \leq 0$ for positive values of s. The first equation may be written as

$$u_t = \varphi'(u_x)u_{xx} - u^{-p}.$$

Thus the problem is equivalent to

$$u_t = \varphi'(u_x)u_{xx} - u^{-p}, \quad x \in (0, 1), \quad t \in (0, T), \tag{1}$$

$$u_x(0, t) = 0, \quad u_x(1, t) = 0, \quad t \in (0, T), \tag{2}$$

$$u(x, 0) = u_0(x) > 0, \quad x \in [0, 1]. \tag{3}$$

When $\varphi(s) = \arctan(s)$, it is not hard to see that the above hypotheses on φ are satisfied and in this case, (1) becomes $u_t = \frac{u_{xx}}{1 + u_x^2} - u^{-p}$. The above problem has a lot of applications in physics (see for instance [5], [8], [10]).

Here $(0, T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is finite, then the solution u develops a singularity in a finite time, namely

$$\lim_{t \to T} \| u(\cdot, t) \|_{\inf} = 0,$$

where $\| u(\cdot, t) \|_{\inf} = \min_{0 \leq x \leq 1} u(x, t)$. In this last case, we say that u quenches in a finite time and the time T is called the quenching time of the solution u.

The theoretical study of solutions which quench in a finite time has been the subject of investigations of many authors (see [2], [4], [10-12] and the references cited therein). In particular in [10], the authors have considered the problem (1)-(3). They have shown that the solution of (1)-(3) quenches in a finite time and the quenching time is estimated.

In this paper, we are interested in the numerical study using a semidiscrete form of (1)-(3). Let I be a positive integer, and define the grid $x_i = ih$, $0 \le i \le I$, where $h = 1/I$. We approximate the solution u of the problem (1)-(3) by the solution $U_h(t) = (U_0(t), U_1(t), \dots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \varphi'(\delta^0 U_i(t))\delta^2 U_i(t) - U_i^{-p}(t), \quad 1 \le i \le I, \quad t \in (0, T_q^h), \tag{4}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{5}$$

where

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2}, \quad 1 \le i \le I-1,$$

$$\delta^0 U_i(t) = \frac{U_{i+1}(t) - U_{i-1}(t)}{2h}, \quad 1 \le i \le I-1, \quad \delta^0 U_0(t) = 0, \quad \delta^0 U_I(t) = 0.$$

Here $(0, T_q^h)$ is the maximal time interval on which $\| U_h(t) \|_{\inf} > 0$ with $\| U_h(t) \|_{\inf} = \min_{0 \le i \le I} U_i(t)$. When the time T_q^h is finite, we say that $U_h(t)$ quenches in a finite time and the time T_q^h is called the quenching time of the solution $U_h(t)$.

Firstly, we give some conditions under which the solution of (4)-(5) quenches in a finite time and estimate its semidiscrete quenching time. Secondly, under some assumptions, we also show that the semidiscrete quenching time converges to the real one when the mesh size goes to zero. An analogous study has been undertaken in [13] by Nabongo and Boni, where they have considered a heat equation with a singular boundary condition. Our work was also motived by the studies in [1], [6], [7], [9], [14], [16], where the authors have obtained comparable results concerning the phenomenon of blow-up (we say that a solution blows up in a finite time if it reaches the values infinity in a finite time). Also in

[3], the phenomenon of extinction is studied by a numerical method (a solution extincts in a finite time if it reaches the value zero in a finite time but there is no singularities on the term of reaction).

Our paper is written in the following manner. In the next section, we prove some results about the discrete maximum principle. In the third section, we show that under some assumptions, the solution of (4)-(5) quenches in a finite time and estimate its semidiscrete quenching time. In the fourth section, we prove the convergence of the semidiscrete quenching time. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Properties of the Semidiscrete Scheme

In this section, we prove some results about the discrete maximum principle. The following lemma is a discrete form of the maximum principle.

Lemma 2.1. *Let* $a_h(t), c_h(t) \in C^0([0, T), \mathbb{R}^{I+1}),\ a_h(t) \geq 0$ *and let* $V_h(t) \in C^1([0, T), \mathbb{R}^{I+1})$ *such that for* $t \in (0, T)$,

$$\frac{dV_i(t)}{dt} - a_i(t)\delta^2 V_i(t) + c_i(t)V_i(t) \geq 0, \quad 0 \leq i \leq I, \tag{6}$$

$$V_i(0) \geq 0, \quad 0 \leq i \leq I. \tag{7}$$

Then we have $V_i(t) \geq 0,\ 0 \leq i \leq I,\ t \in (0, T)$.

Proof. Let $T_0 < T$ and define the vector $Z_h(t) = e^{\lambda t} V_h(t)$, where λ is such that $c_i(t) - \lambda > 0$ for $t \in [0, T_0],\ 0 \leq i \leq I$. Let $m = \min_{0 \leq i \leq I, 0 \leq t \leq T_0} Z_i(t)$. Since for $i \in \{0, ..., I\},\ Z_i(t)$ is a continuous function, there exists $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$ for a certain $i_0 \in \{0, ..., I\}$. It is not hard to see that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \tag{8}$$

$$a_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) \geq 0. \tag{9}$$

From (6), we have

$$\frac{dZ_{i_0}(t_0)}{dt} - a_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + (c_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0, \tag{10}$$

which implies that $(c_{i_0}(t_0) - \lambda)Z_{i_0}(t_0) \geq 0$ because of (8) and (9). Due to the fact that $c_{i_0}(t_0) - \lambda > 0$, we see that $Z_{i_0}(t_0) \geq 0$. We deduce that $V_h(t) \geq 0$ for $t \in [0, T_0]$ and this leads us to the desired result. □

Another version of the discrete maximum principle is the following comparison lemma.

Lemma 2.2. *Let* $V_h(t), U_h(t) \in C^1([0, T), \mathbb{R}^{I+1})$, $f \in C^0(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and* $a_h(t) \geq 0$ *such that for* $t \in (0, T)$,

$$\frac{dV_i}{dt} - a_i(t)\delta^2 V_i + f(V_i(t), t) < \frac{dU_i}{dt} - a_i(t)\delta^2 U_i + f(U_i(t), t), \; 0 \leq i \leq I, \tag{11}$$

$$V_i(0) < U_i(0), \quad 0 \leq i \leq I. \tag{12}$$

Then we have $V_i(t) < U_i(t)$, $0 \leq i \leq I$, $t \in (0, T)$.

Proof. Define the vector $Z_h(t) = U_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_h(t) > 0$ for $t \in [0, t_0)$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, ..., I\}$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \leq 0, \quad \delta^2 Z_{i_0}(t_0) \geq 0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt} - a_{i_0}(t_0)\delta^2 Z_{i_0}(t_0) + f(U_{i_0}(t_0), t_0) - f(V_{i_0}(t_0), t_0) \leq 0.$$

But this inequality contradicts (11) and the proof is complete. □

To end this section, let us give a property on the operator δ^2.

Lemma 2.3. *Let* $U_h \in \mathbb{R}^{I+1}$ *such that* $U_h > 0$. *Then we have*

$$\delta^2(U^{-p})_i \geq -pU_i^{-p-1}\delta^2 U_i \; \textit{for} \; 0 \leq i \leq I.$$

Proof. Applying Taylor's expansion, we get

$$\delta^2(U^{-p})_0 = -pU_0^{-p-1}\delta^2 U_0 + (U_1 - U_0)^2 \frac{p(p+1)}{h^2}\theta_0^{-p-2},$$

$$\delta^2(U^{-p})_i = -pU_i^{-p-1}\delta^2 U_i + (U_{i+1} - U_i)^2 \frac{p(p+1)}{2h^2}\theta_i^{-p-2}$$

$$+ (U_{i-1} - U_i)^2 \frac{p(p+1)}{2h^2}\eta_i^{-p-2} \text{ if } 1 \le i \le I-1,$$

$$\delta^2(U^{-p})_I = -pU_I^{-p-1}\delta^2 U_I + (U_{I-1} - U_I)^2 \frac{p(p+1)}{h^2}\theta_I^{-p-2},$$

where θ_i is an intermediate value between U_i and U_{i+1} and η_i is the one between U_i and U_{i-1}. Use the fact that $U_h > 0$ to complete the rest of the proof. □

3. Quenching Solutions

In this section, under some assumptions, we show that the solution of the semidiscrete problem quenches in a finite time and its semidiscrete quenching time is bounded from above and below.

Theorem 3.1. *Let* $U_h(t)$ *be the solution of* (4)-(5). *Suppose that there exists a positive constant* $A \le 1$ *such that*

$$\varphi'(\delta^0\varphi_i)\delta^2\varphi_i - \varphi_i^{-p} \le -A\varphi_i^{-p}, \quad 0 \le i \le I. \tag{13}$$

Then the solution $U_h(t)$ *of* (4)-(5) *quenches in a finite time* T_q^h *which obeys the following estimate* $T_q^h \le \frac{1}{A}\frac{\|\varphi_h\|_{\inf}^{p+1}}{(1+p)}$.

Proof. Since $(0, T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty > 0$, our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ defined as follows

$$J_i(t) = \frac{dU_i(t)}{dt} + A(U_i(t))^{-p}, \quad 0 \le i \le I, \quad t \in [0, T_q^h). \tag{14}$$

A straightforward computation reveals that

$$\frac{dJ_i}{dt} - \varphi'(\delta^0 U_i)\delta^2 J_i = \frac{d^2 U_i}{dt^2} - ApU_i^{-p-1}\frac{dU_i}{dt} - \varphi'(\delta^0 U_i)\frac{d(\delta^2 U_i)}{dt}$$

$$- A\varphi'(\delta^0 U_i)\delta^2 U_i^{-p}.$$

From Lemma 2.3, we have $\delta^2 U_i^{-p} \geq -pU_i^{-p-1}\delta^2 U_i$, which implies that

$$\frac{dJ_i}{dt} - \varphi'(\delta^0 U_i)J_i \leq \frac{d}{dt}\left(\frac{dU_i}{dt} - \varphi'(\delta^0 U_i)\delta^2 U_i\right)$$

$$- pAU_i^{-p-1}\left(\frac{dU_i}{dt} - \varphi'(\delta^0 U_i)\delta^2 U_i\right)$$

$$+ \varphi''(\delta^0 U_i)\frac{d(\delta^0 U_i)}{dt}\delta^2 U_i.$$

Use (4) to obtain

$$\frac{dJ_i}{dt} - \varphi'(\delta^0 U_i)J_i \leq -pU_i^{-p-1}J_i + \varphi''(\delta^0 U_i)\frac{d(\delta^0 U_i)}{dt}\delta^2 U_i. \tag{15}$$

From the expression of $J_h(t)$ in (14), we derive the following equality

$$\delta^0 J_i = \frac{d(\delta^0 U_i)}{dt} + A\delta^0 U_i^{-p},$$

which allows us to obtain $\frac{d(\delta^0 U_i)}{dt} = \delta^0 J_i - A\delta^0 U_i^{-p}$. Taking into account this equality, the last term on the right hand side on the inequality in (15) can be rewritten in the following manner

$$\varphi''(\delta^0 U_i)\frac{d\delta^0 U_i}{dt}\delta^2 U_i = \varphi''(\delta^0 U_i)\delta^2 U_i \delta^0 J_i - A\varphi''(\delta^0 U_i)\delta^2 U_i \delta^0 U_i^{-p}. \tag{16}$$

On the other hand, from (4) and (14), we find that

$$J_i(t) = \varphi'(\delta^0 U_i)\delta^2 U_i - (1 - A)U_i^{-p}.$$

We deduce from (16) and the above equality that

$$\varphi''(\delta^0 U_i)\frac{d(\delta^0 U_i)}{dt}\delta^2 U_i = \varphi''(\delta^0 U_i)\delta^2 U_i \delta^0 J_i$$

$$- A\varphi''(\delta^0 U_i)\delta^0 U_i^{-p}\left(\frac{J_i(t) + (1-A)U_i^{-p}}{\varphi'(\delta^0 U_i)}\right). \tag{17}$$

Using (17), we arrive at

$$\frac{dJ_i}{dt} - \varphi'(\delta^0 U_i)J_i \le \varphi''(\delta^0 U_i)\delta^2 U_i \delta^0 J_i - \left(pU_i^{-p-1} + \frac{A\varphi''(\delta^0 U_i)\delta^0 U_i^{-p}}{\varphi'(\delta^0 U_i)}\right)J_i$$

$$- A(1-A)\frac{A\varphi''(\delta^0 U_i)\delta^0 U_i^{-p}}{\varphi'(\delta^0 U_i)}.$$

By the mean value theorem, it is not hard to see that $\delta^0 U_i^{-p} = -p\xi_i^{-p-1}\delta^0 U_i$, where ξ_i is an intermediate value between U_{i-1} and U_{i+1}. Using this equality and the fact that $s\varphi''(s) < 0$, we find that

$$\frac{dJ_i}{dt} - \varphi'(\delta^0 U_i)J_i \le \varphi''(\delta^0 U_i)\delta^2 U_i \delta^0 J_i - \left(pU_i^{-p-1} + \frac{A\varphi''(\delta^0 U_i)\delta^0 U_i^{-p}}{\varphi'(\delta^0 U_i)}\right)J_i.$$

From (13), we observe that $J_h(0) \le 0$. We deduce from Lemma 2.1 that $J_h(t) \le 0$ for $t \in (0, T_q^h)$. Obviously $U_i^p(t)dU_i(t) \le -Adt$. Integrating the above inequality over (t, T_q^h), we get

$$T_q^h - t \le \frac{1}{A}\frac{(U_i(t))^{1+p}}{(1+p)} \quad \text{for } 0 \le i \le I, \tag{18}$$

which implies that $T_q^h \le \frac{1}{A}\frac{\| U_h(0)\|_{\inf}^{1+p}}{(1+p)}$. Use the fact that $\| U_h(0)\|_{\inf} = \| \varphi_h \|_{\inf}$ to complete the rest of the proof. □

Remark 3.1. The inequality (18) implies that

$$T_q^h - t_0 \le \frac{1}{A}\frac{\| U_h(t_0)\|_{\inf}^{1+p}}{(1+p)} \quad \text{for } t_0 \in (0, T_q^h),$$

and there exists a constant $C > 0$ such that

$$U_i \ge C(T_q^h - t)^{\frac{1}{1+p}} \quad \text{for } t \in (0, T_q^h), \quad 0 \le i \le I.$$

Theorem 3.2. *Let* $U_h(t)$ *be the solution of* (4)-(5). *Then we have*

$$T_q^h \geq \frac{\| \varphi_h \|_{\inf}^{1+p}}{(p+1)}.$$

Proof. Let i_0 be such that $\| U_h(t) \|_{\inf} = U_{i_0}(t)$. It is not hard to see that

$$\delta^2 U_{i_0}(t) = \frac{U_{i_0+1}(t) - 2U_{i_0}(t) + U_{i_0-1}(t)}{h^2} \geq 0 \text{ if } 1 \leq i_0 \leq I-1,$$

$$\delta^2 U_{i_0}(t) = \frac{2U_1(t) - 2U_0(t)}{h^2} \geq 0 \text{ if } i_0 = 0,$$

$$\delta^2 U_{i_0}(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} \geq 0 \text{ if } i_0 = I.$$

Use the above inequalities and (4) to obtain $\frac{dU_{i_0}}{dt} \geq -U_{i_0}^{-p}$, which implies that $U_{i_0}^p dU_{i_0} \geq -dt$, $0 \leq i \leq I$. Integrating this inequality over $(0, T_q^h)$, we arrive at $T_q^h \geq \frac{(U_{i_0}(0))^{1+p}}{(p+1)}$. Use the fact that $U_{i_0}(0) = \| \varphi_h \|_{\inf}$ to complete the rest of the proof. □

4. Convergence

In this section, under some assumptions, we prove that the quenching time of the solution of the semidiscrete problem converges to the real one when the mesh size tends to zero. Firstly, we show the convergence of our scheme by the following theorem.

Theorem 4.1. *Assume that* (1)-(3) *has a solution* $u \in C^{4,1}([0, 1] \times [0, T-\tau])$ *such that* $\min_{0 \leq t \leq T-\tau} \| u(\cdot, t) \|_{\inf} = \alpha > 0$ *with* $\tau \in (0, T)$. *Suppose that the initial data at* (5) *satisfies*

$$\| \varphi_h - u_h(0) \|_\infty = o(1) \text{ as } h \to 0, \tag{19}$$

where $u_h(t) = (u(x_0, t), \ldots, u(x_I, t))^T$. *Then, for h sufficiently small, the* *problem* (4)-(5) *has a unique solution* $U_h \in C^1([0, T-\tau], \mathbb{R}^{I+1})$ *such that*

$$\max_{0\le t\le T-\tau} \| U_h(t) - u_h(t) \|_\infty = O(\| \varphi_h - u_h(0) \|_\infty + h^2) \text{ as } h \to 0.$$

Proof. The problem (4)-(5) has for each h, a unique solution $U_h \in C^1([0, T_q^h), \mathbb{R}^{I+1})$. Let $t(h)$ be the greatest value of $t > 0$ such that

$$\| U_h(t) - u_h(t) \|_\infty < \frac{\alpha}{2} \text{ for } t \in (0, t(h)). \tag{20}$$

The relation (19) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h), T-\tau\}$. By the triangle inequality, we obtain

$$\| U_h(t) \|_{\inf} \ge \| u_h(t) \|_{\inf} - \| U_h(t) - u_h(t) \|_\infty \ge \frac{\alpha}{2} \text{ for } t \in (0, t^*(h)). \tag{21}$$

Applying Taylor's expansion, we get

$$\frac{d}{dt} u(x_i, t) = \varphi'(\delta^0 u(x_i, t))\delta^2 u(x_i, t) - u^{-p}(x_i, t) + M(x_i, t)h^2, \quad 0 \le i \le I,$$

where $M(x, t)$ is a bounded function. Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. A routine computation yields

$$\frac{de_i(t)}{dt} = (\varphi'(\delta^0 u(x_i, t)) - \varphi'(\delta^0 U_i))\delta^2 u(x_i, t) + \varphi'(\delta^0 U_i)(\delta^2 u(x_i, t) - \delta^2 U_i)$$
$$+ M(x_i, t)h^2 - (u^{-p}(x_i, t) - U_i^{-p}), \quad 0 \le i \le I.$$

Use the mean value theorem to obtain

$$\frac{de_i(t)}{dt} = \varphi''\delta^2 u(x_i, t)e_i(t) + \varphi'(\delta^0 U_i)\delta^2 e_i(t)$$
$$+ M(x_i, t)h^2 - p\theta_i^{-p-1} e_i(t), \quad 0 \le i \le I,$$

where θ_i, ξ_i and χ_i are intermediate values between $U_i(t)$ and $u(x_i, t)$, which implies that

$$\frac{de_i(t)}{dt} \le \varphi'(\delta^0 U_i)\delta^2 e_i(t) + Kh^2 + pK| e_i(t) |, \quad 0 \le i \le I, \quad t \in (0, t^*(h)).$$

Consider the vector Z_h defined as follows

$$Z_i(t) = e^{(K+1)t}(\| \varphi_h - u_h(0) \|_\infty + Kh^2), \quad 0 \le i \le I. \tag{22}$$

A direct calculation yields

$$\frac{dZ_i(t)}{dt} > \varphi'(\delta^0 U_i)\delta^2 Z_i(t) + pK| Z_i(t) | + Kh^2, \quad 0 \le i \le I,$$

$$Z_i(0) > e_i(0), \quad 0 \le i \le I.$$

It follows from comparison Lemma 2.2 that

$$Z_i(t) \ge e_i(t) \text{ for } t \in (0, t^*(h)), \quad 0 \le i \le I.$$

By the same way, we also prove that

$$Z_i(t) \ge -e_i(t) \text{ for } t \in (0, t^*(h)), \quad 0 \le i \le I,$$

which implies that

$$\| U_h(t) - u_h(t) \|_\infty \le e^{(K+1)t}(\| \varphi_h - u_h(0) \|_\infty + Kh^2), \quad t \in (0, t^*(h)).$$

Let us show that $t^*(h) = T - \tau$. Suppose that $T - \tau > t(h)$. From (20), we obtain

$$\frac{\alpha}{2} = \| U_h(t(h)) - u_h(t(h)) \|_\infty \le e^{(K+1)T}(\| \varphi_h - u_h(0) \|_\infty + Kh^2). \tag{23}$$

Since term on the right hand side of the above inequality goes to zero as h goes to zero, we deduce that $\frac{\alpha}{2} \le 0$, which is impossible. Consequently $t^*(h) = T - \tau$, and the proof is complete. □

Now, we are able to prove the main theorem of this section.

Theorem 4.2. *Suppose that the problem* (1)-(3) *has a solution u which quenches in a finite time T such that* $u \in C^{4,1}([0, 1] \times [0, T))$ *and the initial condition at* (5) *satisfies* $\| \varphi_h - u_h(0) \|_\infty = o(1)$ *as* $h \to 0$. *Under the assumption of Theorem* 3.1, *the problem* (4)-(5) *has a unique solution* U_h *which quenches in a finite time* T_q^h *and we have* $\lim_{h \to 0} T_q^h = T$.

Proof. Let $\varepsilon > 0$. There exists $\varrho > 0$ such that

$$\frac{1}{A}\frac{y^{1+p}}{(1+p)} \leq \frac{\varepsilon}{2}, \quad 0 < y \leq \varrho. \tag{24}$$

Since the solution u quenches at the time T, there exists a time $T_0 \in \left(T - \frac{\varepsilon}{2}, T\right)$ such that $0 < \| u(\cdot, t)\|_{\inf} \leq \frac{\varrho}{2}$ for $t \in [T_0, T)$. Setting $T_1 = \frac{T_0 + T}{2}$, it is not hard to see that $0 < \| u(\cdot, t)\|_{\inf} < \frac{\varrho}{2}$ for $t \in [0, T_1]$. From Theorem 4.1, the problem (4)-(5) has a solution $U_h(t)$ and the following estimate holds $\| U_h(t) - u_h(t)\|_\infty \leq \varrho$ for $t \in [0, T_1]$, which implies that $\| U_h(T_1) - u_h(T_1)\|_\infty \leq \frac{\varrho}{2}$. Applying the triangle inequality, we get

$$\| U_h(T_1)\|_{\inf} \leq \| u_h(T_1)\|_{\inf} + \| U_h(T_1) - u_h(T_1)\|_\infty \leq \frac{\varrho}{2} + \frac{\varrho}{2} = \varrho.$$

From Theorem 3.1, U_h quenches in a finite time T_q^h. We deduce from Remark 3.1 and (24) that

$$| T_q^h - T | \leq | T_q^h - T_1 | + | T_1 - T | \leq \frac{1}{A}\frac{\| U_h(T_1)\|_{\inf}^{1+p}}{(1+p)} + \frac{\varepsilon}{2} \leq \varepsilon,$$

which leads us to the desired result. □

5. Numerical Results

In this section, we give some computational results to approximate the quenching time of the continuous problem. We consider the problem (1)-(3) in the case where $\varphi(s) = \arctan(s)$. We approximate the solution u of (1)-(3) by the solution $U_h^{(n)}$ of the following explicit scheme

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} - (U_0^{(n)})^{-p-1} U_0^{(n+1)}, \tag{25}$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{4(U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)})}{4h^2 + (U_{i+1}^{(n)} - U_{i-1}^{(n)})^2}$$

$$-(U_i^{(n)})^{-p-1}U_i^{(n+1)}, \quad 1 \le i \le I-1, \tag{26}$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} - (U_I^{(n)})^{-p-1}U_I^{(n+1)}, \tag{27}$$

$$U_i^{(0)} = \frac{2+\cos(i\pi h)}{10}, \quad 0 \le i \le I, \tag{28}$$

where $n \ge 0$, $\Delta t_n = \min\left(\frac{h^2}{2}, \tau\| U_h^{(n)} \|_{\inf}^{1+p}\right)$ with $\tau = \text{const} \in (0, 1)$.

We also approximate the solution u of (1)-(3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} - (U_0^{(n)})^{-p-1}U_0^{(n+1)}, \tag{29}$$

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{4(U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)})}{4h^2 + (U_{i+1}^{(n)} - U_{i-1}^{(n)})^2}$$

$$-(U_i^{(n)})^{-p-1}U_i^{(n+1)}, \quad 1 \le i \le I-1, \tag{30}$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} - (U_I^{(n)})^{-p-1}U_I^{(n+1)}, \tag{31}$$

$$U_i^{(0)} = \frac{2+\cos(i\pi h)}{10}, \quad 0 \le i \le I, \tag{32}$$

where $n \ge 0$, $\Delta t_n = \tau\| U_h^n \|_{\inf}^{1+p}$ with $\tau = \text{const} \in (0, 1)$.

We need the following definition.

Definition 5.1. We say that the solution $U_h^{(n)}$ of the explicit or the implicit scheme quenches in a finite time if $\lim_{n\to+\infty}\| U_h^{(n)} \|_{\inf} = 0$ and the series $\sum_{n=0}^{+\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{+\infty} \Delta t_n$ is called the *numerical quenching time* of the solution $U_h^{(n)}$.

In the following tables, in rows, we present the numerical quenching times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. We take for the numerical quenching time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when $\Delta t_n = | T^{n+1} - T^n | \leq 10^{-16}$. The order ($s$) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Table 1. Numerical quenching time, number of iterations, CPU time (seconds) and order (s) of the approximations obtained with the explicit scheme

I	T^n	n	CPU time	s
16	0.005539	3495	0.8	-
32	0.005524	13300	1.2	-
64	0.005520	50538	4	1.91
128	0.005518	191497	32	1.00
256	0.005517	723210	240	1.00

Table 2. Numerical quenching time, number of iterations, CPU time (seconds) and order (s) of the approximations obtained with the implicit scheme

I	T^n	n	CPU time	s
16	0.005540	3495	0.9	-
32	0.005525	13301	2	-
64	0.005521	50538	8	1.91
128	0.005519	191497	60	1.00
256	0.005518	723210	454	1.00

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