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GENERALIZED INTUITIONISTIC FUZZY MATRICES

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Abstract

In this paper, generalized intuitionistic fuzzy matrices (GIFMs) are defined. In fact, all GIFMs are intuitionistic fuzzy matrices (IFMs) but all IFMs are not GIFMs. Also, some operations are valid for IFMs but they are not valid for GIFMs. The relational properties, i.e., four types of reflexivity and irreflexivity, symmetricity and transitivity are studied here. Finally, some new operations over GIFMs are studied here.

1. Introduction

In 1965, Zadeh [28] introduced the concept of fuzzy subsets. Latter many authors generalized the concept of fuzzy subsets in different directions, like veg set [8], rough set [22], etc. After two decades, Atanassov [1] introduced the concept of intuitionistic fuzzy sets, which is a generalization of fuzzy subsets. Recently, Mondal and Samanta [15] further generalized the concept of IFSs to generalized intuitionistic fuzzy sets (GIFSs).

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Several authors presented a number of results on fuzzy matrices. Kim and Roush [14] studied the canonical form of an idempotent matrix. Hashimoto [9] studied the canonical form of a transitive matrix, Xin [27] studied controllable fuzzy matrices. Hemasinha et al. [10] investigated iterates of fuzzy circulant matrices. Thomason [26] defined the adjoint of square fuzzy matrix. Pal and Shyamal [20] defined two new operators on fuzzy matrices. Ragab and Emam [25] give min-max composition of fuzzy matrix. Emam [7] investigate some results on circulant fuzzy matrices. Pal [16] introduced intuitionistic fuzzy determinant. Pal et al. [18] introduced intuitionistic fuzzy matrices. Pal and Shyamal [19] define the distance between two intuitionistic fuzzy matrices. Khan and Pal [11] studied some operations on intuitionistic fuzzy matrices. Pal and Khan [17] introduced intuitionistic fuzzy tautological matrices and studied several properties.

In this paper, we introduce generalized intuitionistic fuzzy matrices and studied their various results.

In Section 2, we recall the definitions of IFMs and generalized intuitionistic fuzzy sets. We define GIFM and some relevant basic preliminaries. Also, we studied some relational properties. In Section 3, we shown by examples that some operations are valid for IFMs but they are not valid for GIFMs. In Section 4, we define some new operations on GIFMs with examples for the implementation of the operation in real life problem.

2. Preliminaries and Definitions

Here we define the intuitionistic fuzzy set invented by Atanassov and generalized intuitionistic fuzzy sets introduced by Mondal and Samanta [15].

Definition 1. Let E be a fixed set. An intuitionistic fuzzy set A of E is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\}$, where the function $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define respectively the degree of membership and degree of nonmembership of the element $x \in E$ to the set A, which is a subset of E and for every $x \in E$, $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

Definition 2 [15]. Let E be a fixed set. A generalized intuitionistic fuzzy set A of E is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\}$, where the function $\mu_A : E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define respectively the degree of membership and degree of nonmembership of the element $x \in E$ to the set A, which is a subset of E and for every $x \in E$ satisfy the condition

$$\mu_A(x) \wedge \nu_A(x) \leq 0.5 \text{ for all } x \in E.$$

This condition is called *generalized intuitionistic condition* (GIC). In fact, all GIFs are IFSs but all IFSs are not GIFSs.

Based on the definition of IFSs Pal et al. [18] have defined IFM which is given below.

Definition 3 [18]. An IFM A of order $m \times n$ is defined as $A = [\langle x_{ij}, a_{ij\mu}, a_{ij\nu}\rangle]_{m\times n}$, where $a_{ij\mu}$ and $a_{ij\nu}$ are called *membership* and *nonmembership functions* respectively of the element x_{ij} in the IFM A with the condition $0 \leq a_{ij\mu} + a_{ij\nu} \leq 1$.

For simplicity we referred the IFM A as $A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$ instead of $A = [\langle x_{ij}, a_{ij\mu}, a_{ij\nu}\rangle]$.

Now, we define two operators $\vee$ and $\wedge$. The operators $\vee$ and $\wedge$ are defined as $a \vee b = \min(a, b)$ and $a \wedge b = \max(a, b)$.

Definition 4. A GIFM A of order $m \times n$ is defined as $A = [\langle x_{ij}, a_{ij\mu}, a_{ij\nu}\rangle]_{m\times m}$, where $a_{ij\mu}$ and $a_{ij\nu}$ are called *membership* and *nonmembership functions* respectively of the element x_{ij} in the IFM A with the GIC condition

$$a_{ij\mu} \wedge a_{ij\nu} \leq 0.5.$$

Here also we referred the IFM A as $A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$ instead of $A = [\langle x_{ij}, a_{ij\mu}, a_{ij\nu}\rangle]$.

In the following we define some basic operations on GIFMs.

Definition 5. If $A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$ and $B = [\langle b_{ij\mu}, b_{ij\nu}\rangle]$ are two GIFMs of same order, then

(i) $A = B$ iff $a_{ij\mu} = b_{ij\mu}$ and $a_{ij\nu} = b_{ij\nu}$.

(ii) $A \leq B$ iff $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$.

(iii) $A \geq B$ iff $B \leq A$.

(iv) $A \prec B$ iff $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \leq b_{ij\nu}$.

(v) $A \cap B = [\langle \min(a_{ij\mu}, b_{ij\mu}), \max(a_{ij\nu}, b_{ij\nu})\rangle]$.

(vi) $A \cup B = [\langle \max(a_{ij\mu}, b_{ij\mu}), \min(a_{ij\nu}, b_{ij\nu})\rangle]$.

Two operations $\circ$ and $*$ are define on GIFMs below.

Definition 6. Let $A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$ and $B = [\langle b_{ij\mu}, b_{ij\nu}\rangle]$ be two GIFMs of order $m \times n$ and $n \times p$. Then

$$A \circ B = \left[\left\langle \sum_k (a_{ik\mu} \cdot b_{kj\mu}), \prod_k (a_{ik\nu} + b_{kj\nu}) \right\rangle\right],$$

and

$$A * B = \left[\left\langle \prod_k (a_{ik\mu} + b_{kj\mu}), \sum_k (a_{ik\nu} \cdot b_{kj\nu}) \right\rangle\right].$$

Theorem 1. *If A, B and C are three GIFMs of same order, then*

(i) $A \leq B \Rightarrow A \circ B \leq B \circ C$ *and* $C \circ A \leq C \circ B$.

(ii) $A \prec B \Rightarrow A \circ C \prec B \circ C$ *and* $C \circ A \prec C \circ B$.

(iii) $A \leq B \Rightarrow A \circ A \leq B \circ B$.

(iv) $A \circ B \neq B \circ A$.

Proof. (i) Since, $A \leq B$, therefore, $a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\nu} \geq b_{ij\nu}$.

Let $F = C \circ A$ and $G = C \circ B$, so $f_{ij\mu} = \sum_k (c_{ik\mu} \cdot a_{kj\mu})$, $g_{ij\mu} = \sum_k (c_{ik\mu} \cdot b_{kj\mu})$ and $f_{ij\nu} = \prod_k (c_{ik\nu} + a_{kj\nu})$ and $g_{ij\nu} = \prod_k (c_{ik\nu} + b_{kj\nu})$.

Since $a_{ij\mu} \leq b_{ij\mu}$, so, $a_{kj\mu} \leq b_{kj\mu}$ for any $k \in \{1, 2, ..., n\}$.

Therefore, $c_{ik\mu} \cdot a_{kj\mu} \leq c_{ik\mu} \cdot b_{kj\mu}$ or $\sum_k c_{ik\mu} \cdot a_{kj\mu} \leq \sum_k c_{ik\mu} \cdot b_{kj\mu}$.

Thus, $f_{ij\mu} \leq g_{ij\mu}$.

Also, $a_{ij\nu} \geq b_{ij\nu}$ so $a_{kj\nu} \geq b_{kj\nu}$ and hence $c_{ik\nu} + a_{kj\nu} \geq c_{ik\nu} + b_{kj\nu}$, i.e., $\prod_k (c_{ik\mu} + a_{kj\mu}) \geq \prod_k (c_{ik\mu} + b_{kj\mu})$. So, $f_{ij\nu} \geq g_{ij\nu}$.

Hence, $C \circ A \leq C \circ B$.

Similarly, $A \circ C \leq B \circ C$.

The proofs of remaining results are similar. □

Theorem 2. *If B and* A_n, $n \in I$ *are GIFMs of same order, then*

(i) $(\bigcup A_n) \circ B = \bigcup_n (A_n \circ B)$.

(ii) $(\bigcap A_n) \circ B = \bigcap_n (A_n \circ B)$.

Proof. (i) Let $C = (\bigcup A_n) \circ B$. Then

$$\begin{aligned} c_{ij\mu} &= \vee_k \{(\vee_{p=1}^n a_{pik\mu}) \wedge b_{kj\mu}\} \\ &= \vee_k \{\vee_{p=1}^n (a_{pik\mu} \wedge b_{kj\mu})\} \\ &= \vee_{i=1}^n \{\vee_k (a_{pik\mu} \wedge b_{kj\mu})\} \\ &= \vee_{i=1}^n a_{pij\mu}. \end{aligned}$$

Similarly it can be shown that $c_{ij\nu} = \vee_{i=1}^n a_{pij\nu}$.

Hence, $(\bigcup A_n) \circ B = \bigcup_n (A_n \circ B)$.

(ii) Proof is similar to (i). □

Example 1. Let

$$A = \begin{bmatrix} \langle 0.8, 0.3\rangle & \langle 0.6, 0.5\rangle & \langle 0.1, 0.9\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.7, 0.3\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.5\rangle & \langle 0.0, 0.3\rangle & \langle 0.3. 0.6\rangle \end{bmatrix},$$

and

$$B = \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0.4, 0.6\rangle & \langle 0.2, 0.7\rangle \\ \langle 0.4, 0.7\rangle & \langle 0.5, 0.5\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.2, 0.6\rangle \end{bmatrix}.$$

Then

$$A \circ B = \begin{bmatrix} \langle 0.4, 0.5\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.5, 0.3\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.3, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.2, 0.4\rangle \end{bmatrix}$$

and

$$B \circ A = \begin{bmatrix} \langle 0.4, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.4, 0.6\rangle \\ \langle 0.6, 0.5\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.6, 0.5\rangle & \langle 0.6, 0.3\rangle \end{bmatrix}.$$

This shows that $A \circ B \neq B \circ A$.

Example 2. Let

$$A = \begin{bmatrix} \langle 0.3, 0.5\rangle & \langle 0.4, 0.6\rangle & \langle 0.2, 0.7\rangle \\ \langle 0.4, 0.7\rangle & \langle 0.5, 0.5\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.2, 0.6\rangle \end{bmatrix},$$

and

$$B = \begin{bmatrix} \langle 0.7, 0.4\rangle & \langle 0.4, 0.5\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.8, 0.2\rangle & \langle 0.4, 0.5\rangle \end{bmatrix}.$$

Then

$$A \circ A = \begin{bmatrix} \langle 0.4, 0.5\rangle & \langle 0.4, 0.6\rangle & \langle 0.4, 0.6\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.5\rangle \\ \langle 0.4, 0.3\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.4\rangle \end{bmatrix},$$

and

$$B \circ B = \begin{bmatrix} \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.2\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.6, 0.2\rangle & \langle 0.6, 0.2\rangle \end{bmatrix}.$$

This shows that $A \circ A \leq B \circ B$, where $A \leq B$.

Here we define some special types of reflexivity of a GIFM.

Definition 7. Let A be a GIFM of any order. Then

(1) $T_1 : A$ is a reflexive of type-1 if $a_{ii\mu} = 1$ and $a_{ii\nu} = 0$, for all $i = 1, 2, \ldots, n$.

(2) $T_2 : A$ is reflexive of type-2 if $a_{ii\mu} = 1$ and $(a_{ii\nu} \vee a_{jj\nu}) \leq a_{ij\nu}$, for all $i, j = 1, 2, \ldots, n$.

(3) $T_3 : A$ is reflexive of type-3 if $(a_{ii\mu} \wedge a_{jj\mu}) \geq (0.5 \vee a_{ij\mu})$ and $a_{ii\nu} = 0$, for all $i, j = 1, 2, \ldots, n$.

(4) $T_4 : A$ is reflexive of type-4 if $(a_{ii\mu} \wedge a_{jj\mu}) \geq a_{ij\mu}$ and $(a_{ii\nu} \vee a_{jj\nu}) \leq a_{ij\nu}$, for all $i, j = 1, 2, \ldots, n$.

Theorem 3. (i) *Reflexivity of type*-1 $\Rightarrow$ *Reflexivity of type*-2, *type*-3 *and type*-4.

(ii) *Reflexivity of type*-2 $\Rightarrow$ *Reflexivity of type*-4.

(iii) *Reflexivity of type*-3 $\Rightarrow$ *Reflexivity of type*-4.

The proof follows from the definition. Here we are showing by the numerical example that the above theorems is obvious. □

Example 3. Let A be a GIFM of type-1, where

$$A = \begin{bmatrix} \langle 1, 0\rangle & \langle 0.4, 0.6\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.3, 0.6\rangle & \langle 1, 0\rangle & \langle 0.8, 0.4\rangle \\ \langle 0.1, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 1, 0\rangle \end{bmatrix}.$$

Then T_2, T_3 and T_4 are obvious.

Example 4. Let A be a GIFM of type-2, where

$$A = \begin{bmatrix} \langle 1, 0.4 \rangle & \langle 0.3, 0.6 \rangle & \langle 0.4, 0.8 \rangle \\ \langle 0.3, 0.9 \rangle & \langle 1, 0.3 \rangle & \langle 0.2, 0.7 \rangle \\ \langle 0.7, 0.4 \rangle & \langle 0.3, 0.7 \rangle & \langle 1, 0.4 \rangle \end{bmatrix}.$$

Then T_4 are obvious.

Example 5. Let A be a GIFM of type-3, where

$$A = \begin{bmatrix} \langle 0.7, 0 \rangle & \langle 0.6, 0.3 \rangle & \langle 0.8, 0.4 \rangle \\ \langle 0.4, 0.5 \rangle & \langle 0.8, 0 \rangle & \langle 0.2, 0.7 \rangle \\ \langle 0.6, 0.4 \rangle & \langle 0.7\ 0.3 \rangle & \langle 1, 0 \rangle \end{bmatrix}.$$

Then T_4 are obvious.

Here we define some special types of irreflexivity of a GIFM.

Definition 8. Let A be a GIFM. Then

(1) T_1' : A is a irreflexive of type-1 if $a_{ii\mu} = 0$ and $a_{ii\nu} = 1$, for all $i = 1, 2, \ldots, n$.

(2) T_2' : A is irreflexive of type-2 if $a_{ii\mu} = 0$ and $(a_{ii\nu} \wedge a_{jj\nu}) \geq (0.5 \vee a_{ij\nu})$, for all $i, j = 1, 2, \ldots, n$.

(3) T_3' : A is irreflexive of type-3 if $(a_{ii\mu} \vee a_{jj\mu}) \leq a_{ij\mu}$ and $a_{ii\nu} = 1$, for all $i, j = 1, 2, \ldots, n$.

(4) T_4' : A is irreflexive of type-4 if $(a_{ii\mu} \vee a_{jj\mu}) \leq a_{ij\mu}$ and $(a_{ii\nu} \wedge a_{jj\nu}) \geq a_{ij\nu}$, for all $i, j = 1, 2, \ldots, n$.

Theorem 4. (i) *Irreflexivity of type*-1 $\Rightarrow$ *irreflexivity of type*-2, *type*-3 *and type*-4.

(ii) *Irreflexivity of type*-2 $\Rightarrow$ *irreflexivity of type*-4.

(iii) *Irreflexivity of type*-3 $\Rightarrow$ *irreflexivity of type*-4.

Remark 1. It can easily be shown by constructing examples that reflexive (irreflexive) of type-4 $\not\Rightarrow$ reflexive (irreflexive) of type-3 $\not\Rightarrow$ reflexive (irreflexive) of type-2 $\not\Rightarrow$ reflexive (irreflexive) of type-1.

Theorem 5. (i) *If A is reflexive of any type, then* $A \leq A \circ A$.

(ii) *If A is irreflexive of any type, then* $A \geq A * A$.

Proof. (i) Let $A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]_{m\times n}$ and $R = A \circ A$. Then

$$R = \left[\left\langle \sum_k (a_{ik\mu} \cdot a_{kj\mu}), \prod_k (a_{ik\mu} + a_{kj\mu})\right\rangle\right].$$

Therefore,

$$\begin{aligned} r_{ij\mu} &= \vee_k (a_{ik\mu} \wedge a_{kj\mu}) \\ &= (a_{ii\mu} \wedge a_{ij\mu}) \vee \{\vee_{i\neq k}(a_{ik\mu} \wedge a_{kj\mu})\} \\ &= a_{ij\mu} \vee \{\vee_{i\neq k}(a_{ik\mu} \wedge a_{kj\mu})\}, \text{ for any type of reflexivity} \\ &\geq a_{ii\mu}, \\ r_{ij\nu} &= \wedge_k (a_{ik\nu} \vee a_{kj\nu}) \\ &= (a_{ii\nu} \vee a_{ij\nu}) \wedge \{\wedge_{k\neq i}(a_{ik\nu} \vee a_{kj\nu})\} \\ &= a_{ij\nu} \wedge \{\wedge_{k\neq i}(a_{ik\nu} \vee a_{kj\nu})\}, \text{ for any type of reflexivity} \\ &\leq a_{ij\mu}. \end{aligned}$$

Hence $A \leq R$, i.e., $A \leq A \circ A$.

(ii) Proof is similar to above case. □

Example 6. Let A be a GIFM of any type, where

$$A = \begin{bmatrix} \langle 0.8, 0.3\rangle & \langle 0.6, 0.5\rangle & \langle 0.1, 0.9\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.7, 0.3\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.5\rangle & \langle 0, 0.3\rangle & \langle 0.3, 0.6\rangle \end{bmatrix}.$$

Then

$$A \circ A = \begin{bmatrix} \langle 0.8, 0.3\rangle & \langle 0.6, 0.5\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.7, 0.3\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.6, 0.3\rangle & \langle 0.3, 0.3\rangle \end{bmatrix}.$$

This shows that $A \leq A \circ A$, but A is not reflexive of any special type.

Example 7. Let A be a GIFM of any type, where

$$A = \begin{bmatrix} \langle 0.4,\ 0.3\rangle & \langle 0.6,\ 0.5\rangle & \langle 0.3,\ 0.6\rangle \\ \langle 0.6,\ 0.3\rangle & \langle 0.7,\ 0.3\rangle & \langle 0.6,\ 0.3\rangle \\ \langle 0.3,\ 0.5\rangle & \langle 0.4,\ 0.5\rangle & \langle 0.3,\ 0.6\rangle \end{bmatrix}.$$

Then

$$A * A = \begin{bmatrix} \langle 0.3,\ 0.5\rangle & \langle 0.4,\ 0.5\rangle & \langle 0.3,\ 0.6\rangle \\ \langle 0.6,\ 0.3\rangle & \langle 0.6,\ 0.3\rangle & \langle 0.6,\ 0.3\rangle \\ \langle 0.3,\ 0.5\rangle & \langle 0.4,\ 0.5\rangle & \langle 0.3,\ 0.6\rangle \end{bmatrix}.$$

This shows that $A \geq A * A$, but A is not reflexive of any special type.

Theorem 6. (i) *If the GIFM A is reflexive (irreflexive) of any type, then* $A \circ A$ $(A * A)$ *is reflexive (irreflexive) of same type.*

(ii) *If the GIFMs A and B are reflexive (irreflexive) of a particular type, then* $A \cap B$ $(A \cup B)$ *is reflexive (irreflexive) of same type.*

(iii) *If the GIFMs A and B are reflexive (irreflexive) of the type say i, then* $A \cup B$ $(A \cap B)$ *is reflexive (irreflexive) of same type.*

(iv) *If the GIFM A is reflexive (irreflexive) of type-*1 *and B is a GIFM, then* $A \cup B$ $(A \cap B)$ *is reflexive (irreflexive) of same type.*

Proof. (i) Let A be a GIFM and $R = A \circ A$. Then

$$\begin{aligned} r_{ii\mu} &= \vee_k (a_{ik\mu} \wedge a_{kj\mu}) \\ &= (a_{ii\mu} \wedge a_{ii\mu}) \vee \{\vee_{i \neq k} (a_{ik\mu} \wedge a_{kj\mu})\} \\ &= a_{ii\mu} \vee \{\vee_{i \neq k} (a_{ik\mu} \wedge a_{ki\mu})\} \\ &= a_{ii\mu}, \text{ for } i = 1, 2, \ldots, n. \end{aligned} \tag{1}$$

Similarly,

$$r_{ii\nu} = a_{ii\nu}. \tag{2}$$

If A is reflexive of type-3 or type-4, then we have for $k \neq i$.

$$
\begin{aligned}
r_{ij\mu} &= \vee_k (a_{ik\mu} \wedge a_{kj\mu}) \\
&= (a_{ii\mu} \wedge a_{ij\mu}) \vee (a_{ij\mu} \wedge a_{jj\mu}) \vee \{\vee_{k \neq i, k \neq j}(a_{ik\mu} \wedge a_{kj\mu})\} \\
&= a_{ij\mu} \vee (a_{ii\mu} \wedge a_{jj\mu}) \vee \{\vee_{i \neq k, k \neq j}(a_{ik\mu} \wedge a_{ki\mu})\} \\
&\leq a_{ii\mu} \wedge a_{jj\mu}, \text{ since } a_{ii\mu} \wedge a_{jj\mu} \geq a_{ij\mu} \\
&= r_{ii\mu} \wedge r_{jj\mu} \text{ [by (1)].} \qquad (3)
\end{aligned}
$$

If A is reflexive of type-3, then

$$r_{ii\mu} = a_{ii\mu} \geq 0.5, \text{ also } r_{ii\mu} \geq 0.5.$$

So, $r_{ii\mu} \wedge r_{jj\mu} \geq 0.5$.

Therefore,

$$0.5 \vee r_{ij\mu} \leq (r_{ij\mu} \wedge r_{jj\mu}). \qquad (4)$$

If A is reflexive of type-2 or type-4, then we have for $i \neq j$.

$$
\begin{aligned}
r_{ij\nu} &= \wedge_k (a_{ik\nu} \vee a_{kj\nu}) \\
&= (a_{ii\nu} \vee a_{ij\nu}) \wedge (a_{ij\nu} \vee a_{jj\nu}) \wedge \{\wedge_{k \neq i, k \neq j}(a_{ik\nu} \vee a_{kj\nu})\} \\
&= a_{ij\nu} \wedge (a_{ij\nu} \vee a_{jj\nu}) \wedge \{\vee_{i \neq k, k \neq j}(a_{ik\nu} \vee a_{ki\nu})\} \\
&\geq a_{ii\nu} \vee a_{jj\nu}, \text{ since } a_{ii\nu} \vee a_{jj\nu} \leq a_{ij\nu} \\
&= r_{ii\nu} \vee r_{jj\nu} \text{ [by (2)].} \qquad (5)
\end{aligned}
$$

Hence by equations (1-5), $A \circ A$ is reflexive of the type as that of A.

(ii) Let $C = A \cap B$. Then $c_{ij\mu} = a_{ij\mu} \wedge b_{ij\mu}$, for all $i = 1, 2, \ldots, n$.

Now, for $i \neq j$

$$
\begin{aligned}
c_{ij\mu} &= a_{ij\mu} \wedge b_{ij\mu} \\
&\leq (a_{ii\mu} \wedge a_{jj\mu}) \wedge (b_{ii\mu} \wedge b_{jj\mu}) \\
&= (a_{ii\mu} \wedge b_{ii\mu}) \wedge (a_{jj\mu} \wedge b_{jj\mu}) \\
&= (a_{ij\mu} \wedge b_{jj\mu}).
\end{aligned}
$$

Similarly, $c_{ij\mu} \geq 0.5$.

So, $c_{ii\mu} \vee c_{jj\mu} \geq 0.5$.

Therefore, $(0.5 \vee c_{ij\mu}) \leq (c_{ii\mu} \vee c_{jj\mu})$.

Similarly, we can establish the result for $c_{ij\nu}$.

Hence, from above $C = A \cap B$ of the type as that of A and B.

The proof of irreflexivity is similarly to that of reflexivity.

(iii) The result is obvious for the type-1 and type-2.

Let

$$
\begin{aligned}
C &= a_{ij\mu} \vee b_{ij\mu} \\
&\leq (a_{ii\mu} \wedge a_{jj\mu}) \vee (b_{ii\mu} \vee b_{jj\mu}) \\
&= \{(a_{ii\mu} \wedge a_{jj\mu}) \vee (b_{ii\mu})\} \vee \{(a_{ii\mu} \wedge a_{jj\mu}) \vee (b_{jj\mu})\} \\
&= \{(a_{ii\mu} \vee b_{jj\mu}) \wedge (a_{jj\mu} \vee b_{ii\mu})\} \wedge \{(a_{ii\mu} \vee b_{jj\mu}) \wedge (a_{jj\mu} \vee b_{jj\mu})\} \\
&= (a_{ii\mu} \vee b_{jj\mu}) \vee (a_{jj\mu} \vee b_{jj\mu}) \\
&= c_{ii\mu} \wedge c_{jj\mu}.
\end{aligned}
$$

For reflexivity of type-3,

$$c_{ii\mu} = a_{ii\mu} \vee b_{jj\mu} \geq 0.5, \text{ (since } a_{ii\mu} \geq 0.5).$$

Similarly $c_{jj\nu} \geq 0.5$, so, $c_{ii\mu} \wedge c_{jj\mu} \geq 0.5$.

Therefore, $0.5 \vee c_{ij\mu} \leq c_{ii\mu} \vee c_{jj\mu}$.

Similarly, we can establish the property for $c_{ij\nu}$.

Hence, from the above $A \cup B$ is reflexive of the same type as that of both A and B.

The proof of irreflexivity is similar to that of reflexivity.

(iv) The proof is similar to above cases. □

Example 8. Let A be reflexive of type-1 and B be any GIFMs of same order, where

$$A = \begin{bmatrix} \langle 1, 0.0\rangle & \langle 0.4, 0.6\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.3, 0.6\rangle & \langle 1, 0.0\rangle & \langle 0.8, 0.4\rangle \\ \langle 0.1, 0.8\rangle & \langle 0.4, 0.8\rangle & \langle 1, 0.0\rangle \end{bmatrix},$$

$$B = \begin{bmatrix} \langle 0.4, 0.5\rangle & \langle 0.5, 0.5\rangle & \langle 0.6, 0.5\rangle \\ \langle 0.5, 0.4\rangle & \langle 0.5, 0.3\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.3, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 0.2, 0.4\rangle \end{bmatrix}.$$

Then

$$A \cup B = \begin{bmatrix} \langle 1, 0.0\rangle & \langle 0.5, 0.5\rangle & \langle 0.7, 0.3\rangle \\ \langle 0.5, 0.4\rangle & \langle 1, 0.0\rangle & \langle 0.8, 0.4\rangle \\ \langle 0.3, 0.5\rangle & \langle 0.4, 0.5\rangle & \langle 1, 0.0\rangle \end{bmatrix}.$$

This shows that $A \cup B$ is reflexive of type-1.

Remark 2. It can be easily shown by examples for the reflexivity of type-2, type-3 and type-4, the condition that only one of GIFMs A and B is reflexive does not imply reflexivity of $A \cup B$.

Example 9. Let

$$A = \begin{bmatrix} \langle 1, 0.4\rangle & \langle 0.3, 0.6\rangle & \langle 0.4, 0.8\rangle \\ \langle 0.3, 0.9\rangle & \langle 1, 0.3\rangle & \langle 0.2, 0.7\rangle \\ \langle 0.7, 0.4\rangle & \langle 0.3, 0.7\rangle & \langle 1, 0.4\rangle \end{bmatrix}$$

and

$$B = \begin{bmatrix} \langle 0.4, 0.6\rangle & \langle 0.6, 0.3\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.6, 0.2\rangle & \langle 0.3, 0.8\rangle & \langle 0.2, 0.6\rangle \\ \langle 0.5, 0.6\rangle & \langle 0.2, 0.7\rangle & \langle 0.7, 0.5\rangle \end{bmatrix}.$$

It may be observed that A is reflexive of type-2 and B is any type of GIFM.

Therefore,

$$A \cup B = \begin{bmatrix} \langle 1, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.6, 0.2\rangle & \langle 1, 0.3\rangle & \langle 0.2, 0.6\rangle \\ \langle 0.7, 0.4\rangle & \langle 0.3, 0.7\rangle & \langle 1, 0.4\rangle \end{bmatrix}.$$

Here, $A \cup B$ is not reflexive of type-2.

Theorem 7. (i) *If* A *is symmetric GIFM, then* $A \circ A$, *i.e.,* $A^{\circ 2}$ *is symmetric GIFM.*

(ii) *If* A *and* B *are two symmetric GIFMs, then* $A \circ B \neq B \circ A$.

Example 10. Let

$$A = \begin{bmatrix} \langle 0.6, 0.4 \rangle & \langle 0.3, 0.2 \rangle & \langle 0.4, 0.7 \rangle \\ \langle 0.3, 0.2 \rangle & \langle 0.3, 0.7 \rangle & \langle 0.7, 0.3 \rangle \\ \langle 0.4, 0.7 \rangle & \langle 0.7, 0.3 \rangle & \langle 0.2, 0.8 \rangle \end{bmatrix},$$

then

$$A^{\circ 2} = \begin{bmatrix} \langle 0.6, 0.2 \rangle & \langle 0.4, 0.4 \rangle & \langle 0.4, 0.3 \rangle \\ \langle 0.4, 0.4 \rangle & \langle 0.7, 0.2 \rangle & \langle 0.3, 0.7 \rangle \\ \langle 0.4, 0.3 \rangle & \langle 0.3, 0.7 \rangle & \langle 0.7, 0.3 \rangle \end{bmatrix}.$$

It follows that $A^{\circ 2}$ is symmetric.

Here we shall define transitive and c-transitive GIFM and some of their properties.

Definition 9. (i) A GIFM A is transitive if $A \geq A \circ A$.

(ii) A GIFM A is c-transitive if $A \leq A * A$.

Remark 3. A GIFM A of order $n \times n$ converges to $A^{\circ c}$, where $c \leq (n-1)$ [where $A^{\circ c} = A \circ A \circ A \cdots c$ times].

Definition 10. Let A be a GIFM of any order. Then

(i) $\hat{A} = A^{\circ 1} \cup A^{\circ 2} \cup A^{\circ 3} \cup \cdots \cup A^{\circ n} = \cup_{i=1}^{n} A^{\circ i}$.

(ii) $\breve{A} = A^{*1} \cap A^{*2} \cap A^{*3} \cap \cdots \cap A^{*n} = \cap_{i=1}^{n} A^{*i}$.

where $A^{\circ 1} = A$, $A^{\circ 2} = A \circ A, \ldots, A^{\circ n} = A \circ A \circ A \cdots n$ times, $A^{*1} = A$, $A^{*2} = A * A, \ldots, A^{*n} = A * A * A \cdots n$ times.

Theorem 8. *If* A *is a GIFM and* B *is a transitive GIFM of order* $n \times n$, *then*

(i) $A \leq \hat{A}$.

(ii) *If* $A \leq B$, *then* $\hat{A} \leq B$.

Proof. (i) Straightforward.

(ii) Let us take $A \leq B$, B is transitive, i.e., $B^{\circ 2} \leq B$. Now using Theorem 1(iii), we get $A^{\circ 2} \leq B^{\circ 2} \leq B$.

Similarly, $A^{\circ 3} \leq B, \cdots$.

Therefore, $\bigcup_{i=1}^{n} A^i \leq B \Rightarrow \hat{A} \leq B$. □

Theorem 9. *If A is a GIFM and B is a c-transitive GIFM of* $n \times n$, *then*

(i) $A \geq \hat{A}$.

(ii) *If* $A \geq B$, *then* $\breve{A} \geq B$.

Proof. The proof is analogous to that of Theorem 8. □

Example 11. Let

$$A = A^{\circ 1} = \begin{bmatrix} \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.4, 0.4\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.5\rangle & \langle 0.4, 0.6\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.5, 0.5\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.4, 0.5\rangle & \langle 0.6, 0.4\rangle & \langle 0.4, 0.3\rangle & \langle 0.6, 0.1\rangle \end{bmatrix}.$$

Then

$$A^{\circ 2} = \begin{bmatrix} \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.3\rangle & \langle 0.6, 0.1\rangle \end{bmatrix},$$

$$A^{\circ 3} = \begin{bmatrix} \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.6, 0.1\rangle \end{bmatrix},$$

and

$$A^{\circ 4} = \begin{bmatrix} \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.6, 0.1\rangle \end{bmatrix}.$$

So, $A^{\circ 3} = A^{\circ 4}$, i.e., A converges for $c = 4$.

Now,

$$\hat{A} = A^{\circ 1} \cup A^{\circ 2} \cup A^{\circ 3} = \begin{bmatrix} \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.4\rangle & \langle 0.5, 0.4\rangle \\ \langle 0.7, 0.3\rangle & \langle 0.5, 0.4\rangle & \langle 0.8, 0.2\rangle & \langle 0.5, 0.3\rangle \\ \langle 0.6, 0.3\rangle & \langle 0.6, 0.4\rangle & \langle 0.6, 0.3\rangle & \langle 0.6, 0.1\rangle \end{bmatrix}.$$

It follows that $A \le \hat{A}$.

Theorem 10. *If A and B are two GIFMs, then*

$$A \le B \Rightarrow \hat{A} \le \hat{B} \text{ and } \breve{A} \ge \breve{B}.$$

Corollary 1. *For any GIFM A,* $\breve{A} \le A \le \hat{A}$.

Corollary 2. (i) *If GIFM A is any type of reflexive and transitive, then* $A = A \circ A$.

(ii) *If* GIFM *A is any type of irreflexive and c-transitive* $A = A * A$.

3. Comparison for the Operations of IFMs and GIFMs

We first recall some operations for IFM defined in [18] and show by means of examples that these operations are not valid for GIFM.

Definition 11 [18]. If $A = [\langle a_{ij\mu}, a_{ij\nu}\rangle]$ and $B = [\langle b_{ij\mu}, b_{ij\nu}\rangle]$ are two GIFMs, then

(1) $A \oplus B = [\langle (a_{ij\mu} + b_{ij\mu} - a_{ij\mu} \cdot b_{ij\mu}), (a_{ij\nu} \cdot b_{ij\nu})\rangle]$.

(2) $A \odot B = [\langle (a_{ij\mu} \cdot b_{ij\mu}), (a_{ij\nu} + b_{ij\nu} - a_{ij\nu} \cdot b_{ij\nu})\rangle]$.

(3) $A \mathbin{@} B = \left[\left\langle \dfrac{a_{ij\mu} + b_{ij\mu}}{2}, \dfrac{a_{ij\nu} + b_{ij\nu}}{2} \right\rangle\right]$.

(4) $A \$ B = [\langle \sqrt{a_{ij\mu} \cdot b_{ij\mu}}, \sqrt{a_{ij\nu} \cdot b_{ij\nu}} \rangle]$.

(5) $A \# B = \left[\left\langle \dfrac{2 \cdot a_{ij\mu} \cdot b_{ij\mu}}{a_{ij\mu} + b_{ij\mu}}, \dfrac{2 \cdot a_{ij\nu} \cdot b_{ij\nu}}{a_{ij\nu} + b_{ij\nu}} \right\rangle \right]$.

Example 12. Let

$$A = \begin{bmatrix} \langle 0.3, 0.7 \rangle & \langle 0.5\ 0.3 \rangle & \langle 0.8, 0.4 \rangle \\ \langle 0.4, 0.8 \rangle & \langle 0.4, 0.6 \rangle & \langle 0.7, 0.4 \rangle \\ \langle 0.3, 0.4 \rangle & \langle 0.8, 0.4 \rangle & \langle 0.4, 0.6 \rangle \end{bmatrix}$$

and

$$B = \begin{bmatrix} \langle 0.2, 0.6 \rangle & \langle 0.4, 0.7 \rangle & \langle 0.8, 0.4 \rangle \\ \langle 0.4, 0.8 \rangle & \langle 0.3, 0.5 \rangle & \langle 0.3, 0.6 \rangle \\ \langle 0.6, 0.2 \rangle & \langle 0.4, 0.8 \rangle & \langle 0.4, 0.4 \rangle \end{bmatrix}.$$

Let $A \oplus B = C$, $A \odot B = D$, $A @ B = E$, $A \$ B = F$ and $A \# B = G$. Then

$$c_{21\mu} = a_{21\mu} + b_{21\mu} - a_{21\mu} \cdot b_{21\mu} = 0.64 > 0.5$$

$$c_{21\nu} = a_{21\nu} \cdot b_{21\nu} = 0.64 > 0.5$$

$$d_{13\mu} = a_{13\mu} \cdot b_{13\mu} = 0.64 > 0.5$$

$$d_{13\nu} = a_{13\mu} + b_{13\mu} - a_{13\mu} \cdot b_{13\mu} = 0.64 > 0.5$$

$$e_{32\mu} = \frac{a_{32\mu} + b_{32\mu}}{2} = 0.6 > 0.5$$

$$e_{32\nu} = \frac{a_{32\nu} + b_{32\nu}}{2} = 0.6 > 0.5$$

$$f_{32\mu} = \sqrt{a_{32\mu} \cdot b_{32\mu}} = \sqrt{0.32} > 0.5$$

$$f_{32\nu} = \sqrt{a_{32\nu} \cdot b_{32\nu}} = \sqrt{0.32} > 0.5$$

$$g_{32\mu} = \frac{2 \cdot a_{32\mu} \cdot b_{32\mu}}{a_{32\mu} + b_{32\mu}} = \frac{0.64}{1.2} > 0.5$$

$$g_{32\nu} = \frac{2 \cdot a_{32\nu} \cdot b_{32\nu}}{a_{32\nu} + b_{32\nu}} = \frac{0.64}{1.2} > 0.5.$$

Since at least one element of $A \oplus B$, $A \odot B$, $A @ B$ and $A \$ B$, $A \# B$ are not satisfy the GIC and thus $A \oplus B$, $A \odot B$, $A @ B$, and $A \$ B$, $A \# B$ are not GIFMs.

4. New Operations for GIFMs

Definition 12. For any two GIFMs A and B, we define

$$A \oplus B = \left[\left\langle \frac{a_{ij\mu} + b_{ij\mu}}{2(a_{ij\mu} \cdot b_{ij\mu} + 1)}, \frac{a_{ij\nu} + b_{ij\nu}}{2(a_{ij\nu} \cdot b_{ij\nu} + 1)} \right\rangle\right],$$

$$A \odot B = [\langle (a_{ij\mu} \cdot b_{ij\mu}), (a_{ij\nu} \cdot b_{ij\nu}) \rangle].$$

Note 1. As for $a, b \in [0, 1]$, $\frac{(a+b)}{2(ab+1)} \leq 0.5$. Therefore $A \oplus B$ is a GIFM.

Definition 13. Let A_i for $i = 1, 2, ..., n$ be a set of GIFMs. Then the product $\overset{n}{\underset{i=1}{\otimes}} A_i = A$ (say), whose membership functions and non-membership functions are respectively defined as follows:

$$\mu_A = \begin{cases} \frac{1}{2}, & \text{if } \mu_{A_i} = 1 \text{ for } i = 1, 2, \ldots, n, \\ \dfrac{\sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (\mu_{A_{i_1}} \cdot \mu_{A_{i_2}} \cdots \mu_{A_{i_k}})\right]}{2[1 + (-1)^n (\mu_{A_1} \cdot \mu_{A_2} \cdots \mu_{A_n})]}, & \text{otherwise,} \end{cases} \tag{6}$$

where $i_k = 1, 2, ..., n$ and

$$\nu_A = \begin{cases} \frac{1}{2}, & \text{if } \nu_{A_i} = 1 \text{ for } i = 1, 2, \ldots, n, \\ \dfrac{\sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (\nu_{A_{i_1}} \cdot \nu_{A_{i_2}} \cdots \nu_{A_{i_k}})\right]}{2[1 + (-1)^n (\nu_{A_1} \cdot \nu_{A_2} \cdots \nu_{A_n})]}, & \text{otherwise,} \end{cases} \tag{7}$$

where $i_k = 1, 2, ..., n$.

Theorem 11. *If* $a_i \in [0, 1]$, *for* $i = 1, 2, \ldots, n$ *and* $a_i \neq 1$ *for at least one i, then following inequality holds good*:

$$\frac{1}{2} \geq \frac{\sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1} \cdot a_{i_2} \cdots a_{i_k})\right]}{2[1 + (-1)^n (a_1 \cdot a_2 \cdots a_n)]}, \quad \text{for } i_k = 1, 2, \ldots, n.$$

Proof. We have

$$\prod_{i=1}^{k} (1 - a_i) \geq 0$$

$$\Rightarrow 1 + \sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1} \cdot a_{i_2} \cdots a_{i_k})\right] \geq 0$$

$$\Rightarrow 1 + (-1)^n (a_1 \cdot a_2 \cdots a_n) \geq \sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1} \cdot a_{i_2} \cdots a_{i_k})\right]$$

$$\Rightarrow 1 \geq \frac{\sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1} \cdot a_{i_2} \cdots a_{i_k})\right]}{2[1 + (-1)^n (a_1 \cdot a_2 \cdots a_n)]} \quad (\text{as } a_1 \cdot a_2 \cdots a_n \neq 1)$$

$$\Rightarrow \frac{1}{2} \geq \frac{\sum_{k=1}^{n-1}\left[(-1)^{k+1} \sum_{i_1 < i_2 < \cdots < i_k} (a_{i_1} \cdot a_{i_2} \cdots a_{i_k})\right]}{2[1 + (-1)^n (a_1 \cdot a_2 \cdots a_n)]}.$$

□

Theorem 12. *Let* $\overset{n}{\underset{i=1}{\otimes}} A_i = A$. *Then A is a GIFM. In fact* $\mu_A \leq 0.5$ *and* $\nu_A \leq 0.5$.

Proof. The proof follows from Theorem 11. Note that $\overset{n}{\underset{i=1}{\otimes}} A_i = A$ is not only a GIFM but also an IFM. □

We take an example in practical field to illustrate this operation.

Example 13. In a company, director wish to give promotion of three employees $X = \{x_1, x_2, x_3\}$. An appraisal report has to be made by three supervisors regarding the dedication, expertness, job satisfactory of these employees ($Y = \{\text{dedication, expertness, job satisfactory}\}$). Each supervisor give his statement as "for and against" of the employees in the form of GIFMs. Then the generalized intuitionistic fuzzy matrix represent the relation R between the sets X and Y is given as follows:

$$A_1 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{dedication} & \text{expertness} & \text{job satisfactory} \end{array} \\ \begin{bmatrix} \langle 0.6, 0.5\rangle & \langle 0.3, 0.1\rangle & \langle 0.2, 0.3\rangle \\ \langle 0.4, 0.7\rangle & \langle 0.2, 0.8\rangle & \langle 0.4, 0.5\rangle \\ \langle 0.2, 0.8\rangle & \langle 0.4, 0.1\rangle & \langle 0.3, 0.7\rangle \end{bmatrix} \end{array},$$

$$A_2 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{dedication} & \text{expertness} & \text{job satisfactory} \end{array} \\ \begin{bmatrix} \langle 0.5, 0.3\rangle & \langle 0.4, 0.6\rangle & \langle 0.2, 0.5\rangle \\ \langle 0.4, 0.4\rangle & \langle 0.5, 0.4\rangle & \langle 0.6, 0.3\rangle \\ \langle 0.3, 0.8\rangle & \langle 0.9, 0.1\rangle & \langle 0.4, 0.5\rangle \end{bmatrix} \end{array},$$

$$A_3 = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{dedication} & \text{expertness} & \text{job satisfactory} \end{array} \\ \begin{bmatrix} \langle 0.5, 0.2\rangle & \langle 0.6, 0.3\rangle & \langle 0.7, 0.4\rangle \\ \langle 0.3, 0.5\rangle & \langle 0.4, 0.6\rangle & \langle 0.4, 0.4\rangle \\ \langle 0.5, 0.6\rangle & \langle 0.3, 0.7\rangle & \langle 0.8, 0.1\rangle \end{bmatrix} \end{array}.$$

Then the final conclusion by the experts are given by $\overset{3}{\underset{i=1}{\otimes}} A_i = A$

$$A = \overset{3}{\underset{i=1}{\otimes}} A_i = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \text{dedication} & \text{expertness} & \text{job satisfactory} \end{array} \\ \begin{bmatrix} \langle 0.44, 0.36\rangle & \langle 0.41, 0.37\rangle & \langle 0.4, 0.38\rangle \\ \langle 0.37, 0.44\rangle & \langle 0.38, 0.47\rangle & \langle 0.42, 0.38\rangle \\ \langle 0.3, 0.49\rangle & \langle 0.48, 0.5\rangle & \langle 0.45, 0.42\rangle \end{bmatrix} \end{array}.$$

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