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INERTIAL MANIFOLDS FOR NONAUTONOMOUS SKEW PRODUCT SEMIFLOWS

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Abstract

The mild solution of a nonlinear nonautonomous evolutionary equation $du/dt + A(t)u = F(u, t)$ can be formulated as a skew product semiflow in a product phase space. Under a spectral gap condition, it is shown that there exists an inertial manifold for this skew product semiflow. Instead of the Lyapunov-Perron method, the proof is fulfilled via the approach of conic invariance and incrementally exponential dichotomy and based on two conic differential inequalities. The construction of inertial manifold is made through an exponentially tracking integral manifold, in which the pullback is achieved also by the incremental dichotomy and a homotopy lemma. An illustration of the applications is shown by nonautonomous reaction-diffusion equations.

1. Introduction

It has been shown in recent two decades that for some nonlinear dissipative evolutionary equations in infinite dimensional Hilbert spaces there exist inertial manifolds, which by definition [7] is a finite-dimensional, positively invariant, and Lipschitz continuous manifold attracting all trajectories at a uniform exponential rate. This discovery [7]

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in 1988 had a strong impact on the studies of long-term and global dynamics of the solution semiflow of the underlying autonomous PDEs, most of them are semilinear parabolic equations and some of them are semilinear hyperbolic equations or even dispersive equations with a weak dissipation, cf. [6, 12, 17-19, 21-24] and more references therein.

The existence of inertial manifolds (IM) implies a *global reduction principle* of the infinite dimensional nonlinear dynamics in the sense of asymptotical completeness. That is, every solution of the nonlinear evolutionary PDE is tracked at a fast exponential rate by a solution trajectory on the IM. The majority of the existence results in this regard have been proved by the Lyapunov-Perron method whose essence is to seek for a fixed point of a nonlinear integral mapping based on spectral gap conditions connecting the dissipative part of the diffusion operator and the growth of the nonlinearity, as seen in the aforementioned references. Few results such as in [10, 12, 16-19, 22-24] were established under weaker conditions or without requiring the spectral gap conditions.

Although the original hope that IM will provide an approach to reducing the asymptotical studies of dissipative nonlinear PDEs to the finite-dimensional studies of ODEs in terms of *inertial form* has been partially justified, the restrictive spectral gap conditions and the lack of regularity information of the solutions on inertial manifolds pose challenges to its applications in a broader scope.

In the past decade, generalization of this seemingly very attractive concept IM and its searching methodology have been arousing the interests of many researchers from different areas. New results keep emerging, though not many in comparison with another popular topic of global attractors for infinite dimensional dynamical systems, and new pursuits are committed in several directions. Here we just mention some of them.

A comprehensive survey of the related topics until 1996 can be found in [17], in which the sharper estimates in a renovated proof of the Lyapunov-Perron mapping and more information about the flow of trajectories in the vicinity of an IM are acquired. The following information provides us with more insights toward IM.

First, an IM is characterized as a collection of all complete orbits defined for all $t \in \Re$, whose growth rates when $t \to -\infty$ do not exceed an exponential rate $e^{\sigma t}$. In this sense, one can roughly recognize an IM as a union of exponentially unstable manifolds. Second, an IM generates a continuous and invariant foliation of the entire phase space. Third, the normal hyperbolicity of an IM addressed in [17] and further generalized in [19, Chapter 7] turns out to imply the robustness of the IM with respect to small perturbations in the nonlinear structure of the original evolutionary PDE.

In [10], the differentiabilty of the solution semiflow of nonlinear parabolic equations in L^p spaces is used to show the existence of IM. In that paper, the existence of IM was also proved for some parabolic equations on a compact Riemannian manifold by using the favorable properties of large spectral gaps for the Laplace-Beltrami operators.

In [16], a mixture of analytical argument and geometric construction is exploited to prove the closure theorems on the existence of IM under the general conditions of approximations of nonlinearity. The applications of the obtained results to the Bubnov-Galerkin approximations of PDEs are interesting, provided the uniform Lipschitz condition on the inertial manifolds of the approximating equations is satisfied.

Besides, approximate inertial manifolds (AIM) are important and useful substitutes for inertial manifolds in many cases, for which the existence of IM is unknown. The results on AIM also provide new algorithms to find numerically some global approximations of the solution trajectories for dissipative PDEs.

Now let us come to the front of nonautonomous evolutionary PDEs. While the existence theory of global attractors has been substantially generalized from the autonomous dissipative equations to nonautonomous dissipative equations at the point level and at the orbit level in [2, 3, 13] and references therein, the investigation into the existence of inertial manifolds for nonautonomous dissipative PDEs seems remaining quite open to pursuits. In [4], the Lyapunov-Perron method is extended to treating the nonautonomous dissipative PDEs, in which a concept of nonautonomous inertial manifold is redefined to be a collection of surfaces in the Hilbert phase space H, in the form

$$\boldsymbol{\mathcal{M}} = \{\mathcal{M}_t = \text{Graph}\{\Phi(\cdot, t)\} : t \in \Re\}.$$

This layer structure that if $u(s) = u_0 \in \mathcal{M}_s$, then $u(t, u_0) \in \mathcal{M}_t$ causes the attraction less intuitive and more complicated. The attraction of such defined inertial manifold reads as follows: for any given bounded set B,

$$\sup\{\text{dist}_{D(A^\theta)}(u(t, t_0, u_0), \mathcal{M}_t) : u_0 \in B\} \leq C_B e^{-\mu(t-t_0)}.$$

In [8, 9], local finite-dimensional integral manifolds with exponential tracking property are constructed for nonautonomous evolutionary equations in a Hilbert space,

$$\frac{du}{dt} + A_0 u = R_0(u) + \varepsilon R_1(u, t),$$

where the nonautonomous part is a small perturbation of the autonomous nonlinearity. The construction is again based on the Lyapunov-Perron-type mapping. The existence is established in a local vicinity of an equilibrium of the associated autonomous equation and then a global uniform approximation of solutions is achieved by pasting the locally exponential approximation segments together on the union of finitely many local integral manifolds. However, the assumption is that the solution semigroup of the associated autonomous equation has only finite equilibrium points.

In [11] a new approach is presented to show the existence of inertial manifolds for *abstract* nonautonomous dynamical systems, which features a combination of the graph transformation mapping [12] and the use of squeezing properties.

In this paper, we shall formulate the mild solutions of a nonlinear nonautonomous evolutionary equation

$$\frac{du}{dt} + A(t)u = F(u, t),$$

as a skew product semiflow in a product space and then prove the existence of an inertial manifold for that skew product semiflow under a spectral condition and by means of construction of an integral manifold with exponentially tracking property.

The method we use to prove the existence of inertial manifold of the skew product semiflow consists of conic invariance and incrementally exponential dichotomy, which is different from and more explicit than the Lyapunov-Perron method. This methodology can be viewed as a generalization of the results in [5], in which a geometrically explicit integral manifold construction was introduced to proving the existence of inertial manifolds for a large class of autonomous dissipative PDEs without abstract fixed-point argument, where the key leverage was the *spectral blocking property* of the semiflow.

Here let us first recall the concept of skew product semiflow, see [19]. Consider a product space $E = W \times M$, where W is a Banach space (called state space) and M is a metric space (called base space). A semiflow $\pi = (\phi, \sigma) : E \times \Re^{+} \to E$ is said to be a *skew product semiflow* on E if the two component mappings ϕ and σ have the form

$$\phi = \phi(w, m, t) \quad \text{and} \quad \sigma = \sigma(m, t),$$

namely, σ does not depend on $w \in W$.

Let H be a separable, real Hilbert space with inner-produce $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let V be a real Hilbert space, which is densely and compactly imbedded in H. The inner-product and norm of the space V will be indicated by the corresponding subscript. Consider an initial value problem of a nonautonomous nonlinear evolutionary equation in H,

$$\frac{du}{dt} + A(t)u = F(u, t), \quad u(t_0) = u_0, \tag{1}$$

where $(t_0, u_0) \in \Re \times H$ is arbitrarily given.

We now introduce the following assumptions on the linear and nonlinear parts in (1):

(H1) The linear operator function $A(t) : \Re \to \mathcal{L}(V, H)$ is bounded and uniformly continuous on $[0, \infty)$. For each $t \in \Re$, $A(t) : D(A)(= V) \to H$ is self-adjoint and positively definite:

$$\exists\theta(\text{const}) > 0, \text{ such that } \langle A(t)u, u \rangle \geq \theta \| u \|^2, \quad \forall u \in V \text{ and } t \in \Re.$$

Moreover, $A(t), t \in \Re$, generates a strongly continuous evolution operator $U(t, \tau)$, $-\infty < \tau \leq t < \infty$, on the space H.

(H2) The nonlinear mapping $F : H \times \Re \to H$ is a continuous mapping with the global and uniform Lipschitz property that there exists a constant $K > 0$ such that

$$\| F(u_1, t) - F(u_2, t) \| \leq K \| u_1 - u_2 \|, \quad \forall u_1, u_2 \in H \text{ and } t \in \Re.$$

Briefly, we write this as $F \in C_{\text{Lip}}(H \times \Re, H)$. Assume that $F(0, t) = 0$.

Remark 1. First, when $A(t) \equiv A$ does not depend on t, the assumption (H1) reduces to that $A : D(A)(= V) \to H$ is a positive, self-adjoint, linear operator with compact resolvent. We refer to [15] and [20] for the generation of strongly continuous evolution operators. The illustration is seen in Section 5. Second, concerning the assumption (H2), it may not be satisfied by the original underlying PDE, since usually the Lipschitz constant K depends on the bounded set where u_1 and u_2 sit in and, moreover, usually the nonlinearity may cause that F does not map $H \times \Re$ into H, but maps $V^{\alpha} \times \Re$ into H, where V^{α} is an interpolation space in between V and H and with higher regularity than H. However, the assumption (H2) here can be validated due to the following reason. If the underlying evolutionary PDE is dissipative, then there exists an absorbing set. For many semilinear parabolic equations, oftentimes one can use the bootstrap method to show that the absorbing property is valid in a well-suited Sobolev space such as $L^{\infty}(\Omega)$. Thus one can modify the original equation by an appropriate truncation so that the assumption (H2) is satisfied by the *modified equation* which preserves the long-term dynamics of all the solutions of the original PDE. Again this will be illustrated in detail in Section 5 for some nonautonomous reaction diffusion equations. Alternatively speaking, (H2) can be validated within an absorbing set for dissipative equations.

Under the assumptions (H1) and (H2), it is easy to confirm that the mild solution, cf. [15, 20], of the initial value problem (1) exists uniquely and globally in H for $t \geq t_0$. We shall denote this solution by $u(t) = u(t; u_0, t_0)$, $t \geq t_0$.

For any solution $u(t)$, $t \in I$ (some interval), of the equation (1), the graph

$$\{(u(t), t) : t \in I\} \text{ in the product space } E = H \times \Re$$

is called an *integral curve*. If an integral curve is defined for all $t \in \Re$, it is called a *globally defined* integral curve. We take $E = H \times \Re$ to be the phase space, where $\Re$ stands for the time axis.

Lemma 1. *The mapping* $\pi : E \times [0, \infty) \to E$ *defined by*

$$\pi((u_0, \tau), t) = (u(t + \tau; u_0, \tau), t + \tau), \tag{2}$$

is a skew product semiflow on the space E. This skew product semiflow is briefly referred to as the SPS π *of the underlying evolution equation* (1).

Proof. By definition, $\pi = (\Phi, \sigma)$, where $\sigma(\tau, t) = \tau + t$ is obviously a semiflow on the space $\Re$, and $\Phi(u_0, \tau, t) = u(t + \tau; u_0, \tau)$ satisfies the following properties. First,

$$\Phi((u_0, \tau), 0) = u(\tau; u_0, \tau) = u_0.$$

Second,

$$\begin{aligned}\Phi(u_0, \tau, t + s) &= u(t + s + \tau; u_0, \tau) \quad \text{(by the uniqueness)}\\ &= u(t + s + \tau; u(t + \tau, u_0, \tau), t + \tau)\\ &= \Phi(u(t + \tau; u_0, \tau), t + \tau, s)\\ &= \Phi((\Phi(u_0, \tau), t), \sigma(\tau, t), s), \ \text{for any } (u_0, \tau) \in E \text{ and}\\ &\qquad t, s \geq 0.\end{aligned}$$

Third, by [19, Theorem 46.4], the mapping $\Phi(u_0, \tau, t) = u(t + \tau, u_0, \tau)$ is continuous in $(u_0, \tau, t) \in E \times [0, \infty)$. Assembling together the two components Φ and σ, one can confirm that the three axioms of semiflow are all satisfied by this mapping $\pi = (\Phi, \sigma)$. Here just check the cocycle property:

$$\begin{aligned}\pi(u_0,\ \tau,\ t+s) &= (\Phi(u_0,\ \tau,\ t+s),\ \sigma(\tau,\ t+s))\\ &= (\Phi(\Phi(u_0,\ \tau,\ t),\ \sigma(\tau,\ t),\ s),\ \sigma(\sigma(\tau,\ t),\ s))\\ &= \pi((\Phi(u_0,\ \tau,\ t),\ \sigma(\tau,\ t)),\ s)\\ &= \pi(\pi((u_0,\ \tau),\ t),\ s),\end{aligned}$$

for any $(u_0,\ \tau) \in E$ and any $t, s \geq 0$.

The official definition of inertial manifold is seen in [7] and [19, Chapter 8]: An inertial manifold for the solution semiflow of an evolutionary equation is a finite-dimensional, Lipschitz continuous, positively invariant manifold which attracts every trajectory at a uniform exponential rate. See also [4, 11, 12, 17] for this concept. Now let us introduce another concept *tracking integral manifold*, which will be briefly referred to as TIM.

Definition 1. A set $\mathfrak{M} \subset E = H \times \Re$ is called a *tracking integral manifold* for the skew product semiflow π defined in (2) generated by the underlying equation (1), if the following three conditions are satisfied:

(i) The set $\mathfrak{M}$ entirely consists of some globally defined integral curves of (1).

(ii) $\mathfrak{M}$ is a finite-dimensional, Lipschitz continuous manifold in E.

(iii) There is a constant $\beta > 0$ such that for every solution $u(t;\, u_0,\, t_0)$, $t \geq t_0$, of (1) there is an integral curve $(v(t),\, t)$, $t \in \Re$, on $\mathfrak{M}$ with the tracking property

$$\| u(t) - v(t) \| \leq C(\| u_0 \|)\exp(-\beta(t - t_0)), \text{ for } t \geq t_0,$$

where $u_0 = u(t_0)$ and $C(\| u_0 \|)$ is a constant depending on the norm of u_0.

Lemma 2. *Suppose that there exists a tracking integral manifold* (*TIM*) $\mathfrak{M}$ *for the skew product semiflow* π *defined by* (2). *Then this* $\mathfrak{M}$ *must be an inertial manifold* (*IM*) *for the skew product semiflow* π.

Proof. We just check all the conditions for IM are satisfied by this TIM $\mathfrak{M}$. By (ii) in Definition 1, $\mathfrak{M}$ is a finite-dimensional, Lipschitz continuous manifold in E. $\mathfrak{M}$ is positively invariant because of (i) in Definition 1. The fact that $\mathfrak{M}$ attracts every trajectory of the skew product semiflow π at a uniform rate is implied by (iii) in Definition 1. In fact, the inequality in (iii) yields

$$\begin{aligned}\operatorname{dist}_E(\pi((u_0, \tau), t), \mathfrak{M}) &= \inf\{\| u(t+\tau) - w \| + | t + \tau - s | : \forall (w, s) \in E\} \\ &\leq \| u(t+\tau) - v(t+\tau) \| + | t + \tau - (t + \tau) | \\ &\leq C(\| u_0 \|) \exp(-\beta(t - \tau)), \text{ for } t \geq \tau.\end{aligned}$$

Therefore this TIM $\mathfrak{M}$ turns out to be an IM for the SPS π in the space E.

2. Conic Inequalities and Incremental Dichotomy

Now we make another assumption which can also be called the *spectral gap condition*.

(H3) There is an orthogonal decomposition of the Hilbert space H,

$$H = P(H) \oplus Q(H), \quad P(H) \perp Q(H), \quad \dim P(H) = N < \infty,$$

where P is the orthogonal projection from H onto $P(H)$, and $Q = I_H - P$ is the complementary orthogonal projection, such that there exist positive constants λ_0 and Λ, with

$$\langle A(t)p, p \rangle \leq \lambda_0, \text{ for any } p \in P(H),$$

$$\langle A(t)q, q \rangle \geq \Lambda, \text{ for any } q \in Q(H),$$

and

$$\Lambda - \lambda_0 > 2K, \tag{3}$$

where K is the uniform Lipschitz constant of the nonlinear mapping F in the assumption (H2).

The main result of this paper is stated in the following theorem.

Theorem 1. *Under the assumptions* (H1), (H2) *and* (H3), *there exists an inertial manifold* $\mathfrak{M} \subset E$ *for the skew product semiflow* π *generated* *by the nonautonomous evolutionary equation, such that*

$$\mathfrak{M} = \text{Graph}(\Psi),$$

where $\Psi : P(H) \times \Re \to Q(H)$ *is a continuous mapping and is uniformly Lipschitz continuous in the component* $p \in P(H)$.

The main theorem will be proved in Sections 2, 3 and 4. We set

$$\lambda = \frac{1}{2}(\Lambda + \lambda_0), \quad \delta = \frac{1}{2}(\Lambda - \lambda_0), \quad \mu = \delta - K. \tag{4}$$

Note that $\mu > 0$ due to (3). In order to prove Theorem 1, we begin with any two solutions of the equation (1),

$$u_i(t) = p_i(t) + q_i(t), \quad i = 1, 2,$$

where $p_i(t) = Pu_i(t)$ and $q_i(t) = Qu_i(t)$. The increment between the two solutions and their components are defined by

$$u(t) = u_1(t) - u_2(t),\ p(t) = p_1(t) - p_2(t),\ q(t) = q_1(t) - q_2(t). \tag{5}$$

The incremental components $p(t)$ and $q(t)$ satisfy the inequalities in the following lemma.

Lemma 3. *There exists a constant* $0 < b < 1$, *such that for any two mild solutions* $u_1(t)$ *and* $u_2(t)$ *of the equation* (1), *the associated* $y(t)$ *and* $z(t)$ *defined by*

$$y(t) = (\| p(t) \|^2 - b\| q(t) \|^2) e^{2\lambda t}, \tag{6}$$

$$z(t) = (\| q(t) \|^2 - b\| p(t) \|^2) e^{2\lambda t} \tag{7}$$

satisfy the inequalities

$$y(t) \geq y(\tau) e^{2\mu(t-\tau)}, \quad t \geq \tau, \tag{8}$$

$$z(t) \leq z(\tau) e^{-2\mu(t-\tau)}, \quad t \geq \tau, \tag{9}$$

namely,

$$(\| p(t) \|^2 - b\| q(t) \|^2) \geq (\| p(\tau) \|^2 - b\| q(\tau) \|^2) e^{2(\mu-\lambda)(t-\tau)}, \quad t \geq \tau, \tag{10}$$

$$(\| q(t) \|^2 - b\| p(t) \|^2) \le (\| q(\tau) \|^2 - b\| p(\tau) \|^2) e^{-2(\mu+\lambda)(t-\tau)}, \quad t \ge \tau, \quad (11)$$

where $\mu + \lambda = \Lambda - K > 0$ *and* $\mu - \lambda = -(\lambda_0 + K) < 0$. *We shall refer to* (10) *and* (11) *as the incremental dichotomic inequalities.*

Proof. Note that for any mild solutions u_i with initial data $u_i(\tau) \in V = D(A(t))$, $i = 1, 2$, they are actually strong solutions of the equation (1). Hence, $p(\cdot)$ and $q(\cdot)$ satisfy the following equations for almost every $t > \tau$,

$$\frac{dp}{dt} + A(t)p = P[F(u_1, t) - F(u_2, t)], \quad (12)$$

$$\frac{dq}{dt} + A(t)q = Q[F(u_1, t) - F(u_2, t)]. \quad (13)$$

For a constant b, $0 < b < 1$, taking the inner-products of the equation (12) with $p(t)$ and the equation (13) with $bq(t)$, respectively, then substracting the latter from the former, one can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\| p(t) \|^2 - b\| q(t) \|^2) \\
&= -\langle A(t)p, p\rangle + b\langle A(t)q, q\rangle \\
&\quad + \langle p, P[F(u_1, t) - F(u_2, t)]\rangle - b\langle q, Q[F(u_1, t) - F(u_2, t)]\rangle \\
&\ge -(\lambda - \delta)\| p \|^2 + b(\lambda + \delta)\| q \|^2 - K\| p - bq \| \| u_1 - u_2 \| \\
&\ge -(\lambda - \delta)\| p \|^2 + b(\lambda + \delta)\| q \|^2 - \frac{1}{2}K(\| p \|^2 + b^2\| q \|^2 + \| p + q \|^2) \\
&\ge -(\lambda - \delta + K)\| p \|^2 + b\left[\lambda + \delta - \frac{1}{2}K\left(b + \frac{1}{b}\right)\right]\| q \|^2. \quad (14)
\end{aligned}$$

Note that

$$\lim_{b\to 1}\frac{1}{2}\left(b + \frac{1}{b}\right) = 1.$$

Since (3) in (H3) implies $\delta > K$, so that

$$\frac{2\delta - K}{K} = \frac{2(\delta - K) + K}{K} > 1,$$

we can choose $0 < b < 1$ and close to 1, such that

$$\frac{1}{2}\left(b + \frac{1}{b}\right) \le \frac{2\delta - K}{K} = 1 + \frac{2\mu}{K}. \tag{15}$$

Fix this constant b in (15). Then we find that

$$\lambda + \delta - \frac{1}{2}K\left(b + \frac{1}{b}\right) \ge \lambda + \delta - K\left(\frac{2\delta - K}{K}\right) = \lambda - \delta + K,$$

which is substituted into the last inequality in (14) to yield

$$\frac{1}{2}\frac{d}{dt}(\| p(t)\|^2 - b\| q(t)\|^2) \ge -(\lambda - \delta + K)(\| p(t)\|^2 - b\| q(t)\|^2). \tag{16}$$

According to (6), we see that $y(t)$ satisfies the first conic differential inequality

$$\frac{dy}{dt} \ge 2\mu y. \tag{17}$$

Similarly one can deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\| q(t)\|^2 - b\| p(t)\|^2) \\
&\le -(\lambda + \delta)\| q\|^2 + b(\lambda - \delta)\| p\|^2 \\
&\quad + \frac{1}{2}K(2\| q\|^2 + (b^2 + 1)\| p\|^2) \\
&= -(\lambda + \delta - K)\| q\|^2 + b\left[\lambda - \delta + \frac{1}{2}K\left(b + \frac{1}{b}\right)\right]\| p\|^2.
\end{aligned} \tag{18}$$

With the same chosen b which satisfies (15), one has

$$\lambda - \delta + \frac{1}{2}K\left(b + \frac{1}{b}\right) \le \lambda + \delta - K,$$

so that

$$\frac{1}{2}\frac{d}{dt}(\| q(t)\|^2 - b\| p(t)\|^2) \le -(\lambda + \delta - K)(\| q(t)\|^2 - b\| p(t)\|^2). \tag{19}$$

According to (7), $z(t)$ satisfies the second conic differential inequality

$$\frac{dz}{dt} \le -2\mu z. \tag{20}$$

It is easy to solve these two differential inequalities (17) and (20) to obtain the incremental dichotomic inequalities (10) and (11). Finally, since the space V is densely imbedded in H, by the continuous dependence of mild solutions on initial data, we can conclude that the inequalities (10) and (11) hold for the incremental components of any two mild solutions of the equation (1).

Remark 2. Here is the comment on the implication of the two incremental dichotomic inequalities (10) and (11). By the first inequality (10), the conic sector

$$S_- = \{p \oplus q \in H : \| p \| \leq \sqrt{b} \| q \|\}$$

is negatively invariant. If $u(t) = u_1(t) - u_2(t) = p(t) + q(t)$ is inside the sector S_- for some t, then for $\tau \leq t$, it holds that $u(\tau) \in S_-$. By the second inequality (11), on the other hand, the conic sector

$$S_+ = \{p \oplus q \in H : \| q \| \leq \sqrt{b} \| p \|\}$$

is positively invariant. If $u(\tau) = u_1(\tau) - u_2(\tau) = p(\tau) + q(\tau)$ is inside the sector S_+ for some τ, then for $t \geq \tau$, it holds that $u(t) \in S_+$.

We can interplay the inequalities (10) and (11) to derive more useful properties of the incremental components $p(t)$ and $q(t)$ of any pair of solutions of (1).

Lemma 4. *Let* $u(t) = p(t) + q(t)$ *be given as in* (5). *If* $p(T) = 0$ *for some* $T \in \Re$, *then for any t satisfying* $\tau \leq t \leq T$, *the following properties are satisfied,*

$$\| p(t) \| \leq \sqrt{b} \| q(t) \|, \tag{21}$$

$$\| q(T) \| e^{(\mu+\lambda)(T-t)} \leq \| q(t) \| \leq \frac{1}{\sqrt{1-b^2}} \| q(\tau) \| e^{-(\mu+\lambda)(t-\tau)}, \tag{22}$$

and

$$\| u(t) \| \leq \frac{1}{\sqrt{1-b}} \| q(\tau) \| e^{-(\mu+\lambda)(t-\tau)} \leq \frac{1}{1-b} \| u(\tau) \| e^{-(\mu+\lambda)(t-\tau)}. \tag{23}$$

Proof. Since $p(T) = 0$ and $\tau \le t \le T,$ we get (21) directly from the inequality (10). Then from the inequality (11), in which letting $t = T$ and $\tau = t,$ we obtain

$$\| q(T) \|^2 e^{2(\mu+\lambda)(T-t)} \le \| q(t) \|^2 - b \| p(t) \|^2 \le \| q(t) \|^2.$$

So the first inequality in (22) is shown. In order to show the second inequality in (22), we see from (6) and (7) that

$$by(t) + z(t) = (1 - b^2) \| q(t) \|^2 e^{2\lambda t}.$$

It follows that

$$\| q(t) \|^2 = \frac{1}{1-b^2} (by(t) + z(t)) e^{-2\lambda t} \le \frac{1}{1-b^2} z(t) e^{-2\lambda t}$$

$$= \frac{1}{1-b^2} (\| q(t) \|^2 - b \| p(t) \|^2),$$

because (21) implies that $y(t) \le 0$ for $\tau \le t \le T.$ Now using (11), we obtain

$$\| q(t) \|^2 \le \frac{1}{1-b^2} (\| q(\tau) \|^2 - b \| p(\tau) \|^2) e^{-2(\mu+\lambda)(t-\tau)}$$

$$\le \frac{1}{1-b^2} \| q(\tau) \|^2 e^{-2(\mu+\lambda)(t-\tau)}.$$

Thus the second inequality in (22) is valid. Finally, from (21) and (22) it follows that

$$\| u(t) \|^2 = \| p(t) \|^2 + \| q(t) \|^2 \le (1 + b) \| q(t) \|^2$$

$$\le \frac{1}{1-b} \| q(\tau) \|^2 e^{-2(\mu+\lambda)(t-\tau)},$$

and (23) is shown.

Lemma 5. *Let* $u(t) = p(t) + q(t)$ *be given as in* (5). *If* $q(\tau) = 0$ *for some* $\tau \in \Re,$ *then for any* $T \ge t \ge \tau,$ *the following properties are satisfied,*

$$\| q(t) \| \le \sqrt{b} \| p(t) \|, \tag{24}$$

$$\| p(\tau) \| e^{(\mu-\lambda)(t-\tau)} \leq \| p(t) \| \leq \frac{1}{\sqrt{1-b^2}} \| p(T) \| e^{-(\mu-\lambda)(T-t)}, \tag{25}$$

and

$$\| u(t) \| \leq \frac{1}{\sqrt{1-b}} \| p(T) \| e^{-(\mu-\lambda)(T-t)} \leq \frac{1}{\sqrt{1-b}} \| u(T) \| e^{-(\mu-\lambda)(T-t)}. \tag{26}$$

Proof. Since $q(\tau) = 0$ and $t \geq \tau$, we can get (24) directly from (11). By (10), we can get

$$\| p(t) \|^2 \geq \| p(t) \|^2 - b\| q(t) \|^2 \geq \| p(\tau) \|^2 e^{2(\mu-\lambda)(t-\tau)},$$

from which the first inequality of (25) follows. In order to show the second inequality of (25), we find that

$$y(t) + bz(t) = (1-b^2) \| p(t) \|^2 e^{2\lambda t}$$

and then by (10) we have

$$\begin{aligned}
\| p(t) \|^2 &= \frac{1}{1-b^2} (y(t) + bz(t)) e^{-2\lambda t} \ \text{(where } z(t) \leq z(\tau) e^{-2\mu(t-\tau)} \leq 0) \\
&\leq \frac{1}{1-b^2} y(t) e^{-2\lambda t} = \frac{1}{1-b^2} (\| p(t) \|^2 - b\| q(t) \|^2) \\
&\leq \frac{1}{1-b^2} (\| p(T) \|^2 - b\| q(T) \|^2) e^{-2(\mu-\lambda)(T-t)} \\
&\leq \frac{1}{1-b^2} \| p(T) \|^2 e^{-2(\mu-\lambda)(T-t)}.
\end{aligned}$$

This shows that the second inequality of (25) is valid. Finally, from (24) and (25) it follows that

$$\begin{aligned}
\| u(t) \|^2 &= \| p(t) \|^2 + \| q(t) \|^2 \leq (1+b) \| p(t) \|^2 \\
&\leq \frac{1}{1-b} \| p(T) \|^2 e^{-2(\mu-\lambda)(T-t)},
\end{aligned}$$

so (26) is proved.

3. Construction of Inertial Manifold

In this section, we shall construct a tracking integral manifold in the phase space $E = H \times \Re$, which by Lemma 2 must be an inertial manifold for the skew product semiflow π.

Definition 2. Define a set in the space E to be

$$\mathfrak{M} = \{\text{globally defined integral curves } (u(t), t) \in E, \\ t \in \Re \mid \limsup_{t \to -\infty} \| e^{\lambda t} u(t) \| < \infty\}. \tag{27}$$

Note that $F(0, t) = 0$ in the assumption (H2) implies that the set $\mathfrak{M}$ is nonempty.

The first objective is to show that $\mathfrak{M} = \mathrm{Graph}(\Psi)$ for some Lipschitz mapping $\Psi : P(H) \times \Re \to Q(H)$.

Lemma 6. *Let* $(u_i(t), t)$, $i = 1, 2$, $t \in \Re$, *be any two integral curves on the set* $\mathfrak{M}$ *defined by* (27). *Let* $p_i(t) = Pu_i(t)$ *and* $q_i(t) = Qu_i(t)$, $i = 1, 2$. *Then*

$$\| q_1(t) - q_2(t) \| \le \sqrt{b} \| p_1(t) - p_2(t) \|. \tag{28}$$

Proof. Since we have

$$\limsup_{t \to -\infty} \| e^{\lambda t} u(t) \| < \infty,$$

it follows that, by (7),

$$\begin{aligned} \| z(t) \| &\le (\| q_1(t) - q_2(t) \|^2 + b \| p_1(t) - p_2(t) \|^2) e^{2\lambda t} \\ &\le (1 + b) \| u_1(t) - u_2(t) \|^2 e^{2\lambda t} \\ &\le 2(1 + b)(\| u_1(t) \|^2 + \| u_2(t) \|^2) e^{2\lambda t} \end{aligned}$$

and consequently

$$\limsup_{t \to -\infty} \| z(t) \| < \infty.$$

Since $z(t)$ satisfies (9), in which we can let $\tau \to -\infty$, it is valid that

$$z(t) \le \limsup_{\tau \to -\infty} z(\tau) e^{-2\mu(t - \tau)} = 0.$$

Therefore (28) is proved.

This lemma shows that the increment between any two integral curves on the set $\mathfrak{M}$ is in the cylindrical sector $S_+ \times \Re$. Based on this property, we show that the set $\mathfrak{M}$ can be expressed as the graph of a continuous mapping. Let the orthogonal projection from E onto its finite-dimensional subspace $P(H) \times \Re$ be denoted by *Proj*.

Lemma 7. *The projection* $Proj : \mathfrak{M} \to P(H) \times \Re$ *is a one-to-one mapping. There exists a continuous mapping* $\Psi : \mathrm{Dom}(\Psi) (\subset P(H) \times \Re) \to Q(H)$ *such that*

$$\| \Psi(p_1, t) - \Psi(p_2, t) \| \leq \sqrt{b} \| p_1 - p_2 \|, \tag{29}$$

and that the set $\mathfrak{M}$ *defined in* (27) *is the graph of* Ψ,

$$\mathfrak{M} = \mathrm{Graph}(\Psi). \tag{30}$$

Moreover,

$$\Psi(0, t) = 0 \text{ and } \| \Psi(p, t) \| \leq \sqrt{b} \| p \|.$$

Proof. Note that any point $w \in E$ can be written as $w = (p \oplus q, t) = (p, q, t)$, where the two components $p = Pw$ and $q = Qw$. If there are two points $w_i = (p_i, q_i, t_t)$, $i = 1, 2$, on the set $\mathfrak{M}$ such that $Proj(w_1) = Proj(w_2)$, then one has $p_1 = p_2$ and $t_1 = t_2$.

By the construction of $\mathfrak{M}$, there must be two integral curves $(u_i(t), t)$, $i = 1, 2$ on $\mathfrak{M}$ such that $u_i(t) = p_i \oplus q_i$, $i = 1, 2$, and $t = t_1 = t_2$. Then (28) implies that $q_1 = q_2$. Thus it is proved that $Proj : \mathfrak{M} \to P(H) \times \Re$ is one-to-one.

As a consequence, there exists a mapping

$$\Psi = Q\, Proj^{-1} : \mathrm{Dom}(\Psi) \to Q(H), \text{ where } \mathrm{Dom}(\Psi) = Proj(\mathfrak{M}),$$

such that (30) holds. Then (28) and (30) imply (29). Moreover, by the continuity of the integral curves and (29), it follows that $\Psi(p, t)$ is a continuous mapping with respect to (p, t) and it is uniformly continuous in p-component. Since $u(t) = 0$ for all $t \in \Re$ is an equilibrium solution, we have $\Psi(0, t) = 0$ for $t \in \Re$. As a consequence, (29) yields $\| \Psi(p, t) \| \leq \sqrt{b} \| p \|$. The proof is completed.

The next objective is to show that the mapping $Proj : \mathfrak{M} \to P(H) \times \Re$ is a surjective mapping. That is,

$$\mathrm{Dom}(\Psi) = Proj(\mathfrak{M}) = P(H) \times \Re,$$

and Ψ is a mapping defined on the entire subspace $P(H) \times \Re$. This will be the main result of this section.

Theorem 2. *For any* $(p, \tau) \in P(H) \times \Re$, *there exists a solution* $u(t)$, $t \in \Re$, *of the equation* (1), *which satisfies the conditions*

(i) $Pu(\tau) = p$, *and*

(ii) $\limsup_{t \to -\infty} \| e^{\lambda t} u(t) \| < \infty$.

Consequently it holds that

$$Proj(\mathfrak{M}) = P(H) \times \Re.$$

The proof of Theorem 2 goes through the following several lemmas in this section. First, let us denote the nonlinear evolution operator associated with the mild solution of the equation (1) by $S(t, \tau)$, $-\infty < \tau \le t < \infty$. It is defined by

$$S(t, \tau) u_0 = u(t; u_0, \tau).$$

Lemma 8. *Suppose that a continuous mapping* $\eta : \Re^n \times [0, T] \to \Re^n$ *satisfies the condition*

$$\eta(x, 0) = x, \text{ for any } x \in \Re^n.$$

Let B_r *be the open ball in* $\Re^n$ *of radius r and centered at the origin. If a point* $p \in B_r$ *satisfies*

$$p \notin \eta(\partial B_r, t) \text{ for any } t \in [0, T],$$

then one has $p \in \eta(B_r, T)$.

Proof. This homotopy result can be shown as follows. Define

$$J = \{t \in [0, T] : p \in \eta(B_r, t)\}.$$

By the given conditions, one can show that J is a nonempty, open, and closed subset of the interval $[0, T]$. Thus $J = [0, T]$ so that $p \in \eta(B_r, T)$. Related results can be found in [1].

The next lemma is a key result in the indispensable *pullback argument* for proving the existence of inertial manifolds. A major difference between the Lyapunov-Perron method and the approach taken in this work lies in the pullback processing. The pullback in the Lyapunov-Perron method is analytically reflected in the nonlinear Lyapunov-Perron integral mapping over the time interval $(-\infty, 0]$, while here the pullback can be seen more explicitly in the orbit geometry.

Lemma 9. *For any given* $(p_0, \tau) \in P(H) \times \Re$ *and any given* $T > 0$, *there exists a unique* $p^* \in P(H)$ *such that*

$$P(S(\tau, \tau - T)p^*) = p_0, \tag{31}$$

and

$$\| p^* \| \leq \| p_0 \| e^{\lambda T}. \tag{32}$$

Proof. Here λ is defined in (4). Define a mapping $\eta : P(H) \times \Re \to P(H)$ by

$$\eta(p, t) = P(S(\tau - T + t, \tau - T)(pe^{\lambda T}))e^{-\lambda(T-t)}, \tag{33}$$

where $pe^{\lambda T} \in P(H)$ and $t \geq 0$. By the strong continuity of the evolution operator $S(\cdot, \cdot)$, it is seen that this mapping η is continuous in (p, t). Below we can verify that the conditions in Lemma 8 are all satisfied. First, for any $p \in P(H)$, we have

$$\eta(p, 0) = P(S(\tau - T, \tau - T)(pe^{\lambda T}))e^{-\lambda T} = (pe^{\lambda T})e^{-\lambda T} = p.$$

Second, using the first inequality in (25), which is valid for the incremental components and here we can take one solution to be the trivial solution (the zero equilibrium), we get

$$\| \eta(p, t) \| \geq \| pe^{\lambda T} \| e^{-\lambda(T-t)} e^{(\mu-\lambda)t} \geq \| p \|, \text{ for any } t \in [0, T].$$

Thus for any given $p_0 \in P(H)$, there is a finite number $r > 0$ such that $\| p_0 \| < r$. The above inequality implies that

$$p_0 \notin \eta(\partial B_r, t) \text{ for any } t \in [0, T],$$

since otherwise we would have $\| p_0 \| \geq r > \| p_0 \|$, which is a contradiction.

Therefore, by Lemma 8, we can claim that $p_0 \in \eta(B_r, T)$. It means that there is a point $p \in B_r \subset P(H)$ such that

$$\eta(p, T) = P(S(\tau, \tau - T)(pe^{\lambda T})) = p_0.$$

Let $p^* = pe^{\lambda T}$. The above equality means that (31) is proved. Moreover we have

$$\| p^* \| = \| e^{\lambda T} \| < re^{\lambda T}, \text{ for any } r > \| p_0 \|.$$

Let $r \to \| p_0 \|$ in the above inequality. Then we obtain (32).

Finally we can show that such a point $p^* \in P(H)$ in (31) is unique. Indeed this follows directly from (21), because if there are $p_1, p_2 \in P(H)$ satisfying

$$P(S(\tau, \tau - T)p_1) = p_0 = P(S(\tau, \tau - T)p_2),$$

then

$$\| p_1 - p_2 \| \leq \sqrt{b} 0 = 0.$$

The proof is completed.

Definition 3. For any given $(p_0, \tau) \in P(H) \times \Re$ and $T > 0$, define the pullback point of p_0 from τ to $\tau - T$ to be

$$u_{\tau - T} = u_{\tau - T}(p_0) = p^*, \tag{34}$$

where $p^* \in P(H)$ is the unique point in (31). Moreover, for $0 < T_1 < T_2$, define

$$u_{\tau - T_2}^{\tau - T_1} = S(\tau - T_1, \tau - T_2) u_{\tau - T_2}. \tag{35}$$

Lemma 10. *For any* $(p_0, \tau) \in P(H) \times \Re$ *and any* $T_2 \geq T_1 \geq 0$, *it holds that*

$$\| Pu_{\tau-T_2}^{\tau-T_1} \| \leq \frac{1}{1-b} \| p_0 \| e^{\lambda T_1}, \tag{36}$$

$$\| Qu_{\tau-T_2}^{\tau-T_1} \| \leq \frac{\sqrt{b}}{1-b} \| p_0 \| e^{\lambda T_1}. \tag{37}$$

Proof. Set

$$u_1(t) = S(t, \tau - T_1) u_{\tau-T_1} \text{ and } u_2(t) = S(t, \tau - T_2) u_{\tau-T_2}.$$

These are two solutions of the equation (1) for $t \geq \tau - T_1$. By (35),

$$u_2 = S(t, \tau - T_1) u_{\tau-T_2}^{\tau-T_1}.$$

According to Definition 3, one has

$$Pu_1(\tau) = Pu_2(\tau) = p_0 \text{ and } P(u_1(\tau) - u_2(\tau)) = 0.$$

By Lemma 4 and (21), we have

$$\| Pu_{\tau-T_2}^{\tau-T_1} - Pu_{\tau-T_1} \| \leq \sqrt{b} \| Qu_{\tau-T_2}^{\tau-T_1} - Qu_{\tau-T_1} \| = \sqrt{b} \| Qu_{\tau-T_2}^{\tau-T_1} \|, \tag{38}$$

since $Qu_{\tau-T_1} = 0$. Furthermore, by Lemma 5 and (24), we have

$$\| Qu_{\tau-T_2}^{\tau-T_1} \| \leq \sqrt{b} \| Pu_{\tau-T_2}^{\tau-T_1} \| = \sqrt{b} \| Pu_2(\tau - T_1) \|, \tag{39}$$

since $Qu_{\tau-T_2} - Q(0) = 0$, where $Q(0)$ stands for the q-component of the trivial solution. Then from (38) and (39) we obtain

$$\| Pu_{\tau-T_2}^{\tau-T_1} \| - \| Pu_{\tau-T_1} \| \leq \| Pu_{\tau-T_2}^{\tau-T_1} - Pu_{\tau-T_1} \| \leq b \| Pu_{\tau-T_2}^{\tau-T_1} \|.$$

By (32), the above inequality implies that

$$\| Pu_{\tau-T_2}^{\tau-T_1} \| \leq \frac{1}{1-b} \| Pu_{\tau-T_1} \| = \frac{1}{1-b} \| u_{\tau-T_1} \| \leq \frac{1}{1-b} \| p_0 \| e^{\lambda T_1}$$

and

$$\| Qu_{\tau-T_2}^{\tau-T_1} \| \leq \sqrt{b} \| Pu_{\tau-T_2}^{\tau-T_1} \| \leq \frac{\sqrt{b}}{1-b} \| p_0 \| e^{\lambda T_1}.$$

Therefore, (36) and (37) are validated.

Lemma 11. *For any given* $(p_0, \tau) \in P(H) \times \Re$ *and* $T > 0$, *define a function* $u(t)$ *by*

$$u(t) = \begin{cases} S(t, \tau) p_0, & t \geq \tau, \\ \lim_{T \to \infty} S(t, \tau - T) u_{\tau - T}(p_0), & t < \tau. \end{cases} \tag{40}$$

Then the limit in (40) *exists in* H *for every* $t < \tau$ *and this* $u(t)$ *is well-defined for all* $t \in \Re$.

Proof. Fix a $t_0 \in (-\infty, \tau)$. We shall prove that the function $w(T)$ defined by

$$w(T) = S(t_0, \tau - T) u_{\tau - T}(p_0), \quad T \in [\tau - t_0, \infty), \tag{41}$$

has the Cauchy property as $T \to \infty$. For any $T_2 > T_1 \geq \tau - t_0$, as in the proof of Lemma 10, let us consider $u_1(t) = S(t, \tau - T_1) u_{\tau - T_1}$ and $u_2(t) = S(t, \tau - T_2) u_{\tau - T_2}$. Note that

$$w(T_1) = u_1(t_0), \quad w(T_2) = u_2(t_0),$$

and

$$u_2(t) = S(t, \tau - T_1) u_{\tau - T_2}^{\tau - T_1}, \text{ for } t \geq \tau - T_1.$$

Since $P(u_1(\tau) - u_2(\tau)) = p_0 - p_0 = 0$, by Lemma 4 and (23), noting that

$$\tau > t_0 \geq \tau - T_1 > \tau - T_2,$$

we have

$$\begin{aligned} \| w(T_2) - w(T_1) \| &= \| u_2(t_0) - u_1(t_0) \| \\ &\leq \frac{1}{\sqrt{1 - b}} \| Q u_1(\tau - T_1) - Q u_2(\tau - T_1) \| e^{-(\mu + \lambda)(t_0 - (\tau - T_1))} \\ &= \frac{1}{\sqrt{1 - b}} \| Q u_2(\tau - T_1) \| e^{-(\mu + \lambda)(t_0 - (\tau - T_1))} \\ &= \frac{1}{\sqrt{1 - b}} \| Q u_{\tau - T_2}^{\tau - T_1} \| e^{-(\mu + \lambda)(t_0 - \tau + T_1)}, \end{aligned} \tag{42}$$

since $Qu_1(\tau - T_1) = 0$. Now substituting (37) into (42), we get

$$\| w(T_2) - w(T_1) \| \leq \frac{1}{\sqrt{1-b}} \frac{\sqrt{b}}{1-b} \| p_0 \| e^{\lambda T_1} e^{-(\mu+\lambda)(t_0 - \tau + T_1)}$$

$$= \frac{\sqrt{b}}{\sqrt{(1-b)^3}} \| p_0 \| e^{(\mu+\lambda)(\tau - t_0)} e^{-\mu T_1}$$

$$= \frac{\sqrt{b}}{\sqrt{(1-b)^3}} \| p_0 \| e^{(\mu+\lambda)(\tau - t_0)} e^{-\mu \min\{T_1, T_2\}}, \tag{43}$$

in which $p_0 \in P(H)$, $\tau \in \Re$, and $t_0 (< \tau)$ are relatively fixed. By (41) and (43), we find that for any $\varepsilon > 0$, there is a $T_0 = T_0(\varepsilon) > 0$, such that

$$e^{-\mu T_0} < \varepsilon.$$

Then we have

$$\| w(T_2) - w(T_1) \| = \| S(t_0, \tau - T_2) u_{\tau - T_2}(p_0) - S(t_0, \tau - T_1) u_{\tau - T_1}(p_0) \| < C\varepsilon,$$

whenever both T_1 and $T_2 \geq T_0(\varepsilon)$, in which the constant C depends on $\| p_0 \|$, τ and t_0,

$$C = \frac{\sqrt{b}}{\sqrt{(1-b)^3}} \| p_0 \| e^{(\mu+\lambda)(\tau - t_0)}.$$

The above inequality shows that $w(T)$, $T \in [\tau - t_0, \infty)$, has the Cauchy property as $T \to \infty$. This implies the limit when $t < \tau$ in (40) exists in H and $u(t)$ in (40) is well-defined for all $t \in \Re$.

Lemma 12. *Let $u(t)$, $t \in \Re$, be the function defined by* (40). *Then*

$$\limsup_{t \to -\infty} \| e^{\lambda t} u(t) \| < \infty. \tag{44}$$

Proof. The inequalities (36) and (37) in Lemma 10 imply

$$\| S(\tau - t, \tau - T) u_{\tau - T} \| \leq (\| P u_{\tau - T}^{\tau - t} \|^2 + \| Q u_{\tau - T}^{\tau - t} \|^2)^{1/2}$$

$$\leq \frac{\sqrt{1+b}}{1-b} \| p_0 \| e^{\lambda t}, \text{ for } T \geq t \geq 0.$$

Therefore, the function $u(t)$ in (40) satisfies

$$\| u(\tau - t) \| = \| \lim_{T \to -\infty} S(\tau - t, \tau - T) u_{\tau - T} \| \le \frac{\sqrt{1+b}}{1-b} \| p_0 \| e^{\lambda t}, \tag{45}$$

for all $t > 0$. Replacing $\tau - t$ in (45) by t, we obtain

$$\| e^{\lambda t} u(t) \| \le e^{\lambda t} \frac{\sqrt{1+b}}{1-b} \| p_0 \| e^{\lambda(\tau - t)} = \frac{\sqrt{1+b}}{1-b} \| p_0 \| e^{\lambda \tau}, \text{ for } \tau > t > -\infty,$$

where $\sqrt{1+b}(1-b)^{-1} \| p_0 \| e^{\lambda \tau}$ is a constant depending only on p_0 and τ. Thus (44) is proved.

Lemma 13. *For any given* $(p_0, \tau) \in P(H) \times \Re$, *there exists a solution* $u(t)$, $t \in \Re$, *of the equation* (1) *with the properties*

(i) $Pu(\tau) = p_0$, *and*

(ii) $\limsup_{t \to -\infty} \| e^{\lambda t} u(t) \| < \infty$.

Proof. We claim that $u(t)$, $t \in \Re$, defined by (40) is such a solution of (1). By (40) and Lemma 12, it is obvious that the properties (i) and (ii) are satisfied by this $u(\cdot)$. It suffices to prove that this u is a mild solution of the equation (1). That $u(t)$ is a solution for $t \ge \tau$ is clear. Consider $t \le \tau$. Let $t_2 \ge t_1 \ge 0$. The strong continuity of the evolution operator of the mild solutions implies that

$$\begin{aligned} S(\tau - t_1, \tau - t_2) u(\tau - t_2) &= S(\tau - t_1, \tau - t_2) \lim_{T \to \infty} S(\tau - t_2, \tau - T) u_{\tau - T} \\ &= \lim_{T \to \infty} S(\tau - t_1, \tau - t_2) S(\tau - t_2, \tau - T) u_{\tau - T} \\ &= \lim_{T \to \infty} S(\tau - t_1, \tau - T) u_{\tau - T} = u(\tau - t_1). \end{aligned} \tag{46}$$

In (46) we can replace $\tau - t_1$ by any t and replace $\tau - t_2$ by any t_0, with $t_0 \le t \le \tau$. It shows that this $u(t)$ is a solution of (1) for $t \le \tau$.

Proof of Theorem 2. By identifying the point p in Theorem 2 with p_0 in Lemma 13, the first statement in Theorem 2 has already been proved by Lemma 13. Consequently, it holds that

$$P(H) \times \Re \subset Proj(\mathfrak{M}).$$

On the other hand, by the original definition, $Proj(\mathfrak{M}) \subset P(H) \times \Re$. Thus the statement

$$Proj(\mathfrak{M}) = P(H) \times \Re$$

of Theorem 2 is proved.

4. The Exponential Tracking Property

It has been shown in Lemma 7 and Theorem 2 that the set $\mathfrak{M}$ defined by (27) is characterized by (30),

$$\mathfrak{M} = \text{Graph}(\Psi),$$

where the domain of Ψ is the entire subspace $P(H) \times \Re$, and $\Psi : P(H) \times \Re \to Q(H)$ is a continuous mapping with the Lipschitz continuity (29).

In view of Definitions 1 and 2, these results show that the first two conditions of a tracking integral manifold (TIM) are satisfied by this manifold $\mathfrak{M} = \text{Graph}(\Psi)$. In this section we shall prove that the third condition in Definition 1, namely, the exponential tracking property, is also satisfied by this manifold $\mathfrak{M}$. Therefore, $\mathfrak{M}$ turns out to be a TIM and, by Lemma 2, an inertial manifold as well for the skew product semiflow π associated with the nonautonomous equation (1).

Lemma 14. *Suppose that* $(w, \tau) \in \mathfrak{M}$, $w = \xi + \varphi$, *where* $\xi = Pw$, $\varphi = Qw$, *and* $u = p + q \in H$, *where* $p = Pu$, $q = Qu$, *such that*

$$\| p - \xi \|^2 \leq b \| q - \varphi \|^2. \tag{47}$$

Then it holds that

$$\| Q(u - w) \| \leq 2 \frac{\sqrt{1 + b}}{1 - b} \| u \|. \tag{48}$$

Proof. Since $(w, \tau) = (\xi + \varphi, \tau) \in \mathfrak{M}$, by Lemma 7 and Theorem 2, we have

$$\| \varphi \| = \| \Psi(\xi, \tau) \| \leq \sqrt{b} \| \xi \|.$$

The condition (47) together with the above inequality implies that

$$\| \xi \| - \| p \| \leq \| p - \xi \| \leq \sqrt{b} \| q - \varphi \| \leq \sqrt{b} \| q \| + b \| \xi \|.$$

Hence we get

$$(1 - b) \| \xi \| \leq \| p \| + \sqrt{b} \| q \|$$

and

$$\| \varphi \| \leq \frac{\sqrt{b}}{1 - b} (\| p \| + \sqrt{b} \| q \|).$$

It follows that

$$\begin{aligned} \| Q(u - w) \| &= \| q - \varphi \| \leq \| q \| + \| \varphi \| \leq \| q \| + \frac{\sqrt{b}}{1 - b} (\| p \| + \sqrt{b} \| q \|) \\ &\leq \frac{\sqrt{b}}{1 - b} \| p \| + \frac{1}{1 - b} \| q \| \\ &\leq 2 \frac{\sqrt{1 + b}}{1 - b} (\| p \|^2 + \| q \|^2)^{1/2} = 2 \frac{\sqrt{1 + b}}{1 - b} \| u \|. \end{aligned}$$

Therefore, (48) is valid.

Lemma 15. *For any mild solution* $u(t) = S(t, \tau) u_0$, $t \geq \tau$, *of the equation* (1), *there exists an integral curve* $(v(t), t)$, $t \in \Re$, *on the manifold* $\mathfrak{M}$, *such that*

$$\| u(t) - v(t) \| \leq C \| u_0 \| e^{-(\mu + \lambda)(t - \tau)}, \quad t \geq \tau, \tag{49}$$

where $u_0 = u(\tau)$, C *is a uniform constant*, μ *and* λ *are the constants specified in* (4).

Proof. For any given positive integer $n = 1, 2, \ldots$, since $Proj(\mathfrak{M}) = P(H) \times \Re$, there is an integral curve $\Gamma_n : (u_n(t), t)$, $t \in \Re$, on the manifold $\mathfrak{M}$ such that

$$P u_n(\tau + n) = P u(\tau + n), \tag{50}$$

because

$$(P u(\tau + n), \tau + n) \in P(H) \times \Re.$$

By Lemma 4 and (23), the relation (50) implies that

$$\| u(t) - u_n(t) \| \leq \frac{1}{\sqrt{1 - b}} \| Q u(\tau) - Q u_n(\tau) \| e^{-(\mu + \lambda)(t - \tau)}, \text{ for } t \in [\tau, \tau + n]. \tag{51}$$

On the other hand, (21) shows that

$$\| Pu(t) - Pu_n(t) \|^2 \leq b \| Qu(t) - Qu_n(t) \|^2, \quad t \in [\tau, \tau + n]. \tag{52}$$

Now we can exploit Lemma 14, where the condition (47) is satisfied due to (52), to infer that

$$\| Qu(\tau) - Qu_n(\tau) \| \leq 2 \frac{\sqrt{1+b}}{1-b} \| u(\tau) \|. \tag{53}$$

Substituting (53) into (51), we obtain

$$\| u(t) - u_n(t) \| \leq 2 \frac{\sqrt{1+b}}{\sqrt{(1-b)^3}} \| u_0 \| e^{-(\mu+\lambda)(t-\tau)}, \text{ for } t \in [\tau, \tau + n]. \tag{54}$$

Taking $t = \tau$ in (54), we find that $\{u_n(\tau) : n = 1, 2, \ldots\}$ is a bounded sequence in H, because for all positive integers n,

$$\| u_n(\tau) \| \leq (1 + C_0) \| u_0 \|, \tag{55}$$

where the constant $C_0 = 2\sqrt{(1+b)/(1-b)^3}$. Since $\mathfrak{M}$ is a finite-dimensional manifold, the sequence $\{u_n(\tau)\}$ is a precompact set in H. Thus there exists a convergent subsequence $\{u_{n_k}(\tau) : k = 1, 2, \ldots\}$ such that

$$\lim_{k\to\infty} u_{n_k}(\tau) = v_0, \text{ where } (v_0, \tau) \in \mathfrak{M}. \tag{56}$$

In fact, denoting $Pu_n(\tau)$ by $p_n(\tau)$ and $Qu_n(\tau)$ by $q_n(\tau)$, since $\{p_n(\tau)\}$ is a bounded sequence in $P(H)$, there exists a convergent subsequence $\{p_{n_k}(\tau)\}$ such that

$$\lim_{k\to\infty} p_{n_k}(\tau) = p_0 \in P(H).$$

Then the continuity of the mapping Ψ and (29) implies that

$$\lim_{k\to\infty} Qu_{n_k}(\tau) = \lim_{k\to\infty} q_{n_k}(\tau) = \lim_{k\to\infty} \Psi(p_{n_k}(\tau), \tau) = \Psi(p_0, \tau) \in Q(H).$$

Hence, the limit in (56) exists and $(v_0, \tau) = (p_0 + \Psi(p_0, \tau), \tau) \in \mathfrak{M}$.

Now consider the integral curve $\Gamma : (v(t), t),\ t \in \Re,$ on the manifold $\mathfrak{M},$ which passes through (v_0, τ). We have $v(t) = S(t, \tau)v_0,$ for $t \geq \tau$. By the fact that $\mathfrak{M}$ consists of globally defined integral curves only and that $(v_0, \tau) \in \mathfrak{M},$ such an integral curve Γ exists on $\mathfrak{M}$. The continuity of the evolution operator of the solutions $S(t, \tau)$ implies that

$$\begin{aligned} v(t) &= S(t, \tau)v_0 = S(t, \tau) \lim_{k\to\infty} u_{n_k}(\tau) \\ &= \lim_{k\to\infty} S(t, \tau)u_{n_k}(\tau) = \lim_{k\to\infty} u_{n_k}(t), \text{ for any } t \geq \tau. \end{aligned} \tag{57}$$

Finally, for an arbitrarily fixed $t \geq \tau$, there is a positive integer k_0 depending on t, such that

$$n_k \geq t - \tau, \quad \text{for } k \geq k_0.$$

It follows from (54) that, for this t,

$$\| u(t) - u_{n_k}(t) \| \leq C_0 \| u_0 \| e^{-(\mu+\lambda)(t-\tau)}, \text{ for } k \geq k_0, \tag{58}$$

because $t \in [\tau, \tau + n_k]$, where C_0 is the same constant specified in (55). Letting $k \to \infty$ in (58), and passing to the limit, we can use (57) to confirm

$$\| u(t) - v(t) \| \leq C_0 \| u_0 \| e^{-(\mu+\lambda)(t-\tau)}.$$

Note that here $t\,(\geq \tau)$ is arbitrary. The exponential tracking inequality (49) is proved.

Now we can finish the proof of the main result Theorem 1.

Proof of Theorem 1. We have already shown through Lemma 7 and Theorem 2 that the set $\mathfrak{M}$ defined by (27) can be expressed as the graph of a mapping Ψ which possesses the properties claimed in Theorem 1. Thus we have shown that this manifold $\mathfrak{M}$ satisfies the first two conditions described in Definition 2 for a tracking integral manifold.

Then Lemma 15 shows that the third condition for a tracking integral manifold is also satisfied by this manifold $\mathfrak{M}$. In fact (49) shows that the tracking property in Definition 1,

$$\| u(t) - v(t) \| \leq C(\| u_0 \|) \exp(-\beta(t - t_0)), \text{ for } t \geq t_0,$$

is satisfied with

$$\beta = \mu + \lambda = \Lambda - K > \frac{1}{2}(\Lambda + \lambda_0) > 0.$$

Therefore, $\mathfrak{M}$ is a tracking integral manifold for the skew product semiflow π in the space E. According to Lemma 2, this $\mathfrak{M}$ is also an inertial manifold. The proof of Theorem 1 is completed.

We can generalize Theorem 1 by dropping the assumption $F(0, t) = 0$ in (H2). Instead we make an additional assumption as follows.

(H4) There is at least one globally defined mild solution $u^*(t)$, $t \in \mathfrak{R}$, of the equation (1), which satisfies

$$\limsup_{t \to -\infty} \| e^{\lambda t} u^*(t) \| < \infty,$$

where λ is the same as specified in (4).

Theorem 3. *Under the assumptions* (H1), (H2) *but without* $F(0, t) = 0$, (H3), *and* (H4), *the conclusion of Theorem* 1 *on the existence and characterization of an inertial manifold for the skew product semiflow* π *remains valid.*

Proof. Define $w(t) = u(t) - u^*(t)$. Then the original nonautonomous equation (1) and the initial value condition are transformed to

$$\frac{dw}{dt} + A(t)w = R(w, t),$$
$$w(t_0) = w_0, \tag{59}$$

where

$$R(w, t) = F(w + u^*(t), t) - F(u^*(t), t),$$
$$w_0 = u_0 - u^*(t_0). \tag{60}$$

It is obvious that the new nonlinear term $R(w, t)$ satisfies the same global and uniform Lipschitz continuous property that

$$\| R(w_1, t) - R(w_2, t) \| \le K \| w_1 - w_2 \|,$$

with the same Lipschitz constant K, for any $w_1, w_2 \in H$ and for any $t \in \Re$. Since $R(0, t) = 0$, the original assumption (H2) is satisfied by (59) with (60).

Then the assumption (H3) implies that the set $\mathfrak{M}_w$ constructed in (27) for the new equation (59) with respect to $w(\cdot)$ is *nonempty*, and all the steps through Section 3 and Section 4 remain valid. Therefore, Theorem 1 can be applied to the corresponding skew product semiflow π_w associated with the modified nonautonomous equation (59). This set $\mathfrak{M}_w$ turns out to be a TIM and an inertial manifold for π_w. Finally, the following manifold

$$\mathfrak{M} = \{(w(t) + u^*(t), t), \ t \in \Re \mid \text{ all } (w(t), t) \in \mathfrak{M}_w\}$$

is a TIM and an inertial manifold for the skew product semiflow π of the original equation (1). The detail of verification is omitted.

Remark 3. We emphasize the following facts which feature this work.

(R1) All the steps in Section 3 and Section 4 in confirming that the constructed $\mathfrak{M}$ is an integral tracking manifold follow directly from the two conic inequalities (10) and (11) as well as the consequential inequalities stated in Lemma 4 and Lemma 5.

(R2) The only two places in the entire proof of the main result Theorem 1 where the spectral gap condition (3) is needed are in the proof of Lemma 3. One place is the acquisition of a constant $b \in (0, 1)$ in the conic inequalities. The other place is the assertion that the two exponential constants

$$\mu + \lambda = \Lambda - K > 0$$

and

$$\mu - \lambda = -(\lambda_0 + K) < 0$$

in the incremental dichotomy. Unlike the Lyapunov-Perron method where the spectral gap conditions are needed in several different stages of the proof, this observation sharply focuses the role played by the spectral gap condition in the existence proof of inertial manifolds.

(R3) Another advantage of this approach is the pullback argument, which becomes more geometrically motivated in this work than in the fixed point search of the Lyapunov-Perron mapping of an integral form.

The two common pillars in any typical approach to proving the existence of inertial manifolds are the incremental analysis of trajectories because of the nature of this topic concerning the attraction among trajectories and the pullback analysis due to that an inertial manifold must contain all the unstable portion of the underlying flow or semiflow. The approach we take in this work indicates that the *conic invariance* reflected by the incremental analysis, which is related to the squeezing property [5, 6, 11, 17] but seems more general, actually *dominates* the pullback analysis. This insight is useful in dealing with further investigation problems.

5. Applications to Reaction-Diffusion Equations

In this section, we shall illustrate the applications of the main result of this work to nonautonomous reaction-diffusion equations. Let $\Omega \subset \Re^n$ be a bounded domain such that the boundary $\partial\Omega$ is locally Lipschitz continuous and Ω lies locally on one side of $\partial\Omega$. Consider the following initial-boundary value problem of a nonautonomous reaction-diffusion equation,

$$\begin{aligned} &\frac{\partial u}{\partial t} + A(t)u + f(u, t) = 0, \quad t \geq \tau, \; x \in \Omega, \\ &u = 0, \quad t \geq \tau, \; x \in \partial\Omega, \\ &u(x, \tau) = u_0(x) \in H = L^2(\Omega), \end{aligned} \tag{61}$$

where $\tau \in \Re$ and $u_0 \in H$ are arbitrarily given. In this section, the inner-product and norm of $H = L^2(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

The linear partial differential operator $A(t)$ is given by

$$A(t)u = -\Delta u + \sum_{i=1}^{n} b_i(t) \frac{\partial u}{\partial x_i} + c(x, t)u, \quad (x, t) \in \Omega \times \Re, \tag{62}$$

where Δ is the n-dimensional Laplacian operator, the functions $b_i(t)$,

$i = 1, 2, \ldots, n,$ are bounded and uniformly continuous functions on $\Re$, and the function $c(x, t)$ is a nonnegative, bounded, and uniformly continuous function on $\Omega \times \Re$.

Assume that the nonlinear function $f(s, t)$ satisfies the following conditions. $f : \Re \times \Re \to \Re$ is a continuous function, $f(0, t) = 0$, and there is a constant $p > 2$ such that

$$a_1 | s |^p - C_1 \le f(s, t)s \le a_2 | s |^p + C_1, \tag{63a}$$

$$| f(s, t) | \le C_2(1 + | s |^{p-1}), \tag{63b}$$

$$\left| \frac{\partial f}{\partial s}(s, t) \right| \le C_3(1 + | s |^{p-2}), \tag{63c}$$

where a_1, a_2, C_1, C_2, and C_3 are positive constants. It is proved in [3] and several references therein that the initial-boundary value problem (61) under the above assumptions admits a unique mild solution

$$u = u(\cdot, t) \in L^2_{\text{loc}}((\tau, \infty), H^1_0(\Omega)).$$

In view of this we make the following definition, cf. [19, Section 2.5].

Definition 4. Let Y and W be Banach spaces, where Y is continuously imbedded into W. A mapping $\rho(u, t)$ is said to be a *singular semiflow* on W with respect to Y, if the following properties are satisfied by ρ:

(i) There is a semiflow $\sigma(u, t)$ on Y, such that if $u \in Y$, then $\rho(u, t) = \sigma(u, t)$ for all $t > 0$.

(ii) For each $(u, t) \in W \times (0, \infty)$, one has $\rho(u, t) \in Y$.

(iii) The mapping $(u, t) \to \rho(u, t)$ is a continuous mapping from $W \times (0, \infty)$ into Y.

We call $\sigma(u, t)$ the *reduced semiflow* of $\rho(u, t)$ on Y. The difference between the singular semiflow $\rho(u, t)$ and the reduced semiflow $\sigma(u, t)$ lies in the points when $t = 0$.

Let $S(t, \tau)u_0 = u(t; u_0, \tau)$, where $u(t; u_0, \tau)$ is the mild solution of the associated initial value problem:

$$\frac{du}{dt} + A(t)u + F(u, t) = 0,$$
$$u(\tau) = u_0, \tag{64}$$

where the linear operator $A(t) : H^2(\Omega) \cap H_0^1(\Omega) \to H$, $t \in \Re$, is defined by (62), and the nonlinear operator $F(u, t)$ is the time-variant Nemytskii mapping induced by $f(u, t)$ which satisfies (63). Here the unknown in (64) is $u(t) = u(\cdot, t)$ in (61). The evolution operator $S : (u_0, \tau, t) \to S(t, \tau)u_0$ is a continuous mapping from $H \times \Re \times [0, \infty)$ into H.

Define the phase space to be $E = H \times \Re$. Then one can check that the assumption (H1) on the linear operator is satisfied by this $A(t)$, in which the space $V = H^2(\Omega) \cap H_0^1(\Omega)$ with the $H^2(\Omega)$ topology. Suppose that the assumptions (H2) and (H3) can be verified for this problem (64), then the mild solution $u(t; u_0, \tau)$, $t \geq \tau$, exists uniquely for any $(u_0, \tau) \in H \times \Re$. Then, just as in (2), we can check that the mapping

$$\pi((u_0, \tau), t) = (u(t + \tau; u_0, \tau), t + \tau), \tag{65}$$

turns out to be a singular skew product semiflow on E with respect to $Y = H_0^1(\Omega) \times \Re$ according to Definition 4.

The objective is to prove the following theorem, which enables us to verify that the global and uniform Lipschitz property in the assumption (H2) is satisfied by the nonlinear term $F(u, t)$ in (64). If so, then the assumption (H3), i.e., the spectral gap condition, can also be verified for any one-dimensional and some two-dimensional bounded domain due to the assumptions on $A(t)$ in (62) and

$$\limsup_{k \to \infty} (\lambda_{k+1} - \lambda_k) = \infty$$

for the eigenvalues of the Laplace operator on 1D bounded domain and on some 2D bounded domain with homogeneous Dirichlet or Neumann boundary conditions, cf. [12] and [19, Section 8.6].

Theorem 4. *There exists an absorbing set in* $L^\infty(\Omega)$,

$$B_\infty = \{w \in H : \| w \|_{L^\infty(\Omega)} \le R_0\} \tag{66}$$

for the solutions $S(t, \tau)u_0$ *of the initial value problem* (64), *where the constant* R_0 *is uniform with respect to* $(u_0, \tau) \in E$.

Note that if the space dimension is $n = 1$, one can prove Theorem 4 by showing that there is an absorbing set in $H_0^1(\Omega)$ and using the Sobolev imbedding that $H_0^1(\Omega)$ is imbedded into $L^\infty(\Omega)$. But for space dimensions $n \ge 2$, we need to do more as shown in the following several lemmas.

Lemma 16. *Under the assumptions* (62) *and* (63), *the following statements hold*:

(i) *The mild solution* $u(t)$ *of the initial value problem* (64) *satisfies* $u \in L^\infty([\tau, \infty), H)$ *and*

$$\sup\{\| u(t) \| : t \ge \tau\} \le K_1(\| u_0 \|), \tag{67}$$

where $K_1(r)$ *is a nondecreasing, nonnegative, scalar function.*

(ii) *It holds that*

$$u \in L^p_{loc}([\tau, \infty), L^p(\Omega)). \tag{68}$$

(iii) *There exists an absorbing set* B_1 *in H for the solution trajectories of the equation* (64), *which is uniform with respect to* $\tau \in \Re$.

Proof. Taking the inner-product in H of the equation (64) with $u(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\| u(t) \|^2 + \| \nabla u(t) \|^2 + \langle F(u(t), t), u(t) \rangle = 0.$$

Using the Poincaré inequality and the first inequality in (63a), we see that there exists a constant $\alpha > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\| u(t) \|^2 + \alpha \| u(t) \|^2 + a_1 \int_\Omega | u(x, t) |^p dx \le C_1 | \Omega |. \tag{69}$$

Consequently,

$$\| u(t) \|^2 e^{2\alpha t} \le \| u_0 \|^2 e^{2\alpha\tau} + \alpha^{-1} C_1 | \Omega | (e^{2\alpha t} - e^{2\alpha\tau}).$$

Thus we obtain

$$\| u(t) \|^2 \le \| u_0 \|^2 e^{-2\alpha(t-\tau)} + \alpha^{-1} C_1 | \Omega | \le \| u_0 \|^2 + \alpha^{-1} C_1 | \Omega |, \quad t \ge \tau. \tag{70}$$

Let

$$K_1(r) = r^2 + \alpha^{-1} C_1 | \Omega |.$$

Then (67) follows from (70). Besides (69) implies that

$$\begin{aligned} 2a_1 \int_t^{t+1} \int_\Omega | u(x, s) |^p dxds &\le \| u(t) \|^2 + 2C_1 | \Omega | \\ &\le K_1(\| u_0 \|) + 2C_1 | \Omega |, \text{ for } t \ge \tau. \end{aligned} \tag{71}$$

This inequality shows that (68) holds. Moreover, the first inequality in (70) also implies that

$$B_1 = \{ w \in H : \| w \| \le 1 + \alpha^{-1} C_1 | \Omega | \}$$

is an absorbing set for the family of mild solutions of the equation (64), which is uniform in $\tau \in \Re$.

Definition 5. For any given $q \ge 1$, define a *Banach space*

$$W_q(t_0) \stackrel{\text{def}}{=} L^\infty([t_0, \infty), L^q(\Omega)) \cap Z_{\text{loc}}^{p+q-2}([t_0, \infty), L^{p+q-2}(\Omega)),$$

where $Z_{\text{loc}}^{p+q-2}([t_0, \infty), L^{p+q-2}(\Omega))$ is the space of functions $u : [t_0, \infty) \to L^{p+q-2}(\Omega)$ such that

$$\sup_{t \ge t_0} \int_t^{t+1} \int_\Omega | u(x, s) |^{p+q-2} dxds < \infty.$$

Then $W_q(t_0)$ with the following norm becomes a Banach space,

$$\| u \|_{W_q(t_0)} = \| u \|_{L^\infty([t_0, \infty), L^q(\Omega))} + \sup_{t \ge t_0} \left(\int_t^{t+1} \int_\Omega | u(x, s) |^{p+q-2} dxds \right)^{1/(p+q-2)}.$$

By this definition, it has been shown in Lemma 16 that the solution of the initial value problem (64) satisfies

$$u \in W_2(\tau) \text{ and } \| u \|_{W_2(\tau)} \leq K_2(\| u_0 \|),$$

where the constant $K_2(r)$ is given by

$$K_2(r) = \left(1 + \frac{1}{2a_1}\right) K_1(r) + \frac{C_1 | \Omega |}{a_1}.$$

Lemma 17. *If the solution u of the initial value problem* (64) *satisfies $u \in W_{q_0}(\tau)$ for some $q_0 \geq 2$, then for any given s, $0 < s \leq 1$, it holds that*

$$u \in W_{q_1}(\tau + s) \text{ and } \| u \|_{W_{q_1}(\tau+s)} \leq K_3(s, \| u \|_{W_{q_0}(\tau)}), \tag{72}$$

where $q_1 = p + q_0 - 2$ and $K_3(s, r)$ is a constant continuously depending on (s, r) and increasing in $r \geq 0$.

Proof. Multiplying the equation (61) by $(t - \tau_0) | u |^{q_1 - 2} u$ and then integrating the two sides in $x \in \Omega$ and in $t \in [\tau_0, \tau_0 + s]$, with $\tau_0 \geq \tau$ and $0 < s \leq 1$ relatively fixed, we can get

$$\int_{\tau_0}^{\tau_0 + s} \int_\Omega (t - \tau) | u |^{q_1 - 2} u u_t dx dt$$

$$- \int_{\tau_0}^{\tau_0 + s} \int_\Omega (t - \tau_0) | u |^{q_1 - 2} u (\Delta u - f(u, t)) dx dt = 0. \tag{73}$$

For each term in (73), we can make an estimate as follows. First we have

$$\int_{\tau_0}^{\tau_0 + s} (t - \tau_0) \int_\Omega | u |^{q_1 - 2} u u_t dx dt$$

$$= \frac{1}{q_1} \int_{\tau_0}^{\tau_0 + s} \frac{d}{dt} \left((t - \tau_0) \int_\Omega | u |^{q_1} dx \right) dt - \frac{1}{q_1} \int_{\tau_0}^{\tau_0 + s} \| u(t) \|_{L^{q_1}(\Omega)}^{q_1} dt$$

$$= \frac{s}{q_1} \| u(\tau_0 + s) \|_{L^{q_1}(\Omega)}^{q_1} - \frac{1}{q_1} \int_{\tau_0}^{\tau_0 + s} \| u(t) \|_{L^{q_1}(\Omega)}^{q_1} dt. \tag{74}$$

Secondly, by using the Gauss Divergence Theorem and due to $u = 0$ on $\partial\Omega$, we have

$$
\begin{aligned}
& -\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{q_1-2} u\Delta u dxdt \\
&= -\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega \operatorname{div}(|u|^{q_1-2} u\nabla u) dxdt \\
&\quad + \int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega \nabla(|u|^{q_1-2} u)\cdot\nabla u dxdt \\
&= \int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega (|u|^{q_1-2}|\nabla u|^2 + u\nabla(|u|^{q_1-2})\cdot\nabla u) dxdt \\
&\geq (q_1-2)\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{q_1-3} u(\nabla|u|\cdot\nabla u) dxdt \\
&= (q_1-2)\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{q_1-4}(|u|\nabla|u|)\cdot(u\nabla u) dxdt \\
&= (q_1-2)\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{q_1-4}|u\nabla u|^2 dxdt \geq 0. \qquad (75)
\end{aligned}
$$

Thirdly, by (63a) we can get

$$
\begin{aligned}
\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{q_1-2} uf(u, t) dxdt &\geq a_1\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{p+q_1-2} dxdt \\
&\quad - C_1\int_{\tau_0}^{\tau_0+s} (t-\tau_0)\int_\Omega |u|^{q_1-2} dxdt.
\end{aligned}
$$

Since for any given $\varepsilon > 0$, there is a constant $C(\varepsilon)$ such that

$$y^{q_1-2} \leq \varepsilon y^{p+q-2} + C(\varepsilon), \text{ for all } y \geq 0,$$

there exists a constant $C_4 = C_4(a_1, C_1, |\Omega|)$ such that

$$C_1\int_\Omega |u|^{q_1-2} dxdt \leq \frac{a_1}{2}\int_\Omega |u|^{p+q_1-2} dx + C_4.$$

Therefore, we have

$$\int_{\tau_0}^{\tau_0+s}(t-\tau_0)\int_\Omega |u|^{q_1-2}uf(u,t)dxdt$$

$$\geq \frac{a_1}{2}\int_{\tau_0}^{\tau_0+s}(t-\tau_0)\int_\Omega |u|^{p+q_1-2}dxdt - C_4s^2. \tag{76}$$

Substituting (74), (75) and (76) into (73), we obtain

$$\frac{s}{q_1}\|u(\tau_0+s)\|_{L^{q_1}(\Omega)}^{q_1} + \frac{1}{8}sa_1\int_{\tau_0+s/2}^{\tau_0+s}\int_\Omega |u|^{p+q_1-2}dxdt$$

$$+\frac{1}{4}a_1\int_{\tau_0}^{\tau_0+s/2}(t-\tau_0)\int_\Omega |u|^{p+q_1-2}dxdt$$

$$\leq C_4s^2 + \frac{1}{q_1}\int_{\tau_0}^{\tau_0+s}\|u(t)\|_{L^{q_1}(\Omega)}^{q_1}dt. \tag{77}$$

By the assumption of this lemma, $u \in W_{q_0}(\tau)$ for some $q_0 \geq 2$. For any fixed $s \in (0, 2]$, we make the following two assertions from the estimate (77). The first assertion is

$$\sup_{t\geq\tau+s}\|u(t)\|_{L^{q_1}(\Omega)} = \sup_{\tau_0\geq\tau}\|u(\tau_0+s)\|_{L^{q_1}(\Omega)}$$

$$\leq \left(C_4sq_1 + \frac{1}{s}\int_{\tau_0}^{\tau_0+s}\|u(t)\|_{L^{q_1}(\Omega)}^{q_1}dt\right)^{1/q_1}$$

$$\leq \left(C_4sq_1 + \frac{1}{s}\|u\|_{W_{q_0}(\tau)}^{q_1}\right)^{1/q_1}. \tag{78}$$

The second assertion is

$$\sup_{t\geq\tau+s}\int_t^{t+1}\int_\Omega |u(x,\xi)|^{p+q_1-2}dxd\xi$$

$$\leq \left(\left[\frac{2}{s}\right]+1\right)\sup_{\tau_0\geq\tau}\int_{\tau_0+s/2}^{\tau_0+s}\int_\Omega |u(x,\xi)|^{p+q_1-2}dxd\xi$$

$$\leq \frac{8}{a_1}\left(\left[\frac{2}{s}\right]+1\right)\left[C_4s + \frac{1}{q_1s}\|u\|_{W_{q_0}(\tau)}^{q_1}\right],$$

so that

$$\sup_{t \geq \tau+s}\left(\int_t^{t+1}\int_\Omega |u(x, \xi)|^{p+q_1-2} dx d\xi\right)^{1/(p+q_1-2)}$$

$$\leq \left(\frac{8}{a_1}\left(\left[\frac{2}{s}\right]+1\right)\left[C_4 s + \frac{1}{q_1 s}\| u \|_{W_{q_0}(\tau)}^{q_1}\right]\right)^{1/(p+q_1-2)}. \tag{79}$$

From (78) and (79) we can conclude that (72) holds with

$$K_3(s, r) = \left(C_4 s q_1 + \frac{1}{s} r^{q_1}\right)^{1/q_1}$$

$$+ \left\{\frac{8}{a_1}\left(\left[\frac{2}{s}\right]+1\right)\left[C_4 s + \frac{1}{q_1 s} r^{q_1}\right]\right\}^{1/(p+q_1-2)}. \tag{80}$$

The proof is completed.

Lemma 18. *For the solution trajectories of* (64), *there exists an absorbing set* B_2 *in the space* $L^{p_1}(\Omega)$, *where*

$$p_1 = 1 + \frac{1}{2} n(p-1).$$

Proof. Based on (70) and (71), for every solution trajectory u of the equation (64), there is a time

$$\tau_1 = \tau_1(u_0) \geq \tau + 1$$

such that

$$\| u(t) \| \leq C_5 = 1 + \alpha^{-1} C_1 |\Omega|, \text{ for } t \geq \tau_1,$$

and

$$\int_t^{t+1}\int_\Omega |u(x, t)|^p dx ds \leq \frac{1}{2a_1}(K_1(C_5) + 2C_1|\Omega|), \text{ for } t \geq \tau_1.$$

The above two inequalities allow us to apply Lemma 17 to the solutions of (64), since $u \in W_{q_0}(\tau_1)$ for $q_0 = 2$. Therefore we can assert that for a given $s \in (0, 2]$, for any initial data $u_0 \in H$, the solution of (64) satisfies

$$u \in W_p(\tau_1 + s) \text{ and } \| u \|_{W_p(\tau_1+s)} \leq K_3(s, \| u \|_{W_2(\tau_1)}),$$

where $K_3(s, r)$ is given in (80) with $q_1 = p$ here. In particular, we can simply take $s = 2$ to obtain

$$u \in W_p(\tau_1 + 2) \text{ and } \| u \|_{W_p(\tau_1 + 2)} \le K_3(2, C_6),$$

where

$$C_6 = C_5 + \left\{ \frac{1}{2a_1} [K_1(C_5) + 2C_1 |\, \Omega \,|] \right\}^{1/p}.$$

By the bootstrap argument, there is an integer $m \ge 0$ such that

$$1 + \frac{1}{2} n(p - 1) \le p + m(p - 2),$$

where $m = 0$ for dimension $n = 1$ or 2, and $m \ge 1$ for $n \ge 3$. Accordingly we can apply Lemma 17 up to m times, if necessary, to reach the following statement,

$$u \in W_{p_1}(\tau_1 + 2(1 + m)) \text{ and } \| u \|_{W_{p_1}(\tau_1 + 2(1+m))} < K_4, \tag{81}$$

where $K_4 = K_4(m, K_3(2, C_6))$ is a uniform constant because m only depends on the parameter p and C_6 is a uniform constant. As a consequence of (81), we find that

$$u \in L^\infty([\tau_1 + 2(1 + m), \infty), L^{p_1}(\Omega)) \tag{82}$$

and

$$B_2 = \{w \in L^{p_1}(\Omega) : \| w \|_{L^{p_1}(\Omega)} \le K_4\} \tag{83}$$

is an absorbing set in the space $L^{p_1}(\Omega)$ for the solutions of the equation (64).

Lemma 19. *For the solution trajectories of the equation* (64), *there exists an absorbing set* B_∞ *in the space* $L^\infty(\Omega)$, *as stated in Theorem* 4.

Proof. Here we use the (L^p, L^∞) regularity property of analytic semigroups stated in [19, Theorem 38.10]. According to that, the analytic semigroup $T(t)$, $t \ge 0$, generated by $A_0 = \Delta : H^2(\Omega) \cap H_0^1(\Omega) \to H$ has the regularity: for any given $1 \le p < \infty$, $1 < q \le \infty$,

$$T(t) : L^p(\Omega) \to L^q(\Omega) \text{ for all } t > 0,$$

and there is a constant $C = C(p, q)$ such that for any $t > 0$ and $u_0 \in L^p(\Omega)$,

$$\| T(t)u_0 \|_{L^q(\Omega)} \leq Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \| u_0 \|_{L^p(\Omega)}. \tag{84}$$

The linear evolution operator $U(t, \tau)$, $t \geq \tau$, $\tau \in \Re$, generated by the nonautonomous linear operator $A(t) = -A_0 + A_1(t)$, where the lower order perturbation (with the assumptions on the coefficient functions aforementioned in this section) is given by

$$A_1(t)u = \sum_{i=1}^{n} b_i(t) \frac{\partial u}{\partial x_i} + c(x, t)u,$$

and possesses the same regularity property, which can be shown in detail by the approach provided in [19, Section 4.4]. Namely, for any $t \geq \tau$, $\tau \in \Re$, and $u_0 \in L^p(\Omega)$, one has

$$\| U(t, \tau)u_0 \|_{L^q(\Omega)} \leq C(t - \tau)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \| u_0 \|_{L^p(\Omega)}, \tag{85}$$

where $C = C(p, q)$ again is a constant. In (84) and (85), if $q = \infty$, then $1/q = 0$.

The mild solution of the initial value problem (64) for any given $u_0 \in H$ satisfies the integral equation

$$u(t) = U(t, \tau)u_0 - \int_{\tau}^{t} U(t, s)f(u(s), s)ds, \quad t \geq \tau.$$

Let $\tau_2 \geq \tau_1 + 2(1 + m)$, where $\tau_1 \geq \tau + 1$ and $m \geq 0$ have been specified in the proof of Lemma 18. Note that τ_1 depends on the initial data u_0.

For any given $\zeta \in (0, 1]$, we have

$$\| u(\tau_2 + \zeta) \|_{L^\infty(\Omega)}$$

$$\leq \| U(\tau_2 + \zeta, \tau_2) u(\tau_2) \|_{L^\infty(\Omega)} + \int_{\tau_2}^{\tau_2+\zeta} \| U(\tau_2 + \zeta, t) f(u(t), t) \|_{L^\infty(\Omega)} dt$$

$$= \| U(\tau_2 + \zeta, \tau_2) u(\tau_2) \|_{L^\infty(\Omega)}$$

$$+ \int_0^\zeta \| U(\tau_2 + \zeta, t + \tau_2) f(u(t + \tau_2), t + \tau_2) \|_{L^\infty(\Omega)} dt. \tag{86}$$

By Lemma 18 and using (85), we get

$$\| U(\tau_2 + \zeta, \tau_2) u(\tau_2) \|_{L^\infty(\Omega)} \leq C\zeta^{-n/(2p_1)} \| u(\tau_2) \|_{L^{p_1}(\Omega)} \leq CK_4 \zeta^{-n/(2p_1)} \tag{87}$$

due to (82). Moreover, (63b) and (82) imply that

$$f(u(\cdot), \cdot) \in L^\infty([\tau_1 + 2(1 + m), \infty), L^\theta(\Omega)),$$

where

$$\theta = \frac{p_1}{p-1} = \frac{1}{p-1} + \frac{n}{2} > \frac{n}{2}. \tag{88}$$

Therefore, by (85) and (88), we have the following estimate,

$$\int_0^\zeta \| U(\tau_2 + \zeta, t + \tau_2) f(u(t + \tau_2), t + \tau_2) \|_{L^\infty(\Omega)} dt$$

$$\leq \int_0^\zeta C(\zeta - t)^{-n/(2\theta)} \| f(u(t + \tau_2), t + \tau_2) \|_{L^\theta(\Omega)} dt$$

$$\leq K_5(C_2, K_4) \int_0^\zeta C(\zeta - t)^{-n/(2\theta)} dt$$

$$= CK_5(C_2, K_4) \left(1 - \frac{n}{2\theta}\right)^{-1} \zeta^{1-n/(2\theta)}, \tag{89}$$

where C_2 is specified in (63b) and

$$K_5(C_2, K_4) = \| f(u(\cdot), \cdot) \|_{L^\infty([\tau_1 + 2(1+m), \infty), L^\theta(\Omega))}.$$

Note that $0 < n/(2\theta) < 1$ so that the constant $1 - n/(2\theta) > 0$. In view of (63b), (81) and (88), $K_5(C_2, K_4)$ is a uniform constant.

Substituting (87) and (89) into (86), we end up with

$$\| u(\tau_2 + \zeta) \|_{L^\infty(\Omega)} \leq K_6(\zeta), \quad \text{for } \tau_2 \geq \tau_1 + 2(1+m) \text{ and } 0 < \zeta \leq 1, \tag{90}$$

where

$$K_6(\zeta) = CK_4 \zeta^{-n/(2p_1)} + CK_5(C_2, K_4)\left(1 - \frac{n}{2\theta}\right)^{-1} \zeta^{1-n/(2\theta)}.$$

Finally, the result (90) shows that there exists an absorbing set in the space $L^\infty(\Omega)$,

$$B_\infty = \{w \in L^\infty(\Omega) : \| w \|_{L^\infty(\Omega)} \leq 1 + K_6(1)\},$$

for the solutions of the equation (64). In fact, for each initial status $u_0 \in H$, the solution $u(t) = u(t; u_0, \tau)$ will enter this absorbing set B_∞ and stay in B_∞ forever when

$$t \geq \tau_2 + 1 \geq \tau_1(u_0) + 2(1+m) + 1 = \tau_1(u_0) + 2m + 3.$$

The proof of this lemma is completed.

Thus the proof of Theorem 4 is also completed with the constant in (66) determined by

$$R_0 = 1 + K_6(1).$$

Now let us continue to address the existence of an inertial manifold for the skew product semiflow π associated with the equation (64) on $E = H \times \Re$. Let K be the constant

$$K = C_3(1 + (R_0)^{p-2}), \tag{91}$$

where R_0 is the constant given in (66) and shown above. By (63c) it holds that

$$\sup\left\{ \left| \frac{\partial f}{\partial u}(u, t) \right| : |u| \leq R_0,\ t \in \Re \right\} \leq K. \tag{92}$$

We can modify the nonautonomous reaction-diffusion equation in (61) by replacing the original nonlinear term $f(u, t)$ by the truncated nonlinear term

$$g(u, t) = \begin{cases} f(u, t) & \text{if } |u| \le R_0, \\ f\left(\dfrac{R_0 u}{|u|}, t\right) & \text{if } |u| > R_0. \end{cases}$$

Consider the modified nonautonomous equation and the associated initial-boundary value problem:

$$\frac{\partial u}{\partial t} - A(t)u + g(u, t) = 0, \quad t \ge \tau,\ x \in \Omega,$$

$$u = 0, \quad t \ge \tau,\ x \in \partial\Omega,$$

$$u(x, \tau) = u_0(x) \in H = L^2(\Omega),\ x \in \Omega. \tag{93}$$

The initial value problem of the corresponding nonautonomous evolutionary equation is

$$\frac{du}{dt} + A(t)u + G(u, t) = 0,$$

$$u(\tau) = u_0, \tag{94}$$

where the linear operator $A(t)$ is the same as in (64), while the nonlinear mapping $G(u, t)$ is the corresponding time-variant Nemytskii mapping induced by $g(u, t)$ and defined on the same phase space E. Based on Theorem 4, we can invoke the proved absorbing property of the solution semiflow of the original problem (64) and then apply Theorem 1 to this modified problem (94), which shares exactly the same dynamics with the original problem (64) in the absorbing set B_∞, to establish the following result.

Theorem 5. *Under the assumptions made on* $A(t)$ *and* $f(u, t)$ *in this section, for any one-dimensional bounded domain and some two-dimensional bounded domain* [19, *Section* 8.6], *such as* $\Omega = (0, \ell_1) \times (0, \ell_2)$ *with* $(\ell_1/\ell_2)^2$ *being rational, the assumption* (H3) *and the spectral gap condition* (3) *are satisfied with* K *given by* (91). *Thus there* *exists an inertial manifold* $\mathfrak{M}$ *in the space* $E = L^2(\Omega) \times \Re$ *for the skew product semiflow* π *defined in* (65) *and generated by the solutions of the nonautonomous reaction-diffusion equation* (64).

Proof. For the truncated reaction-diffusion equation (93) and the associated nonautonomous evolutionary equation (94), the global and uniform Lipschitz condition in (H2) is satisfied:

$$\| G(u_1, t) - G(u_2, t) \| = \| g(u_1(\cdot), t) - g(u_2(\cdot), t) \|_{L^2(\Omega)}$$

$$= \left(\int_\Omega \left| \frac{\partial g}{\partial u}(\xi, t) \right|^2 | u_1(x) - u_2(x) |^2 dx \right)^{1/2}$$

$$\leq \left(\sup_{\substack{|\xi| \leq R_0 \\ t \in \Re}} \left| \frac{\partial f}{\partial u}(\xi, t) \right| \right) \| u_1 - u_2 \| \leq K \| u_1 - u_2 \|,$$

where $\xi = \kappa u_1(x) + (1 - \kappa) u_2(x)$ for some $\kappa \in [0, 1]$, which is valid for any $u_1, u_2 \in L^\infty(\Omega) \subset L^2(\Omega)$ and for any $t \in \Re$ due to (92). Under the spectral gap condition as stated in this theorem for the specified 1D and 2D domains which can be confirmed by the assumptions on (62) and this finite K given in (91), Theorem 1 is applicable to this problem (94) and we can assert that an inertial manifold $\mathfrak{M} \subset E$ exists for the skew product semiflow π_g associated with (94) which is confined in the subset $B_\infty \times \Re \subset E$.

We claim that this manifold $\mathfrak{M}$ turns out to be an inertial manifold also for the original skew product semiflow π defined in (65) associated with the solutions of (64). In order to prove this claim, it suffices to show that this $\mathfrak{M}$ satisfies the required exponential attraction property:

$$dist_E((u(t), t), \mathfrak{M}) \leq C(u_0) \exp(-\gamma(t - \tau)), \quad t \geq \tau, \tag{95}$$

where $\gamma > 0$ is a uniform attraction rate for all solutions $u(t) = u(t; u_0, \tau)$ of (64). Here the Hausdorff distances in the space $E = H \times \Re$ and in the space H are denoted by $dist_E$, and $dist_H$, respectively.

In fact, by the definition of this manifold $\mathfrak{M}$, such a uniform attraction rate is guaranteed after any solution trajectory $u(t; u_0, \tau)$ eventually and permanently enters the absorbing set $B_\infty \subset H$, in which the dynamics of the truncated evolutionary equation (94) and the original equation (64) are the same. Let this uniform attraction rate valid in the absorbing set B_∞ be $\beta(> 0)$. It holds that

$$dist_E((u(t), t), \mathfrak{M}) \le C^*(\| u(\tau_\infty) \|) \exp(-\beta(t - \tau_\infty)), \quad t \ge \tau_\infty, \tag{96}$$

where $C^*(r)$ is a continuous positive function and $\tau_\infty (\ge \tau)$ is the time when the solution $u(t; u_0, \tau)$ permanently enters B_∞.

At the end of the proof of Lemma 19, we have shown that the time τ_∞ can be taken as

$$\tau_\infty = \tau_\infty(u_0) = \tau_1(u_0) + 2m + 3,$$

where $\tau_1(u_0)$ is the time when $u(t; u_0, \tau)$ permanently enters the absorbing set $B_1 \subset H$ as specified at the beginning of the proof of Lemma 18. Let

$$d_M = dist_H(B_1, Proj_H \mathfrak{M}),$$

where $Proj_H \mathfrak{M}$ is the orthogonal projection of $\mathfrak{M}$ on H. Since B_1 is a closed, bounded ball in H, this d_M must be a finite constant. In view of (70) and the specification of B_1, we see that

$$dist_H(u(t; u_0, \tau), B_1) \le \| u_0 \| \exp(-\alpha(t - \tau)), \text{ for } t \in [\tau, \tau_1(u_0)]. \tag{97}$$

When $t \ge \tau_1(u_0)$, $u(t)$ will stay inside B_1 forever, so that

$$\begin{aligned} &dist_E((u(t), t), \mathfrak{M}) \le \| u_0 \| \exp(-\alpha(t - \tau)) + d_M \\ &\le \{\| u_0 \| + d_M \exp(\alpha[\tau_\infty(u_0) - \tau])\} \exp(-\alpha(t - \tau)), \text{ for } t \in [\tau, \tau_\infty(u_0)]. \end{aligned} \tag{98}$$

Assembling together the result (98) during the time interval $[\tau, \tau_\infty(u_0)]$ and the result (96) for $t \ge \tau_\infty(u_0)$, and noting that $\| u(\tau_\infty) \| \le 1 + \alpha^{-1} C_1 | \Omega |$, we finally proved the exponential attraction inequality (95) with the uniform constant attraction rate

$$\gamma = \min\{\alpha, \beta\}$$

and the coefficient constant

$$C(u_0) = \max\{\| u_0 \| + d_M \exp(\alpha[\tau_\infty(u_0) - \tau]), C^*(1 + \alpha^{-1}C_1|\Omega|)\exp(\beta[\tau_\infty(u_0) - \tau])\}.$$

Thus the proof is completed.

Furthermore we can extend the existence result on inertial manifolds shown in Theorem 5 to more general nonautonomous reaction-diffusion equations

$$\frac{\partial u}{\partial t} - A(t)u + f(u, x, t) = h(x, t), \; t \geq \tau, \; x \in \Omega$$

$$u = 0, \; t \geq \tau, \; x \in \partial\Omega,$$

$$u(x, \tau) = u_0(x) \in H = L^2(\Omega),$$

with the same assumptions on Ω and $A(t)$ and similar assumptions on $f(u, x, t)$ as in Theorem 5, where $h(x, t)$ is any given bounded measurable function or satisfying even more general conditions. The proof will be similar to what we have shown above with some modifications to take into account the additional inhomogeneous term $h(x, t)$.

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