Far East J. Math. Sci. (FJMS) **20**(2) (2006), 179-188

GLOBAL STABILITY FOR NONLINEAR DELAY DIFFERENTIAL EQUATIONS

YUEHUI PENG

(Received September 22, 2005)

Submitted by Sui Sun Cheng

Abstract

In this paper, a sufficient condition is established for the global asymptotic stability of the nonlinear delay differential equation

$$x'(t) + a(t)x(t-\tau) = b(t)f(x(t-\sigma)), \quad t \geq 0,$$

which generalizes and improves some existing results in the literature.

1. Introduction

It is well known [1, 2, 6] that every solution of the first order nonlinear delay differential equation with instantaneous term

$$x'(t) + a(t)x(t) = b(t)f(x(t-\sigma)), \quad t \geq 0 \tag{1.1}$$

tends to zero as $t \to \infty$, if there exists a $c \in [0, 1)$ such that

$$|b(t)f(u)| \leq ca(t)|u|, \tag{1.2}$$

and

$$\int_0^\infty a(s)ds = \infty, \tag{1.3}$$

2000 Mathematics Subject Classification: 34K11, 34K20.

Keywords and phrases: asymptotic stability, nonlinear delay differential equation.

This work was supported by NNSF of China.

where $\sigma \in [0, \infty)$, $a, b \in C([0, \infty), \mathbf{R})$, $f \in C(\mathbf{R}, \mathbf{R})$. When incorporating delay into the instantaneous term $a(t)x(t)$, we have the following nonlinear pure delay differential equation

$$x'(t) + a(t)x(t - \tau) = b(t)f(x(t - \sigma)), \quad t \geq 0, \tag{1.4}$$

where $\tau \in [0, \infty)$. In paper [3], the authors extended the above result for Eq. (1.1) to Eq. (1.4), i.e., they proved that if (1.2) and (1.3) hold and

$$0 \leq \tau \sup_{t \in [0, \infty)} a(t) < \frac{1}{e}, \tag{1.5}$$

then every solution of Eq. (1.4) tends to zero as $t \to \infty$.

When $c = 0$, Eq. (1.4) reduces to

$$x'(t) = -a(t)x(t - \tau), \quad t \geq 0. \tag{1.6}$$

In this case, [4, 7-9] proved that if (1.3) holds and that

$$\limsup_{t \to \infty} \int_{t-\tau}^{t} a(s)ds < \frac{3}{2}, \tag{1.7}$$

then every solution of Eq. (1.6) tends to zero as $t \to \infty$. Obviously, condition (1.7) is weaker than (1.5) when $c = 0$. So, one would naturally expect that (1.5) can be also weakened when $c > 0$ is small. This constitutes the purpose of this paper. In fact, we establish the following theorem by using the basic ideas of [4, 5, 6] and some new techniques.

Theorem 1.1. *Assume that* (1.2) *and* (1.3) *hold, and that*

$$\limsup_{t \to \infty} \int_{t-\tau}^{t} a(s)ds < \begin{cases} (3 - c)/2(1 + c), & \text{if } 0 \leq c < 1/3, \\ \sqrt{2(1 - c)/(1 + c)}, & \text{if } 1/3 \leq c < 1. \end{cases} \tag{1.8}$$

Then every solution of Eq. (1.4) *tends to zero as* $t \to \infty$.

Compare (1.5) with (1.8), we see that condition (1.8) is better than (1.5) when $0 \leq c \leq (2e^2 - 1)/(2e^2 + 1) \approx 0.87324$. And one easily sees that (1.8) reproduces (1.7) when $c = 0$.

2. Proof of Theorem 1.1

Lemma 2.1. *Assume that* (1.2) *holds, and that*

$$\int_{t-\tau}^{t} a(s)ds < \begin{cases} (3-c)/2(1+c), & \text{if } 0 \le c < 1/3, \\ \sqrt{2(1-c)/(1+c)}, & \text{if } 1/3 \le c < 1. \end{cases} \tag{2.1}$$

Then every solution of Eq. (1.4) *is bounded.*

Proof. If not, assume that $\limsup_{t\to\infty}|x(t)| = \infty$, then there exists a large $T > 2(\tau + \sigma)$ such that $|x(T)| > |x(t)|$ for $t \in [\min\{-\tau, -\sigma\}, T)$. Without loss of the generality, we may assume that $x(T) = |x(T)|$. Note that $x(T) > cx(T)$. We can prove that $x(T-\tau) \le cx(T)$. Otherwise, $x(T-\tau) > cx(T)$. By the continuous of $x(t)$, there exists a $T_1 < T$ such that $x(t-\tau) > cx(T)$ for $T_1 \le t \le T$. Hence, from (1.2) and (1.4), we have

$$x'(t) = -a(t)x(t-\tau) + b(t)f(x(t-\sigma)) \le a(t)[-x(t-\tau) + cx(T)] \le 0, \ \ T_1 \le t \le T,$$

which implies that $x(t)$ is not increasing on $[T_1, T]$. This contradicts to the definition of T. Hence, there exists a $\xi \in [T-\tau, T)$ such that $x(\xi) = cx(T)$. From (1.2) and (1.4), we have

$$x'(t) \le -a(t)x(t-\tau) + cx(T)a(t) \le (1+c)x(T)a(t), \ \ t \le T. \tag{2.2}$$

For $\xi \le t \le T$, by (2.2), we have

$$cx(T) - x(t-\tau) \le (1+c)x(T)\int_{t-\tau}^{\xi} a(\mu)d\mu, \ \ \xi \le t \le T.$$

Substituting this into the first inequality in (2.2), we have

$$x'(t) \le (1+c)x(T)a(t)\int_{t-\tau}^{\xi} a(s)ds, \ \ \xi \le t \le T. \tag{2.3}$$

Let

$$A = \begin{cases} (3-c)/2(1+c), & \text{if } 0 \le c < 1/3, \\ \sqrt{2(1-c)/(1+c)}, & \text{if } 1/3 \le c < 1. \end{cases} \tag{2.4}$$

There are three possible cases to consider:

Case 1. $c < 1/3$ and $\int_{\xi}^{T} a(s)ds < 1 \le A$. In this case, integrating (2.3) from ξ to T and using (2.1), we have

$$\begin{aligned} x(T) &= x(\xi) + \int_{\xi}^{T} x'(t)dt \\ &\le cx(T) + (1+c)x(T)\int_{\xi}^{T} a(t)\int_{t-\tau}^{\xi} a(s)dsdt \\ &\le cx(T) + (1+c)x(T)\int_{\xi}^{T} a(t)\left(A - \int_{\xi}^{t} a(s)ds\right)dt \\ &\le cx(T) + (1+c)x(T)\left[A\int_{\xi}^{T} a(s)ds - \frac{1}{2}\left(\int_{\xi}^{T} a(s)ds\right)^2\right] \\ &< cx(T) + (1+c)x(T)\left(A - \frac{1}{2}\right) \\ &= x(T). \end{aligned}$$

Case 2. $c < 1/3$ and $\int_{\xi}^{T} a(s)ds \ge 1$. Then there exists an $\eta \in [\xi, T)$ such that $\int_{\eta}^{T} a(s)ds = 1$. Integrating (2.2) and (2.3) and using (2.1), we have

$$\begin{aligned} x(T) &= x(\xi) + \int_{\xi}^{\eta} x'(t)dt + \int_{\eta}^{T} x'(t)dt \\ &\le cx(T) + (1+c)x(T)\left[\int_{\xi}^{\eta} a(s)ds + \int_{\eta}^{T} a(t)\int_{t-\tau}^{\xi} a(s)dsdt\right] \\ &= cx(T) + (1+c)x(T)\left[\int_{\eta}^{T} a(t)\int_{\xi}^{\eta} a(s)dsdt + \int_{\eta}^{T} a(t)\int_{t-\tau}^{\xi} a(s)dsdt\right] \\ &= cx(T) + (1+c)x(T)\int_{\eta}^{T} a(t)\int_{t-\tau}^{\eta} a(s)dsdt \\ &< cx(T) + (1+c)x(T)\left[A\int_{\eta}^{T} a(s)ds - \frac{1}{2}\left(\int_{\eta}^{T} a(s)ds\right)^2\right] \\ &= cx(T) + (1+c)x(T)\left(A - \frac{1}{2}\right) \\ &= x(T). \end{aligned}$$

Case 3. $1/3 \le c < 1$ and $\int_{\xi}^{T} a(s)ds < A \le 1$. In this case, integrating (2.3) from ξ to T and using (2.1), we have

$$\begin{aligned}
x(T) &= x(\xi) + \int_{\xi}^{T} x'(t)dt \\
&\le cx(T) + (1+c)x(T)\int_{\xi}^{T} a(t)\int_{t-\tau}^{\xi} a(s)dsdt \\
&\le cx(T) + (1+c)x(T)\int_{\xi}^{T} a(t)\left(A - \int_{\xi}^{t} a(s)ds\right)dt \\
&\le cx(T) + (1+c)x(T)\left[A\int_{\xi}^{T} a(s)ds - \frac{1}{2}\left(\int_{\xi}^{T} a(s)ds\right)^2\right] \\
&< cx(T) + \frac{1}{2}(1+c)x(T)A^2 \\
&= x(T).
\end{aligned}$$

Combining Case 1, Case 2 and Case 3, we have concluded a contradiction, and so the proof is complete.

We are now in a position to show our main result.

Proof of Theorem 1.1. When $c = 0$, Theorem 1.1 is known, so we assume that $c \in (0, 1)$ in the sequel. Set $\mu = \limsup_{t\to\infty} |x(t)|$. It follows from Lemma 2.1 that $\mu \in [0, \infty)$. We shall prove $\mu = 0$ in two cases.

Case 1. $x'(t)$ is nonoscillatory. Then $x(t)$ is increasing or decreasing eventually. This implies that the limit $\lim_{t\to\infty} |x(t)| = \mu$ exists. There are two possible subcases.

Subcase 1. $\limsup_{t\to\infty} x(t) = -\mu$. Then from (1.2) and (1.4),

$$\begin{aligned}
-\mu - x(t) &= \int_{t}^{\infty} [-a(s)x(s-\tau) + b(s)(x(s-\sigma))]ds \\
&\ge \int_{t}^{\infty} [-a(s)x(s-\tau) - |b(s)f(x(s-\sigma))|]ds \\
&\ge \int_{t}^{\infty} a(s)[-x(s-\tau) - c|x(s-\sigma)|]ds, \quad t \ge T.
\end{aligned}$$

Note that

$$\lim_{s\to\infty}[-x(s-\tau)-c|\,x(s-\sigma)\,|] = (1-c)\mu.$$

It follows from (1.3) that $\mu = 0$.

Subcase 2. $\limsup_{t\to\infty} x(t) = \mu$. Then from (1.2) and (1.4),

$$\mu - x(t) = \int_t^{\infty}[-a(s)x(s-\tau)+b(s)(x(s-\sigma))]ds$$

$$\leq \int_t^{\infty}[-a(s)x(s-\tau)+|\,b(s)f(x(s-\sigma))\,|]ds$$

$$\leq \int_t^{\infty}a(s)[-x(s-\tau)+c|\,x(s-\sigma)\,|]ds, \quad t \geq T.$$

Note that

$$\lim_{s\to\infty}[-x(s-\tau)+c|\,x(s-\sigma)\,|] = -(1-c)\mu.$$

It follows from (1.3) that $\mu = 0$. Combining both Subcase 1 and Subcase 2, we have $\mu = 0$.

Case 2. $x'(t)$ is oscillatory. Assume that $\mu > 0$ and let

$$1-c < A < \begin{cases} (3-c)/2(1+c), & \text{if } 0 \leq c < 1/3, \\ \sqrt{2(1-c)/(1+c)}, & \text{if } 1/3 \leq c < 1, \end{cases}$$

and let $\varepsilon \in (0, (1-c)\mu/2(1+c))$ be any positive given number. Then it follows from (1.8) and the definition of μ that there exists a $T > t_0$ such that

$$\int_{t-\tau}^{t} a(s)ds \leq A, \quad t \geq T, \tag{2.5}$$

and

$$|\,x(t)\,| < (\mu+\varepsilon), \quad t \geq T. \tag{2.6}$$

Choose an increasing sequence $\{t_n\}$ with $t_n \geq T + \tau + \sigma$, $t_n \to \infty$, $n \to \infty$ such that $\lim_{n\to\infty} |x(t_n)| = \mu$, $|x(t_n)| > c(\mu + \varepsilon)$, $x'(t_n) = 0$ and $|x(t_n)|$ is left local maximum point for $n = 1, 2, \ldots$. Similar to the proof of Lemma 2.1, it is easy to prove that there exists $\xi_n \in [t_n - \tau, t_n)$ such that $x(\xi_n) = c(\mu + \varepsilon)$. By (1.2), (1.4) and (2.6), we have

$$x'(t) \leq -a(t)x(t-\tau) + c(\mu+\varepsilon)a(t) \leq (1+c)(\mu+\varepsilon)a(t), \quad t \geq T. \tag{2.7}$$

For $\xi_n \leq t \leq t_n$, by (2.7), we have

$$c(\mu+\varepsilon) - x(t-\tau) \leq (1+c)(\mu+\varepsilon)\int_{t-\tau}^{\xi_n} a(\mu)d\mu, \quad \xi_n \leq t \leq t_n.$$

Substituting this into the first inequality in (2.7), we have

$$x'(t) \leq (1+c)(\mu+\varepsilon)a(t)\int_{t-\tau}^{\xi_n} a(s)ds, \quad \xi_n \leq t \leq t_n. \tag{2.8}$$

There are three possible subcases to consider:

Subcase 1. $c < 1/3$ and $\int_{\xi_n}^{t_n} a(s)ds < 1 \leq A$. In this case, integrating (2.8) from ξ_n to t_n and using (2.5), we have

$$\begin{aligned} x(t_n) - c(\mu+\varepsilon) &= \int_{\xi_n}^{t_n} x'(t)dt \\ &\leq (1+c)(\mu+\varepsilon)\int_{\xi_n}^{t_n} a(t)\int_{t-\tau}^{\xi_n} a(s)dsdt \\ &= (1+c)(\mu+\varepsilon)\int_{\xi_n}^{t_n} a(t)\left(\int_{t-\tau}^{t} a(s)ds - \int_{\xi_n}^{t} a(s)ds\right)dt \\ &\leq (1+c)(\mu+\varepsilon)\left[A\int_{\xi_n}^{t_n} a(s)ds - \frac{1}{2}\left(\int_{\xi_n}^{t_n} a(s)ds\right)^2\right] \\ &\leq (1+c)\left(A - \frac{1}{2}\right)(\mu+\varepsilon). \end{aligned}$$

Subcase 2. $c < 1/3$ and $\int_{\xi_n}^{t_n} a(s)ds \geq 1$. Then there exists an $\eta_n \in (\xi_n, t_n)$ such that $\int_{\eta_n}^{t_n} a(s)ds = 1$. Integrating (2.7) and (2.8) and using (2.5), we have

$$x(t_n) - c(\mu + \varepsilon)$$

$$= \int_{\xi_n}^{\eta_n} x'(t)dt + \int_{\eta_n}^{t_n} x'(t)dt$$

$$\leq (1 + c)(\mu + \varepsilon)\left[\int_{\xi_n}^{\eta_n} a(s)ds + \int_{\eta_n}^{t_n} a(t)\int_{t-\tau}^{\xi_n} a(s)dsdt\right]$$

$$= (1 + c)(\mu + \varepsilon)\left[\int_{\eta_n}^{t_n} a(t)\int_{\xi_n}^{\eta_n} a(s)dsdt + \int_{\eta_n}^{t_n} a(t)\int_{t-\tau}^{\xi_n} a(s)dsdt\right]$$

$$= (1 + c)(\mu + \varepsilon)\int_{\eta_n}^{t_n} a(t)\int_{t-\tau}^{\eta_n} a(s)dsdt$$

$$= (1 + c)(\mu + \varepsilon)\left[\int_{\eta_n}^{t_n} a(t)\left(\int_{t-\tau}^{t} a(s)ds - \int_{\eta_n}^{t} a(s)ds\right)dt\right]$$

$$\leq (1 + c)(\mu + \varepsilon)\left[A\int_{\eta_n}^{t_n} a(s)ds - \frac{1}{2}\left(\int_{\eta_n}^{t_n} a(s)ds\right)^2\right]$$

$$= (1 + c)\left(A - \frac{1}{2}\right)(\mu + \varepsilon).$$

Subcase 3. $1/3 \leq c < 1$ and $\int_{\xi_n}^{t_n} a(s)ds < A \leq 1$. In this case, integrating (2.8) from ξ_n to t_n and using (2.5), we have

$$x(t_n) - c(\mu + \varepsilon) = \int_{\xi_n}^{t_n} x'(t)dt$$

$$\leq (1 + c)(\mu + \varepsilon)\int_{\xi_n}^{t_n} a(t)\int_{t-\tau}^{\xi_n} a(s)dsdt$$

$$\le (1+c)(\mu+\varepsilon)\int_{\xi_n}^{t_n} a(t)\left(A - \int_{\xi_n}^{t} a(s)ds\right)dt$$

$$\le (1+c)(\mu+\varepsilon)\left[A\int_{\xi_n}^{t_n} a(s)ds - \frac{1}{2}\left(\int_{\xi_n}^{t_n} a(s)ds\right)^2\right]$$

$$\le \frac{1}{2}(1+c)A^2(\mu+\varepsilon).$$

Subcases 1, 2 and 3 imply

$$x(t_n) - c(\mu+\varepsilon) \le \begin{cases}(1+c)(A-1/2)(\mu+\varepsilon), & \text{if } 0 \le c < 1/3,\\ (1+c)A^2(\mu+\varepsilon)/2, & \text{if } 1/3 \le c < 1.\end{cases}$$

Let $n \to \infty$ and $\varepsilon \to 0$. Then we obtain

$$1 - c \le \begin{cases}(1+c)(A-1/2), & \text{if } 0 \le c < 1/3,\\ (1+c)A^2/2, & \text{if } 1/3 \le c < 1,\end{cases}$$

which yields

$$A \ge \begin{cases}(3-c)/2(1+c), & \text{if } 0 \le c < 1/3,\\ \sqrt{2(1-c)/(1+c)}, & \text{if } 1/3 \le c < 1.\end{cases}$$

This is a contradiction, and so $\mu = 0$. The proof is complete.

References

[1] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic Publishers, 1992.

[2] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.

[3] B. Shi, M. J. Gai, D. C. Zhang and J. C. Zhai, Global attractivity in nonlinear differential equations with delays, Nonlinear Anal. 47 (2001), 4071-4082.

[4] J. W. H. So, J. S. Yu and M. P. Chen, Asymptotic stability for scalar delay differential equations, Funkcial. Ekvac. 39 (1996), 1-17.

[5] X. H. Tang, Asymptotic behavior of a differential equation with distributed delay, J. Math. Anal. Appl. 301 (2005), 313-335.

[6] X. H. Tang, Asymptotic behavior of a differential equation with instantaneously terms, J. Math. Anal. Appl. 302 (2005), 342-359.

[7] T. Yoneyama, Uniform stability for one-dimensional delay-differential equations with dominant delayed term, Tohoku Math. J. (2) 41(2) (1989), 217-236.

[8] T. Yoneyama, On the 3/2 stability theorem for one-dimensional delay-differential equations with unbounded delay, J. Math. Anal. Appl. 165 (1992), 133-143.

[9] T. Yoneyama and J. Sugie, On the stability region of scalar delay-differential equations, J. Math. Anal. Appl. 134 (1988), 408-425.

Department of Mathematics
Shaoyang University
Shaoyang, Hunan 422000, P. R. China
e-mail: pengyh2004@sina.com